GRAPH COLORING

Project report submitted to

The Kannur University

for the award of the degree

of

Bachelor of Science

by

AGNUS CHACKO

DB18CMSR01

Under the guidance of

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March 2021

Examiners 1: Examiner 2:

CERTIFICATE

It is to certify that this project report ‘**GRAPH COLORING**’ is the bona fide project of **AGNUS CHACKO** who carried out the project under my supervision.

Mrs. Riya Baby

Supervisor, HOD

Department of Mathematics

Don Bosco Arts and Science College

Angadikadavu

DECLARATION

I, AGNUS CHACKO, hereby declare that this project report entitled "**GRAPH COLORING**" is an original record of studies and bona fide project carried out by me during the period from November 2019 to March 2020, under the guidance of **Mrs. Riya Baby**, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not been submitted by me elsewhere for the award of any degree, diploma, title or recognition, before.

AGNUS CHACKO

Department of Mathematics

Don Bosco Arts and Science College

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ACKNOWLEDGEMENT

I sincerely express my deep sense of gratitude to all who have been of great help to me during the course of my dissertation. First and foremost I thank the almighty, for his blessing and protection during the period of this work. I express my thanks to Dr. Fr. Francies Karakkatt, Principal, for support in the completion of this dissertation. I express my gratitude to Mrs. Riya Baby, my project supervisor, for the constant encouragement, valuable guidance and timely corrections, which made this work a success.

I am also indebted to all my classmates and friends who supported me throughout the study. I would like to express my thanks to my parents and dear ones for their constant encouragement and support. I also thank all those who helped me directly or indirectly to complete this project.

AGNUS CHACKO

CONTENTS

INTRODUCTION

A proper coloring of a graph is an assignment of colors to the vertices of the graph so that no two adjacent vertices have the same color.

Usually we drop the word “proper” unless other types of coloring are also under discussion. Of course, the “colors” don’t have to be actual colors ; may can be any distinct labels - integers ,for examples , if a graph is not connected , each connected component can be colored independently; except where otherwise noted , we assume graphs are connected. We also assume graphs are simple in this section. Graph coloring has many applications in addition to its intrinsic interest.

In the same way the most important concept of graph coloring is utilized in resource allocation, scheduling. Also, paths, walks and circuits in graph theory are used in tremendous applications say travelling salesman problem, database design concepts, resource networking.

This project deals with coloring which is one of the most important topics in graph theory. In this project there are three chapters. First chapter is coloring . The second chapter is chromatic number. The last chapter deals with application of graph coloring.

BASIC CONCEPTS

1. GRAPH

A graph is an ordered triplet. G=(V(G), E(G), I(G)); V(G) is a non empty set, E(G) is a set disjoint from V(G) and I(G) is an incidence map that associates each element of E(G) and unrecorded pair of element of V(G). The elements of V(G) are called vertices (or nodes or points) of G and the elements of E(G) are Called edges or lines of G.

2. MULTIPLE EDGE / PARALLEL EDGE

A set of 2 or more edges of a graph G is called a multiple edge or parallel edge if they have the same end vertices.

3. LOOP

An edge for which the 2 end vertices are same is called a loop.

4. SIMPLE GRAPH

A graph is simple if it has no loop and no multiple edges.

5. DEGREE

Let G be a graph and v € V the number of edge incident at V in G is called the degree or vacancy of the vertex v in G.

CHAPTER - 1

COLORING

Graph coloring is nothing but a simple way of labeling graph components such as vertices , edges and regions under some constraints. In a graph, no two adjacent vertices, adjacent edges , or adjacent regions are colored with minimum number of colors .This number is called the chromatic number and the graph is called properly colored graph.

In graph theory coloring is a special case of graph labeling; it is an assignment of labels traditionally called “colors” to elements of a graph subject to certain constraints. In it simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color, it is called vertex coloring. Similarly, edge coloring assigns a color to each edge so that no two adjacent edges share the common color.

While graph coloring , the constraints that are set on the graph are colors , order of coloring , the way of assigning color , etc. A coloring is given to a vertex or a particular region . Thus, the vertices or regions having same colors form independent sets.

VERTEX COLORING

Vertex coloring is an assignment of colors to the vertices of a graph ‘G’ such that no two adjacent vertices have the same color .Simply put , no two vertices of an edge should be of the same color.

The most common type of vertex coloring seeks to minimize the number of colors for a given graph . Such a coloring is known as a minimum vertex coloring , and the minimum number of colors which with the vertices of a graph may be colored is called the chromatic number .

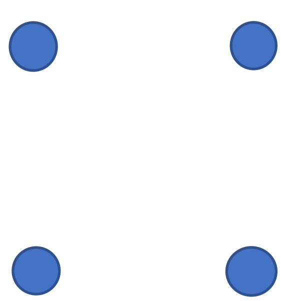
CHROMATIC NUMBER:

The minimum number of colors required for vertex coloring of graph ‘G’ is called as the chromatic number of G , denoted by X (G) .

X(G) = 1 iff ‘G’ is a null graph. If ‘G’ is not a null graph , then X(G) ≥ 2.

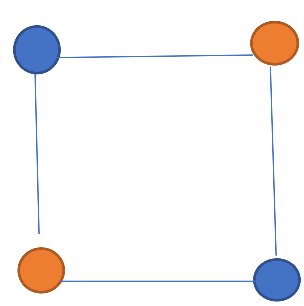
EXAMPLES;

1.



Null Graph (X (G) = 1)

2.

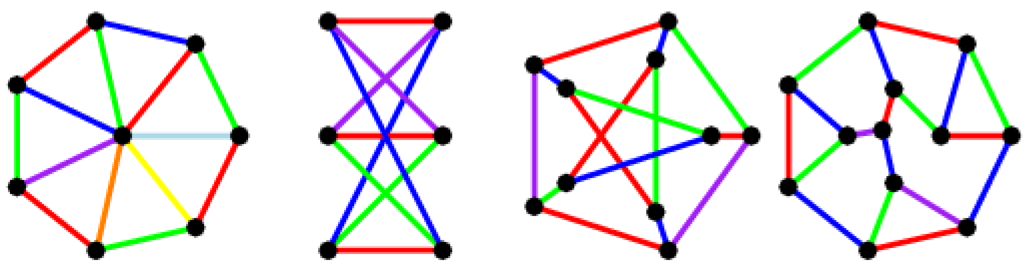


Not Null Graph (X (G) = 2)

EDGE COLORING

An edge coloring of a graph G is a coloring of the edges of G such that adjacent edges (or the edges bounding different regions) receive different colors. An edge coloring containing the smallest possible number of colors for a given graph is known as a minimum edge coloring.

The edge chromatic number gives the minimum number of colours with which graph's edges can be colored.



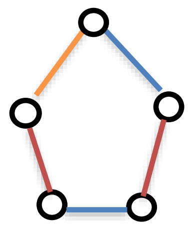
CHROMATIC INDEX

The minimum number of colors required for proper edge coloring of graph is called chromatic index.

A complete graph is the one in which each vertex is directly connected with all other vertices with an edge. If the number of vertices of a complete graph is n, then the chromatic index for an odd number of vertices will be n and the chromatic index for even number of vertices will be n-1.

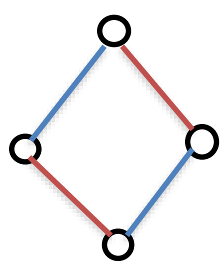
EXAMPLES;

1.



The given graph will require 3 unique colors so that no two incident edges have the Same color. So its chromatic index will be 3.

2.



The given graph will require 2 unique colors so that no two incident edges have the same color. So its chromatic index will be 2.

CHAPTER 2

Chromatic Number

The chromatic number of a graph is the smallest number of colors needed to color the vertices of so that no two adjacent vertices share the same color. That is the smallest value of possible to obtain a k-coloring.

- Graph Coloring is a process of assigning colors to the vertices of a graph.
- It ensures that no two adjacent vertices of the graph are colored with the same color.
- Chromatic Number is the minimum number of colors required to properly color any graph.

Graph Coloring Algorithm

- There exists no efficient algorithm for coloring a graph with minimum number of colors.

However, a following greedy algorithm is known for finding the chromatic number of any given graph.

Greedy Algorithm

<u>Step-01:</u>

Color first vertex with the first color.

Step-02:

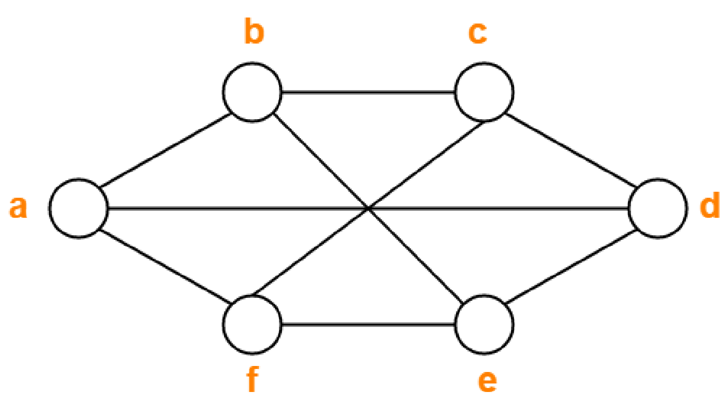
Now, consider the remaining (V-1) vertices one by one and do the following-

- Color the currently picked vertex with the lowest numbered color if it has not been used to color any of its adjacent vertices.
- If it has been used, then choose the next least numbered color.
- If all the previously used colors have been used, then assign a new color to the currently picked vertex.

Problems Based On Finding Chromatic Number of a Graph

Problem-01:

Find chromatic number of the following graph-



Solution-

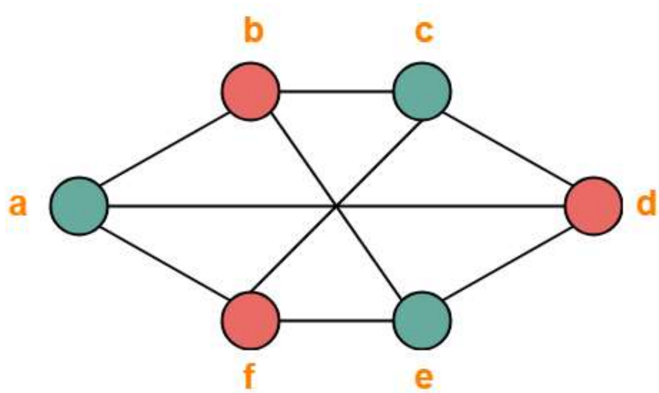
Applying Greedy Algorithm, we have

Vertex	a	B	C	d	e	f
Color	C1	C2	C1	C2	C1	C2

From here,

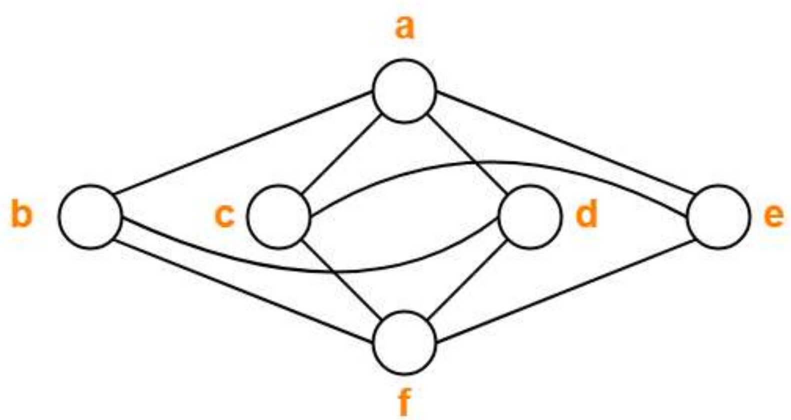
- Minimum numbers of colors used to color the given graph are 2.
- Therefore, Chromatic Number of the given graph = 2.

The given graph may be properly colored using 2 colors as shown below-



Problem-02:

Find chromatic number of the following graph-



Solution-

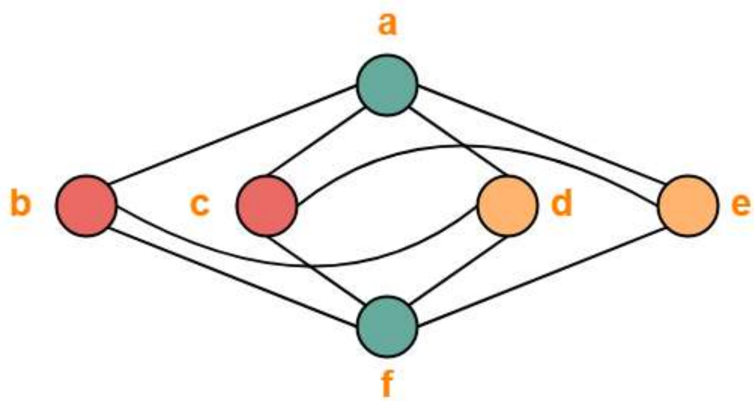
Applying Greedy Algorithm, we have-

Vertex	a	b	C	d	e	f
Color	C1	C2	C2	C3	C3	C1

From here,

- Minimum numbers of colors used to color the given graph are 3.
- Therefore, Chromatic Number of the given graph = 3.

The given graph may be properly colored using 3 colors as shown below-



Chromatic Number of Graphs

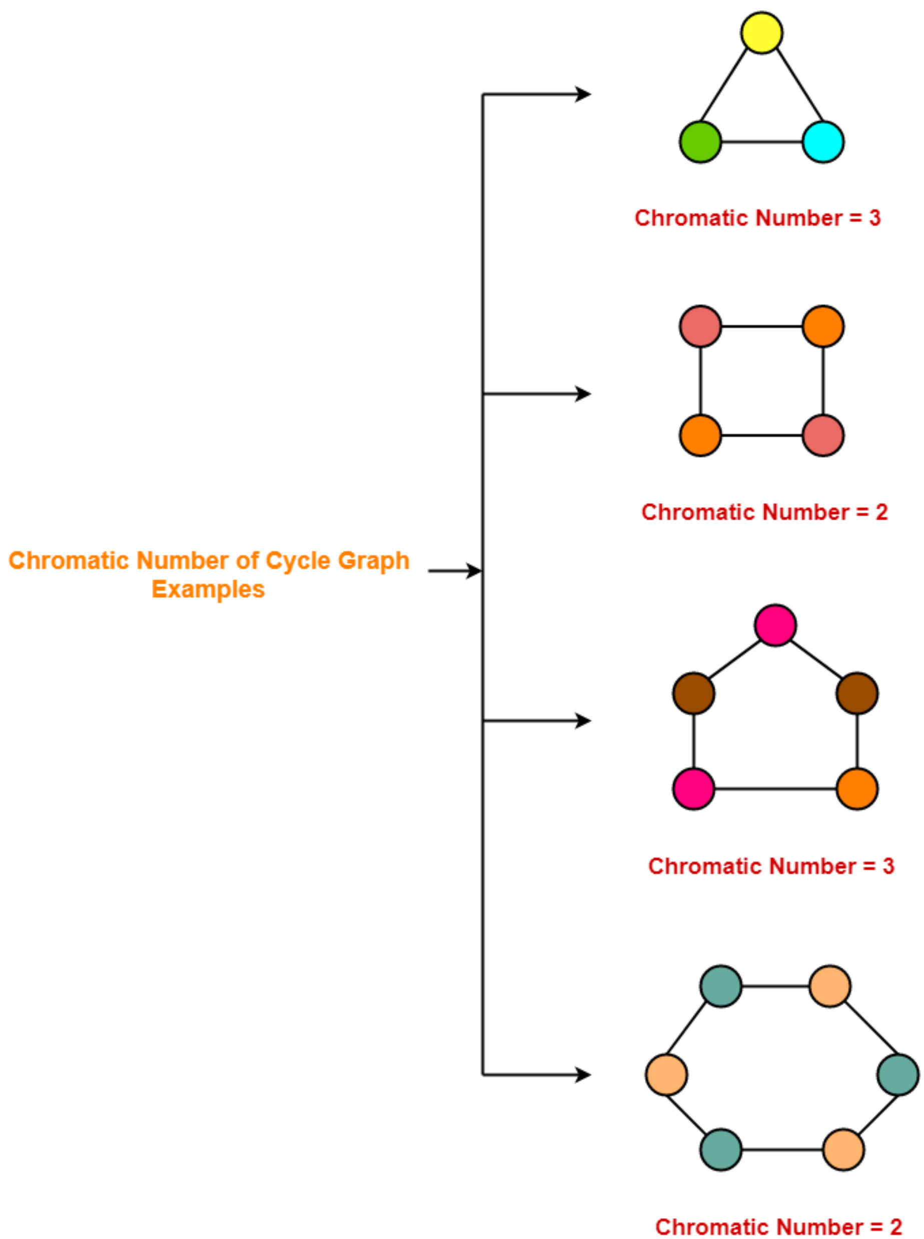
Chromatic Number of some common types of graphs are as follows-

1. Cycle Graph-

- A simple graph of 'n' vertices (n>=3) and 'n' edges forming a cycle of length 'n' is called as a cycle graph.
- In a cycle graph, all the vertices are of degree 2.

Chromatic Number
• If number of vertices in cycle graph is even, then its chromatic number = 2. • If number of vertices in cycle graph is odd, then its chromatic number = 3.

Examples-



2. Planar Graphs-

A planar graph is a graph that can be embedded in the plane, that is it can be drawn on the plane in such a way that its edges intersect only at their endpoint. In other words, it can be drawn in such a way that no edges cross each other.

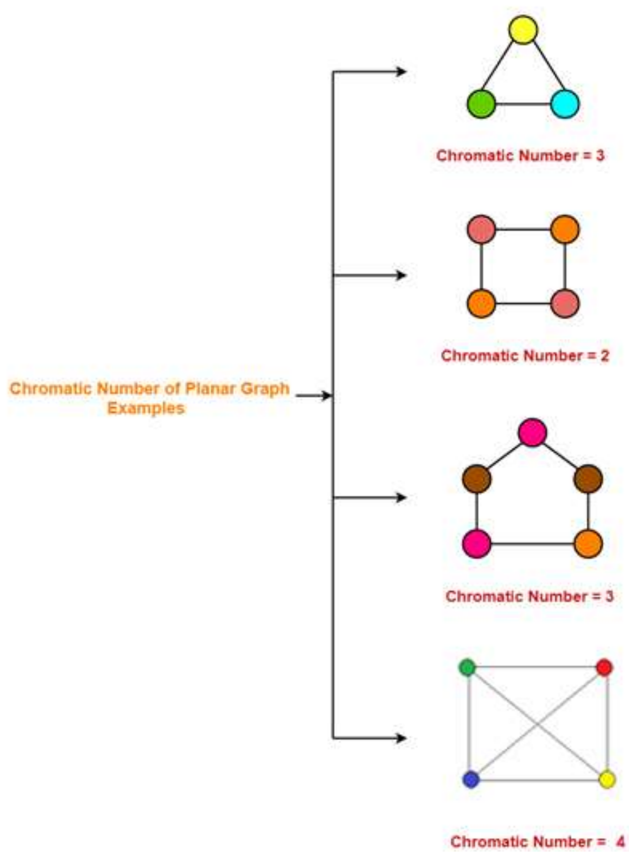
A **Planar Graph** is a graph that can be drawn in a plane such that none of its edges cross each other.

Chromatic Number Chromatic Number of any Planar Graph is less than or equal to 4

Examples-

+

- All the above cycle graphs are also planar graphs.
- Chromatic number of each graph is less than or equal to 4.

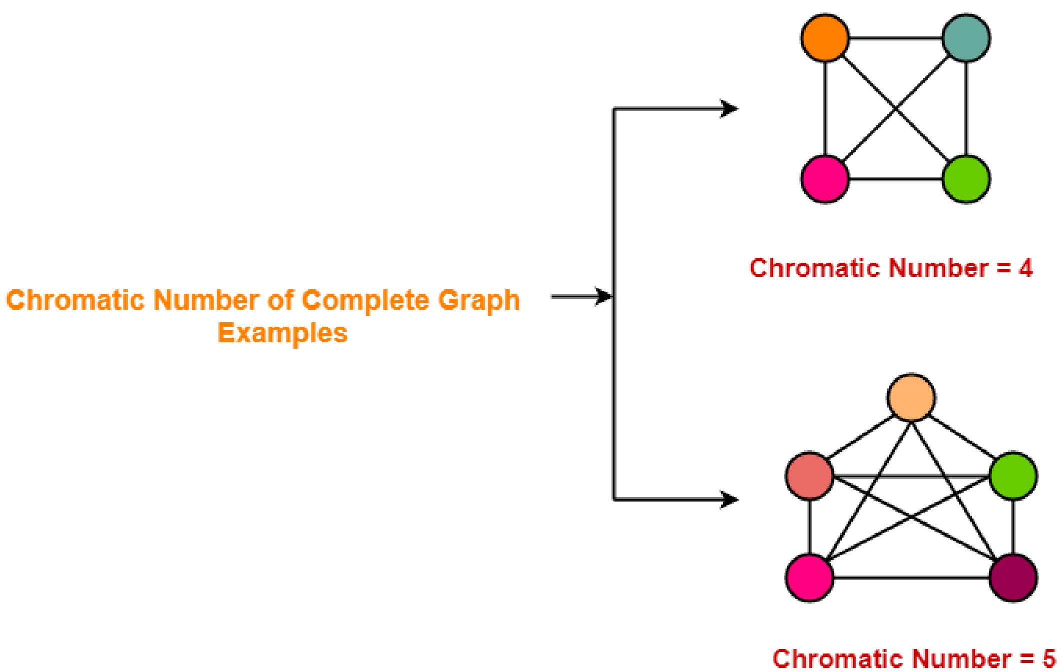


3. Complete Graphs-

- A complete graph is a graph in which every two distinct vertices are joined by exactly one edge.
- In a complete graph, each vertex is connected with every other vertex.
- So to properly it, as many different colors are needed as there are number of vertices in the given graph.

Chromatic Number Chromatic Number of any Complete Graph = Number of vertices in that Complete Graph

Examples-



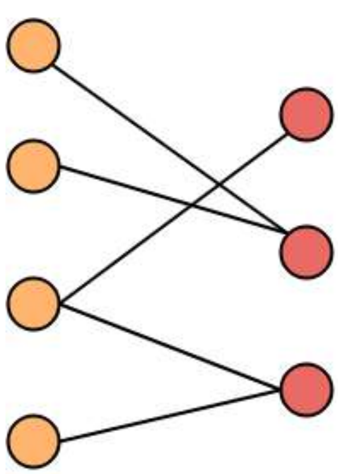
4. Bipartite Graphs-

A bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to one in V. Vertex sets U and V are usually called the parts of the graph.

- A **Bipartite Graph** consists of two sets of vertices X and Y.
- The edges only join vertices in X to vertices in Y, not vertices within a set.

Chromatic Number Chromatic Number of any Bipartite Graph = 2

Example-



Chromatic Number = 2

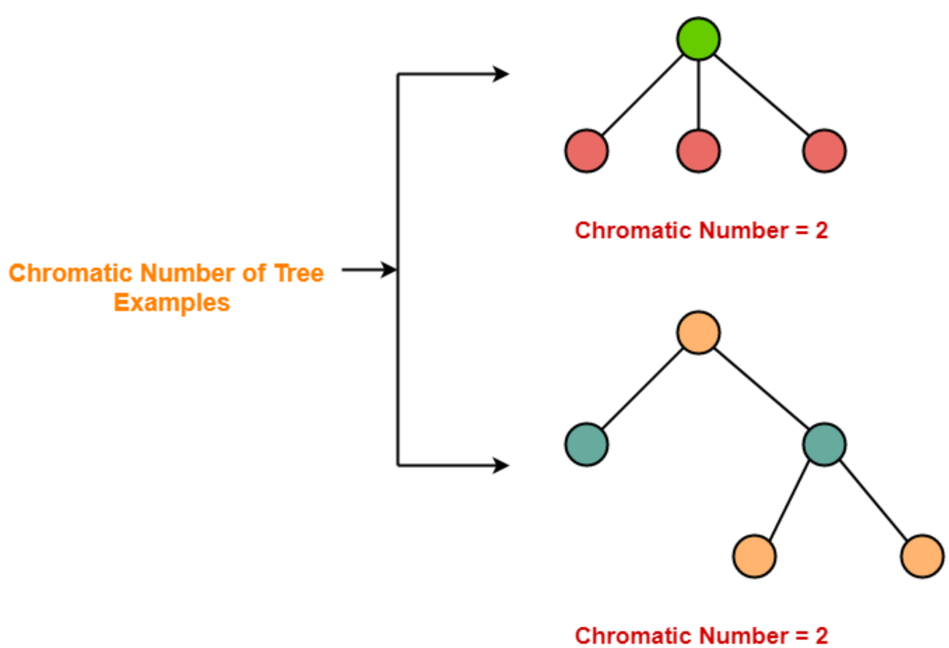
5. Trees-

A tree is an undirected graph in which any two vertices are connected by exactly one path, or equivalently a connected acyclic undirected graph.

- A **Tree** is a special type of connected graph in which there are no circuits.
- Every tree is a bipartite graph.
- So, chromatic number of a tree with any number of vertices = 2.

Chromatic Number Chromatic Number of any tree = 2

Examples-



CHAPTER-3

APPLICATIONS OF GRAPH COLORING

1) Making Schedule or Time Table:

Suppose we want to make an exam schedule for a university. We have list different subjects and students enrolled in every subject. Many subjects would have common students (of same batch, some backlog students, etc). How do we schedule the exam so that no two exams with a common student are scheduled at same time? How many minimum time slots are needed to schedule all exams? This problem can be represented as a graph where every vertex is a subject and an edge between two vertices mean there is a common student. So this is a graph coloring problem where minimum number of time slots is equal to the chromatic number of the graph.

2) Mobile Radio Frequency Assignment:

When frequencies are assigned to towers, frequencies assigned to all towers at the same location must be different. How to assign frequencies with this constraint? What is the minimum number of frequencies needed? This problem is also an instance of graph coloring problem where every tower represents a vertex and an edge between two towers represents that they are in range of each other.

3) Register Allocation:

In compiler optimization, register allocation is the process of assigning a large number of target program variables onto a small number of CPU registers. This problem is also a graph coloring problem.

4) Sudoku:

Sudoku is also a variation of Graph coloring problem where every cell represents a vertex. There is an edge between two vertices if they are in same row or same column or same block.

5) Map Coloring:

Geographical maps of countries or states where no two adjacent cities cannot be assigned same color. Four colors are sufficient to color any map.

6) Bipartite Graphs:

We can check if a graph is bipartite or not by coloring the graph using two colors. If a given graph is 2-colorable, then it is Bipartite, otherwise not. See this for more details.

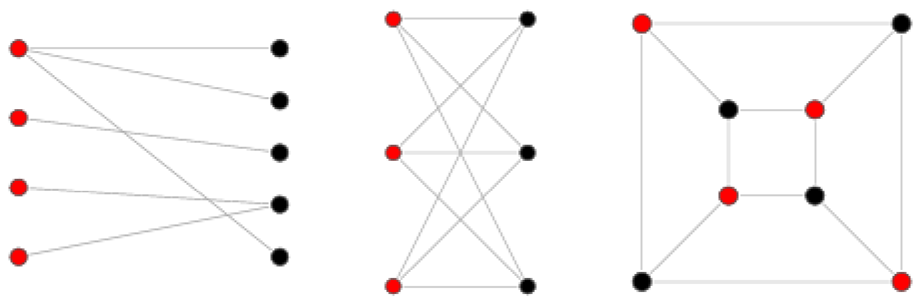
Explanation;

Algorithm:

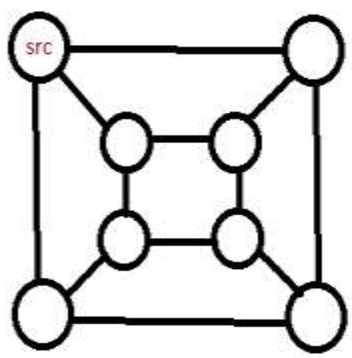
A bipartite graph is possible if it is possible to assign a color to each vertex such that no two neighbour vertices are assigned the same color. Only two colors can be used in this process.

Steps:

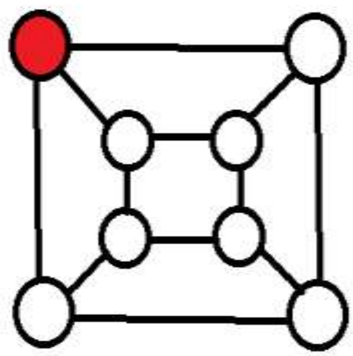
1. Assign a color (say red) to the source vertex.
2. Assign all the neighbours of the above vertex another color (say blue).
3. Taking one neighbour at a time, assign all the neighbour's neighbours the color red.
4. Continue in this manner till all the vertices have been assigned a color.
5. If at any stage, we find a neighbour which has been assigned the same color as that of the current vertex, stop the process. The graph cannot be colored using two colors. Thus the graph is not bipartite.



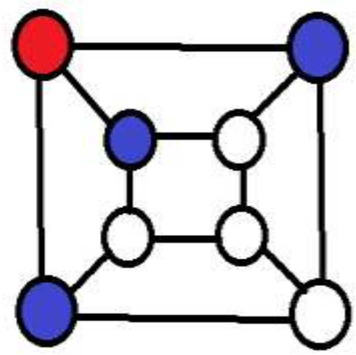
Example:



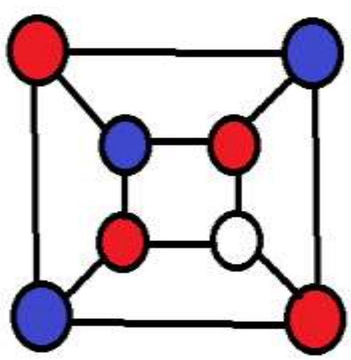
given a graph with source vertex



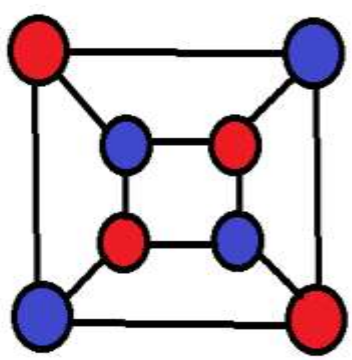
colour src vertex, say red



assign another colour to the neighbours, say blue



assign the neighbours of the vertices of the previous step the colour red



repeat till all vertices are coloured, or a conflicting colour assignment occurs.

set U: red colour
set V: blue colour

CONCLUSION

This project aims to provide a solid background in the basic topics of graph coloring. Graph coloring problem is to assign colors to certain elements of a graph subject to certain constraints. The nature of coloring problem depends on the number of colors but not on what they are.

The study of this topic gives excellent introduction to the subject called “Graph Coloring”.

This project includes two important topics such as vertex coloring and edge coloring and came to know about different ways and importance of coloring.

Graph coloring enjoys many practical applications as well as theoretical challenges. Besides the applications, different limitations can also be set on the graph or on the away a color is assigned or even on the color itself. It has been reached popularity with the general public in the form of the popular number puzzle Sudoku and it is also use in the making of time management which is an important application of coloring. So graph coloring is still a very active field of research.

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PLANARITY IN GRAPH THEORY

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AMAL THOMAS

DB18CMSR11

Under the guidance of

Mrs. PRIJA V

Department of Mathematics
Don Bosco Arts and Science College
Angadikkadavu
June 2021

CERTIFICATE

Certified that this project **'Planar Graph'** is a bona fide project of **Amal Thomas** carried out the project work under my supervision.

Mrs. Riya Baby

Head of Department

Mrs. Prija V

Supervisor

Department of Mathematics

Don Bosco Arts and Science College

Angadikkadavu

DECLARATION

I **Amal Thomas** hereby declare that the project **‘Planar Graph’** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Mrs. Prija V, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu and has not submitted by me elsewhere for the award of my degree, diploma, title, or recognition, before.

Amal Thomas

DBI8CMSR11

Department of Mathematics

Don Bosco Arts and Science College,

Angadikkadavu

ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express my deepest gratitude to people along the way.

No words can adequately express the sense of gratitude; still, I try to express my heartfelt thanks through words. At the outset, I am deeply indebted to my project supervisor Mrs. Prija V, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu, for the invaluable guidance, loving encouragement, and meticulous care towards me throughout my career. I express my deep sense of gratitude to all the faculty members of the Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu.

I can never forget the support and encouragement rendered by the principal and the staff of Don Bosco Arts and Science College, Angadikkadavu.

I could not name many who sincerely supported and helped for the successful completion of this project. It is my pleasure and duty to thank each and every one of them who walked with me.

My greatest debt is always, to God Almighty.

CONTENTS

INTRODUCTION

In recent years, Graph Theory has established itself as an important mathematical tool in a wide variety of subjects, ranging from Operational Research and Chemistry to Genetics and Linguistics, and from Electrical Engineering and Geography to Sociology and Architecture. At the same time, it has also emerged as a worthwhile mathematical discipline in its own right.

A great mathematician, Euler become the Father of Graph Theory, when in 1736, he solved a famous unsolved problem of his days, called Konigsberg Bridge Problem. This is today, called as the First Problem of the Graph theory. This problem leads to the concept of the planar graph as well as Eulerian Graphs, while planar graphs were introduced for practical reasons, they pose many remarkable mathematical properties. In 1936, the psychologist Lewin used planar graphs to represent the life space of an individual.

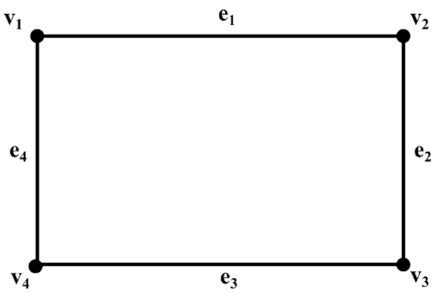
Chapter 1

BASIC CONCEPTS

Graph

A **graph** is an ordered triple $G = \{V(G), E(G), I_G\}$ where $V(G)$ is a non-empty set, $E(G)$ is a set disjoint from $V(G)$ and $I(G)$ is an incidence map that associates each element of $E(G)$ and unordered pair of elements of $V(G)$. The elements of $V(G)$ are called vertices (or nodes or points) of G and the element of $E(G)$ are called edges or lines of G.

Example:



Here $V(G) = \{v_1, v_2, v_3, v_4\}$

$E(G) = \{e_1, e_2, e_3, e_4\}$

$I_G(e_1) = \{v_1, v_2\}$ or $\{v_2, v_1\}$

$\mathrm{I}_G(e_2) = \{v_2, v_3\}$ or $\{v_3, v_2\}$

$I_G(e_3) = \{v_3, v_4\}$ or $\{v_4, v_3\}$

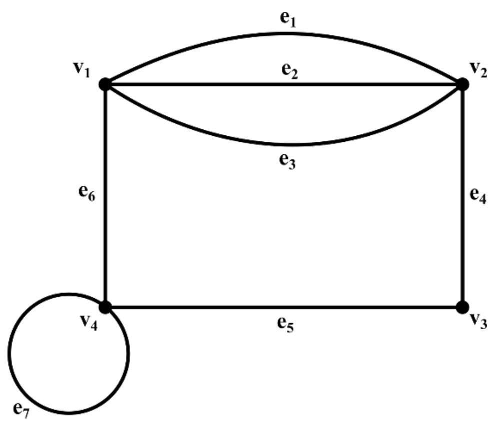
$I_G(e_4) = \{v_4, v_1\}$ or $\{v_1, v_4\}$

Multiple edges

A set of two or more edges of a graph G is called multiple edges or parallel edges if they have the same end vertices.

Loop

An edge for which the two end vertices are same is called a loop.



Here $\{e_1, e_2, e_3, e_4\}$ form the parallel edges.

e_7 is the Loop.

Simple Graph

A graph is simple if it has no loops and no multiple edges.

Finite & Infinite Graphs

A graph is called finite if both $V(G)$ & $E(G)$ are finite. A graph that is not finite is called infinite graph.

Adjacent Vertices

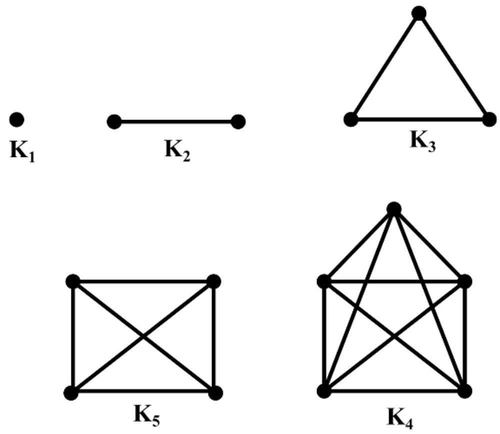
Two vertices u and v are said to be adjacent vertices if and only if there is an edge with u and v as its end vertices.

Adjacent Edges

Two distinct edges are said to be adjacent edges if and only if they have a continuous end vertex.

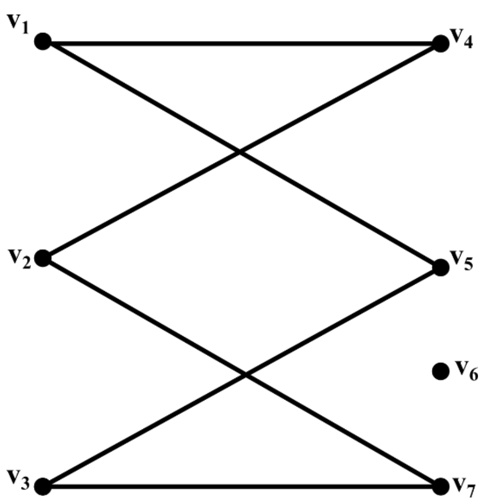
Complete Graph

A simple graph G is said to be complete if every pair of distinct vertices of G are adjacent in G. A complete graph with n vertices is denoted by K_n.



Bipartite Graph

A graph is bipartite if its vertex set can be partitioned into two non-empty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a bipartition of the bipartite graph G. The bipartite graph G with bipartition (X, Y) denoted by $G\ (X, Y)$.



Here $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$

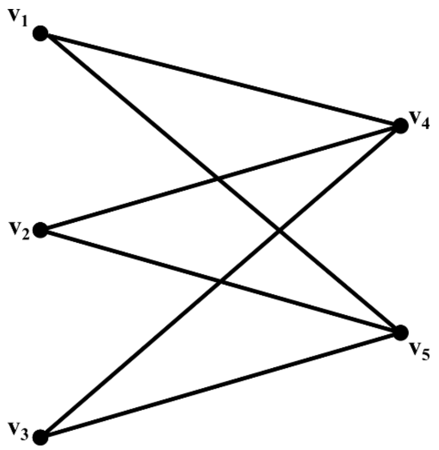
The Bipartition is

$X = \{v_1, v_2, v_3\}$

$Y = \{v_4, v_5, v_6, v_7\}$

Complete Bipartite Graph

A simple bipartite graph $G\ (X, Y)$ is complete if each vertex X is adjacent to all the vertices of Y.

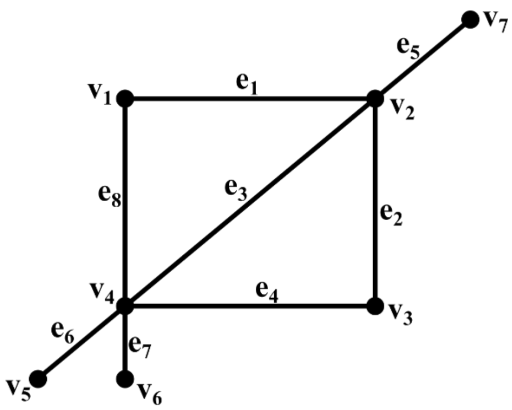


Here $X = \{v_1, v_2, v_3\}$

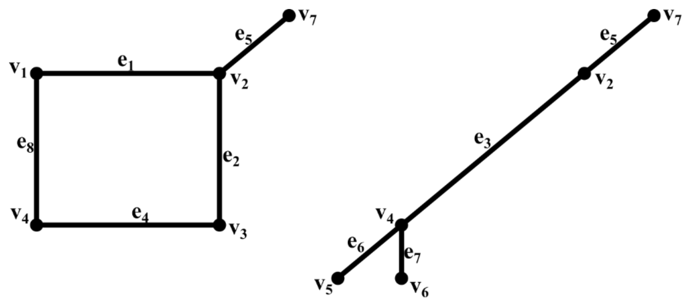
$Y = \{v_4, v_5\}$

Subgraph

A graph H is called subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and I_H is the restriction of I_G to $E(H)$ [ie, $I_H(e) = I_G(e)$ whenever $e \in E(H)$.



G



Subgraphs

Degrees of Vertices

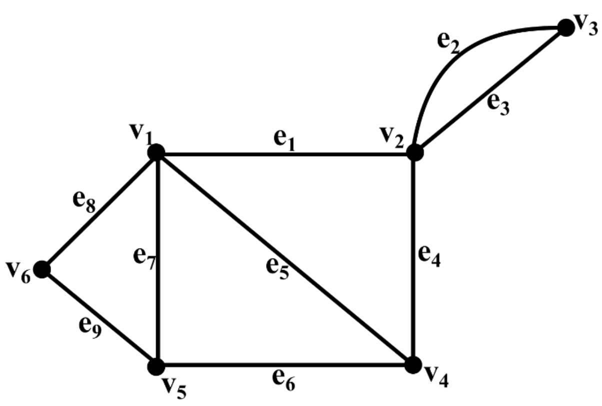
The number of edges incident with vertex V is called degree of a vertex or valency of a vertex and it is denoted by $d(v)$.

Isomorphism of Graph

A graph isomorphism from a graph G to a graph H is a pair (ϕ, θ), where $\phi : V(G) \to V(H)$ and $\theta : E(G) \to E(H)$ are bijection with a property that $I_G(e) = \{u, v\}$ and $I_H(\theta(e)) = \{\phi(u), \phi(v)\}$.

Walk

A walk in a graph G is an alternative sequence $W = v_0 v_1 e_1 v_2 e_2 \dots v_n e_n$ vertices and edges, beginning and ending with vertices where v_0 is the origin and v_n is the terminus of W.



$W = v_6 e_8 v_1 e_1 v_2 e_2 v_3 e_3 v_2 e_1 v_1$

Closed Walk

A walk to begin and ends at the same vertex is called a closed walk. That is, the walk W is closed if $v_0 = v_n$.

Open Walk

If the origin of the walk and terminus of the walk are different vertices, then it is called an open walk.

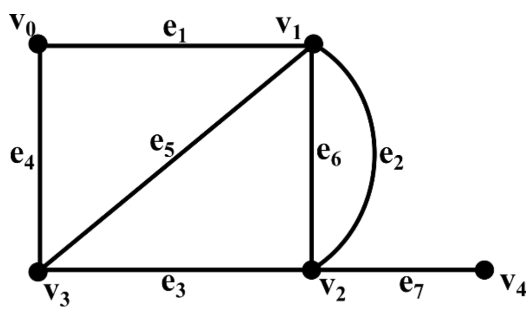
Trail

A walk is called a trail if all the edges in the walk are distinct.

Path

A walk is called a path if all the vertices are distinct.

Example:



$v_0 e_1 v_1 e_2 v_2 e_6 v_1 \rightarrow$ A trail

$v_0 e_1 v_1 e_2 v_2 e_3 v_3 \rightarrow$ A path

$v_0 e_1 v_1 e_2 v_2 e_3 v_3 e_5 v_1 \rightarrow$ A trail, but not a path

Euler's Theorem

The sum of the degrees of the vertices of a graph is equal to the twice the number of edges.

ie: $\sum_{i=1}^{n} d(v_i) = 2m$

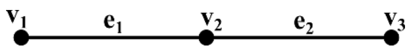
Isomorphic Graph

$$I_H(\theta(e)) = \{\phi(u), \phi(v)\}$$

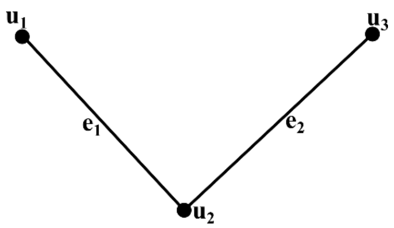
A graph $G_1 = (V_1, E_1)$ is said to be isomorphic to graph $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and v_1 in G_1 then the corresponding edge e_2 in G_2 has its end vertices u_2 and v_2 in G_2. This correspondence is called a graph isomorphism.

Example:

$G =$



$H =$



ie: G and H are isomorphic.

Components

A connected component of a graph is a maximal connected subgraph. The term is also used for maximal subgraph or subset of a graph 's vertices that have some higher order of connectivity, including bi-connected components, tri-connected components and strongly connected components.

Tree

A connected graph without cycles is called a tree.

Vertex Cut

Let G be a connected graph. The set V' subset of $V(G)$ is called a Vertex cut of G, if $G - V'$ is a disconnected graph.

Cut Vertex

If $V' = \{v\}$ is a Vertex cut of the connected Graph G, then the vertex $\boldsymbol{v}$ is called a Cut vertex.

Edge Cut

Let G be a non-trivial connected graph with vertex set V and let S be a non-empty subset of V and $\bar{S} = V - S$. Let $E' = [S, \bar{S}]$ denote the set of all edges of G that have one end vertex is S and the other is $\bar{S}$. Then $G - E'$ is a disconnected graph and $E' = [S, \bar{S}]$ is called an edge cut of G.

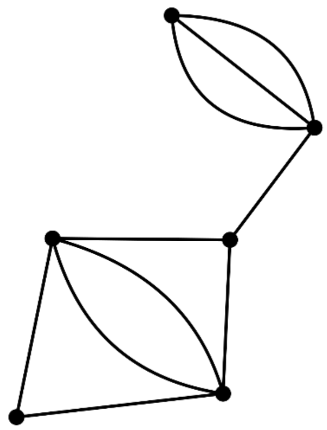
Cut Edge

If $E' = \{e\}$ is an edge cut of G then e is called a cut edge of G.

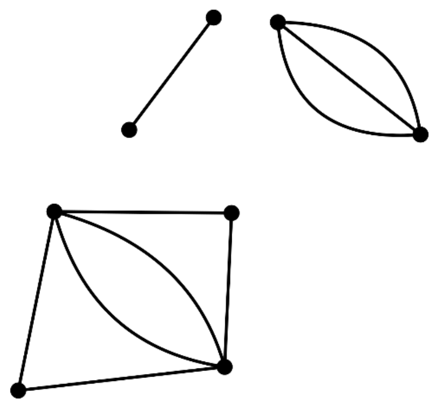
Block

A block is a Connected graph without any cut vertices.

Eg:



Graph G



Blocks of G

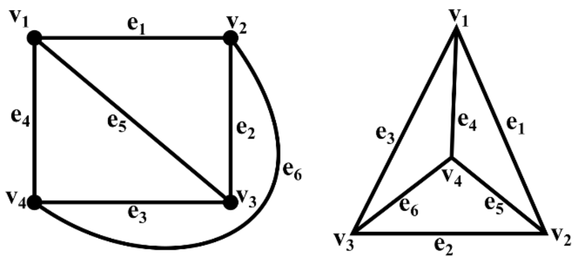
Chapter 2

PLANAR GRAPHS

Plane Graph

A plane graph is a graph drawn in the plane, such a way that any pair of edges meet only at their end vertices.

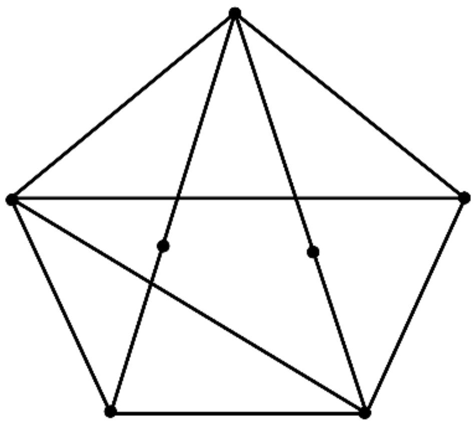
Example:



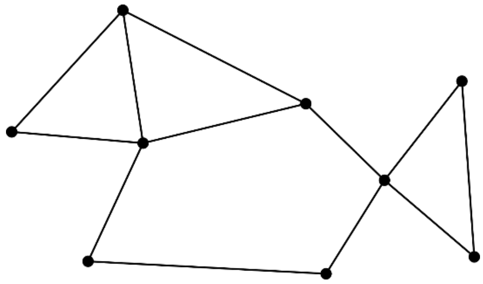
Planar Graph

A planar graph is a graph which is isomorphic to a plane graph, ie: it can be drawn as a plane graph.

A plane graph is a graph that can be drawn in the plane without any edge crossing.



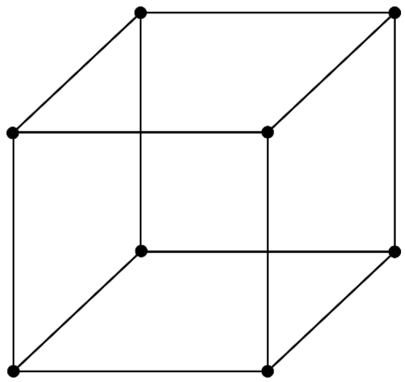
Example of Planar graph:



Planar Representation

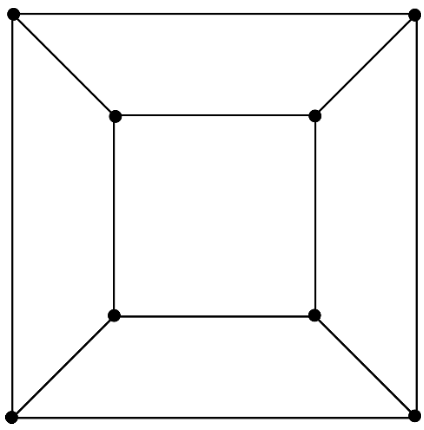
The pictorial representation of a planar graph as a plane graph is called a planar representation.

Eg: Is Q_3 shown below, planar?



The graph Q_3

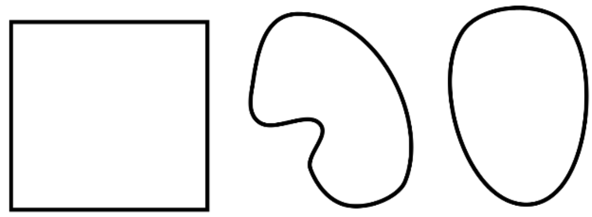
Planar representation of Q_3 is:



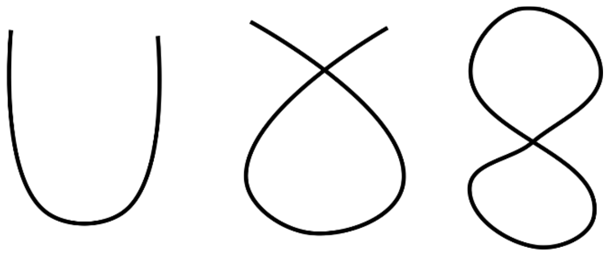
Jordan Curve

A Jordan Curve in the plane is a continuous non-self-intersecting curve where Origin and Terminals coincide.

Example:



Jordan Curves

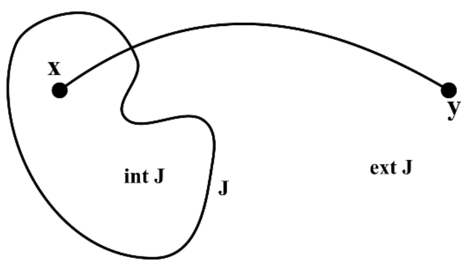


Non-Jordan Curves

Remark

If J is a Jordan Curve in the plane, then the part of the plane enclosed by J is called interior of J and is denoted by 'int J'. We exclude from 'int J' the points actually lying on J. Similarly, the part of the plane lying outside J is called the exterior of J and is denoted by 'ext J'.

Example:



Arc connecting point x in int J with point y in ext J.

Theorem

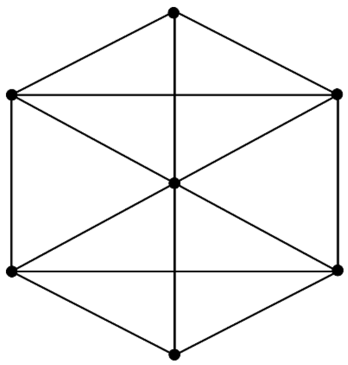
Let J be a Jordan Curve, if x is a point in int J and y is a point in ext J then any line joining x to y must meet J at some point, ie: must cross J. this is called Jordan Curve Theorem.

Boundary

The set of edges that bound a region is called its boundary.

Definition

A graph which is not planar is known as non-planar graph or a graph that cannot be drawn in the plane without any edge crossing is known as non-planar graph.



Theorem

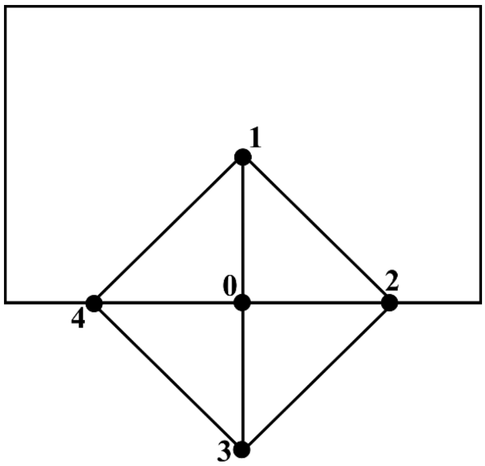
K_5 is nonplanar:

Every drawing of the complex graph K_5 in the plane (or sphere) contains at least one edge crossing.

Proof:

Label the vertices 0, 1, 2, 3, 4. By the Jordan Curve theorem any drawing of the cycle (1, 2, 3, 4, 1) separates the plane into two regions. Consider the region with

vertex 0 in its interior as the 'inside' of the circle. By the Jordan Curve theorem, the edges joining vertex 0 to each of its vertices 1, 2, 3 and 4 must also lie entirely inside the cycle, as illustrated below.



Drawing most of the K_5 in the plane

Moreover, each of the 3-cycles $\{0,1,2,0\}$, $\{0,2,3,0\}$, $\{0,3,4,0\}$ and $\{0,4,1,0\}$ also separates the plane and hence the edges $(2,4)$ must also lie to the exterior of the cycle $\{1,2,3,4\}$ as shown. It follows that the cycle formed by edges $(2,4)$, $(4,0)$ and $(0,2)$ separates the vertices 1 and 3, again by Jordan Curve theorem. Thus, it is impossible to draw edge $(1,3)$ without crossing an edge of that cycle. So, it is proven that the drawing of the K_5 in the plane contains at least one edge-crossing.

Theorem

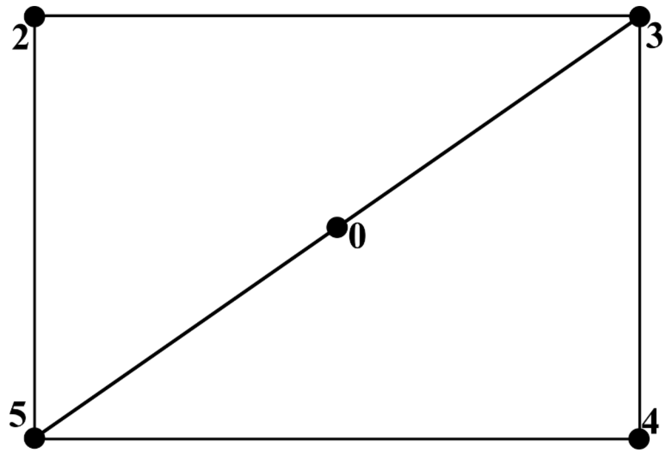
K_{33} is nonplanar:

Every drawing of the complete bipartite graph K_{33} in the plane (or sphere) contains at least one edge crossing.

Proof:

Label the vertices of one partite set 0, 2, 4 and of the order 1, 3, 5. By the Jordan Curve theorem, cycle $\{2,3,4,5,2\}$ separates the plane into two regions,

and as in the previous proof (K_5), we regard the region containing the vertex 0 as the 'inside' of the cycle. By the Jordan Curve theorem, the edges joining vertex 0 to each of the vertices 3 and 5 lie entirely inside that cycle, and each of the cycle $\{0, 3, 2, 5, 0\}$ and $\{0, 3. 4, 5, 0\}$ separates the plane, as illustrated below.



Drawing most of the K_{33} in the plane

Thus, there are 3 regions: the exterior of cycles $\{2, 3, 4, 5, 2\}$ and the inside of each of the other two cycles. It follows that no matter which region contains vertex 1, there must be some even numbered vertex that is not in that region, and hence the edge from vertex 1 to that even-numbered vertex would have to cross some cycle edge.

Corollary

Subgraph of a planar graph is planar.

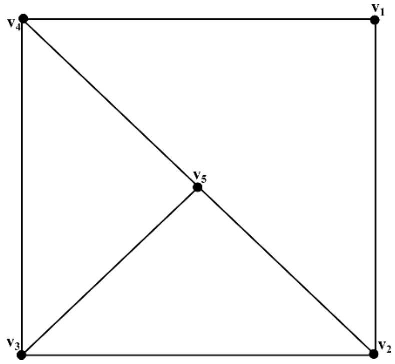
Definition

A plane graph partitions the plane into number of regions called faces.

Let G be plane graph. If x is a point on the plane which is not in G, ie: x is not a vertex of G or a point on any edge of G, then we define the faces of G containing x to be the set of all points on the plane which can be reached from x by a line which does not cross any edge of G or go through any vertex of G.

The number of faces of a plane graph G denoted by $f(a)$ or simply f.

Each plane graph has exactly one unbounded face called the exterior face.



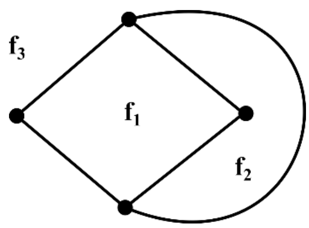
Here $f(G) = 4$

Degree of faces

The degree $d(f)$ of a face f is the number of edges with which it is incident, that is the number of edges in the boundary of a face.

Cut edge being counted twice.

Eg:



$d(f_2) = 3$

$d(f_1) = 4$

$d(f_3) = 3$

Theorem

A graph is planar if and only if each of its blocks is planar.

Proof:

If G is planar, then each of its blocks is planar since a subgraph of planar graph is planar.

Conversely, suppose that each block of G is planar. We now use induction on the number of blocks of G to prove the result. Without loss of generality, we

assume that G is connected. If G has only one block, then G itself is a block, and hence G is planar.

Now suppose G has k planar blocks and that the result has been proved for all connected graph having (k-1) planar blocks. Choose any end block B_0 of G and delete from G all the vertices of B_0 except the unique cut vertex, say v_0 of G in B_0. The resulting connected graph G\` of G contains (k-1) planar blocks. Hence, by the induction hypothesis G\` is planar. Let G˜\` be plane embedded of G\` such that v_0 belongs to the boundary of unbounded face, say f \`. Let $B_0$˜ be a plane embedding of $B_0$ in $f$ \`, so that v_0 is in the exterior face of B_0˜. Then G˜\` and B_0˜ is a plane embedding of G.

Chapter 3

EULER'S FORMULA

Theorems

Euler Formula:

For a connected plain graph G, $n - m + f = 2$ where n, m, and f denote the number of vertices, edges and faces of G respectively.

Proof:

We apply the induction on f.

If $f = 1$ the G is a tree and $m = n - 1$.

Hence $n - m + f = 2$ and suppose that G has f faces.

Since $f \geq 2$, G is not a tree and hence contains a cycle C. Let e be an edge of C. Then e belongs to exactly 2 faces, say $f_1$and f_2 and the deletion of e from G results in the formation of a single face from $f_1$and f_2. Also, since e is not a cut edge of G. $G - e$ is connected.

Further the number of faces of $G - e$ is $f - 1$, number of edges in $G - e$ is $m - 1$ and number of vertices in $G - e$ is n. So, applying induction to $G - e$, we get $n - (m - 1) + (f - 1) = 2$ and this implies that $n - m + f = 2$. This completes the proof of theorem.

Corollary 1

All plane embedding of a planar graph have the same number of faces.

Proof:

Since $f = m - n + 2$ the number of faces depends only on n and m and not on the particular embedding.

Corollary 2

If G is a simple planar graph with at least 3 vertices, then $m \leq 3n - 6$.

Proof:

Without the generality we can assume that G is a simple connected plane graph. Since G is simple and $n \geq 3$, each face of G has degree at least 3. Hence if f denote the set of faces of G $\sum_{f \epsilon F} d(f) \geq 3f$. But $\sum_{f \epsilon F} d(f) = 2m$.

Consequently $2m \geq 3f$ so that $f \leq \frac{2m}{3}$.

By the Euler formula $m = n + f - 2$ now $f \leq \frac{2m}{3}$ implies $\mathrm{m} \leq \mathrm{n} + \left(\frac{2\mathrm{m}}{3}\right) - 2$. This gives. $m \leq 3n - 6$.

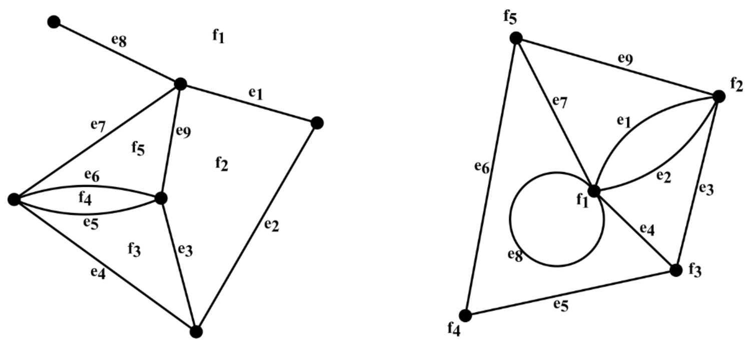
Chapter 4

DUAL OF A PLANE GRAPH

Definition

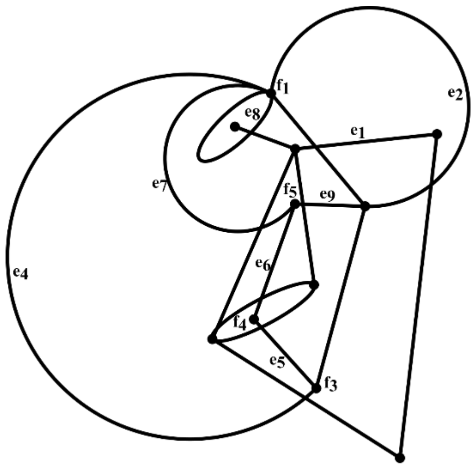
Let G be a plane graph. One can form out of G a new graph H in the following way corresponding to each face f(g), take the vertex f* and corresponding to each edge e(g), take an edge e*. Then edge e* joins vertices f* and g* in H iff edge e is common to the boundaries of faces f and g in G. The graph H is then called dual of G.

Example:



(a) (b)

Plane graph and its Dual



(c)

CONCLUSION

In this project we discussed the topic planar graph in graph theory.

We discussed about Euler formula and verified that some graphs are planar, and some are non-planar. A related important property of planar graphs, maps and triangulations is that they can be enumerated very nicely.

We also discussed about duality of a graph.in mathematical discipline of graph theory, the dual graph of a plane graph G is a graph that has a vertex of each face of G .it has many applications in mathematical and computational study.

In fact, graph theory is being used in our so many routine activities. For eg; using GPS or google maps to determine a route based on used settings.

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2) J. A. Bondy and U. S. R. Murty, Graph Theory with Great Britain: The Macmillian Press Ltd. 1976

POWER SERIES SOLUTIONS AND SPECIAL FUNCTIONS

Project report submitted to
The Kannur University
for the award of the degree
of

Bachelor of Science

by

ANAINA MANIYATH

DB18CMSR17

Under the guidance of

Ms. Athulya P

Department of Mathematics
Don Bosco Arts and Science College
Angadikadavu

CERTIFICATE

Certified that this project **‘Power Series’** is a bona fide project of **ANAINA MANIYATH** carried out the project work under my supervision.

Mrs. Riya Baby
Head of Department

Ms. Athulya P
Supervisor

Department of Mathematics
Don Bosco Arts and Science College
Angadikadavu

DECLARATION

I **ANAINA MANIYATH** hereby declare that the project **'Power Series'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Athulya P, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

Name

ANAINA MANIYATH

Department of Mathematics

Don Bosco Arts and Science College,

Angadikkadavu

ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express my deepest gratitude to people along the way.

No words can adequately express the sense of gratitude; still, I try to express my heartfelt thanks through words. At the outset, I am deeply indebted to my project supervisor Ms. Athulya P, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu, for the invaluable guidance, loving encouragement and meticulous care towards me throughout my career. I express my deep sense of gratitude to all the faculty members of the Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu.

I can never forget the support and encouragement rendered by the Principal and the staff of Don Bosco Arts and Science College, Angadikkadavu.

I could not name many who sincerely supported and helped for the successful completion of this project. It is my pleasure and duty to thank each and every one of them who walked with me.

My greatest debt is always, to God Almighty.

CONTENTS

INTRODUCTION

A power series is a type of series with terms involving a variable. Power series are often used by calculators and computers to evaluate trigonometric, hyperbolic, exponential and logarithm functions. So any application of these kind of functions is a possible application of power series. Many interesting and important differential equations can be found in power series.

.

PRELIMINERY

A. An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad (1)$$

is called a *power series in x*. The series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

is a *power series in* $x - x_0$.

B. The series (1) is said to *converge* at a point x if the limit

$$\lim_{m \to \infty} \sum_{n=0}^{m} a_n x^n$$

exists, and in this case the *sum* of the series is the value of this limit.

Radius of convergence: Series in x has a radius of convergence R, where $0 \le R \le \infty$, with the property that the series converges if $|x| < R$ and diverges if $|x| > R$. It should be noted that if $R = 0$ then no x satisfies $|x| < R$, and if $R = \infty$ then no x satisfies $|x| > R$

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \text{, if the limit exists.}$$

C. Suppose that (1) converges for $|x| < R$ with $R > 0$, and denote its sum by $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Then $f(x)$ is automatically continuous and has derivatives of all orders for $|x| < R$.

D. Let $f(x)$ be a continuous function that has derivatives of all orders for $|x| < R$ with $R > 0$. $f(x)$ be represented as power series using *Taylor's formula*:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x)$$

where the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(\bar{x})}{(n+1)!} x^{n+1}$$

for some point $\bar{x}$ between 0 and x.

E. A function $f(x)$ with the property that a power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is valid in some neighbourhood of the point x_0 is said to be *analytic* at x_0. In this case the a_n are necessarily given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

and is called the *Taylor series* of $f(x)$ at x_0.

Analytic functions: A function f defined on some open subset U of R or C is called analytic if it is locally given by a convergent power series. This means that every a ∈ U has an open neighbourhood V ⊆ U, such that there exists a power series with centre a that converges to f(x) for every x ∈ V.

CHAPTER 1

SERIES SOLUTION OF FIRST ORDER EQUATION

We have studied to solve linear equations with constants coefficient but with variable coefficient only specific cases are discussed. Now we turn to these latter cases and try to find a general method to solve this. The idea is to assume that the unknown function y can be explained into a power series. Our purpose in this section is to explain the procedures by showing how it works in the case of first order equation that are easy to solve by elementary methods.

Example 1: we consider the equation

$$y' = y$$

Consider the above equation as (1). Assume that y has a power series solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Which converges for $|x| < R$, $R > 0$

That is we assume that $y' = y$ has a solution that is analytic at origin. We have

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots \cdots$$

then

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots \ldots \ldots$$

$$\therefore (1) \Rightarrow a_1 + 2a_2 x + 3a_3 x^2 \cdots$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$\Rightarrow a_1 = a_0$

$2a_2 = a_1 \Rightarrow \qquad a_2 = \dfrac{a_1}{2} = \dfrac{a_0}{2}$

$3a_3 = a_2 \Rightarrow \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3} = \frac{a_0}{3!}$

$4a_4 = a_3 \Rightarrow \qquad a_4 = \frac{a_3}{4} = \frac{a_0}{2 \cdot 3 \cdot 4} = \frac{a_0}{4!}$

$\therefore$ we get

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{3!} x^3 + \frac{a_0}{4!} x^4 + \cdots$$

$$= a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right)$$

$$y = a_0 e^x$$

To find the actual function we have $y' = y$

$$\text{i.e.,} \quad \frac{dy}{dx} = y \Rightarrow \quad \frac{dy}{y} = dx$$

integrating

$$\log\, y = x + c$$

$$\text{i.e.,} \quad y = e^{x+c} = e^x \cdot e^c$$

$$\underline{\underline{y = a_0 e^x \text{, where } a_0 = e^c \text{, a constant.}}}$$

Example 2: solve $y' = 2xy$. Also find its actual solution.

<u>Solution:</u> $\qquad y' = 2xy \qquad (1)$

Assume that y has a power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Which converges for $|x| < R, R > 0$

We have

$$y = \sum_{n=0}^{\infty} a_n \; x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= a_1 + 2a_2x + 3a_3x^2 + \cdots$$

Then (1) $\Rightarrow a_1 + 2a_2x + 3a_3x^2 + \cdots = 2x(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)$

$$= 2xa_0 + 2xa_1x + 2xa_2x^2 + 2xa_3x^3 + \cdots$$

$$= 2xa_0 + 2a_1x^2 + 2a_2x^3 + 2a_3x^4 + \cdots \ldots..$$

$\Rightarrow a_1 = 0 \qquad 2a_2 = 2a_0 \Rightarrow a_2 = \frac{2a_0}{z} = a_0$

$$3.a_3 = 2a_1 \Rightarrow a_3 = \frac{2a_1}{3} = 0$$

$$4a_4 = 2a_2 \Rightarrow a_4 = \frac{2a_2}{42} = \frac{a_0}{2}$$

$$5a_5 = 2a_3 = 0 \Rightarrow a_5 = 0$$

$$6a_6 = 2a_4 \Rightarrow a_6 = \frac{2a_4}{6} = \frac{a_4}{3} = \frac{a_0}{2\cdot 3} = \frac{a_0}{3!}$$

We get,

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ &= a_0 + 0 + a_0x^2 + 0x^3 + \frac{a_0}{2}x^4 + \cdots \\ &= a_0 + a_0x^2 + \frac{a_0}{2}x^4 + \cdots \\ &= a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) \end{aligned}$$

$$y = a_0e^{x^2}$$

To find an actual solution

$$y' = 2xy$$

$$\frac{dy}{dx} = 2xy$$

$$\Rightarrow \qquad \frac{dy}{y} = 2x \cdot dx$$

$$\log\, y = x^2 + c$$

$$y = e^{x^2} + c$$

$$\Rightarrow y = a_0e^{x^2}, \text{ where } a_0 = e^c$$

Example 3: Consider $y = (1+x)^p$ where p is an arbitrary constant. Construct a differential equation from this and then find the solution using power series method.

Solution

First, we construct a differential equation

i.e. $y = (1+x)^p$

$$y' = p(1+x)^{p-1} = \frac{p(1+x)^p}{1+x} = \frac{py}{1+x}$$

$$\therefore (1+x)y' = py, \quad y(0) = r$$

Assume that y has a power series solution of the form,

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + \dot{a}_2 x^2 + \cdots \ldots \ldots$$

Which converges for $|x| < \dot{R}, \quad R > 0$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots \ldots \ldots ..$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Then $(1+x)y' = py$

$$\Rightarrow (1+x)a_1 + 2a_2 x + 3a_3 x^2 + \cdots = p(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$\Rightarrow (a_1 + 2a_2 x + 3a_3 x^2 + \cdots) + (a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots)$$

$$= a_0 p + a_1 p x + a_2 p x^2 + \cdots$$

Equating the coefficients of $x, x^2, \ldots$

$$a_1 = a_0 p \text{ i.e. } a_1 = p, (\text{since } a_0 = 1)$$

$$\Rightarrow 2a_2 = a_1(\mathrm{p}-1)$$

$$a_2 = \frac{a_1(\mathrm{p}-1)}{2} = \frac{a_0 P(p-1)}{2}$$

$$3a_3 + 2a_2 = a_2 p$$
$$sa_3 = a_2 p - 2a_2$$
$$= a_2(p-2)$$
$$a_3 = \frac{a_2(p-2)}{3} = \frac{a_0 p(p-1)(p-2)}{2 \cdot 3}$$

$$4a_4 + 3a_3 = a_3 p$$
$$4a_4 = a_3 p - 3a_3$$
$$= a_3(p-3)$$
$$a_4 = \frac{a_3(p-3)}{4} = \frac{a_0 p(p-1)(p-2)(p-3)}{2 \cdot 3 \cdot 4}$$

∴ we get,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= a_0 + a_0 px + \frac{a_0 p(p-1)}{2} x^2 + \frac{a_0 p(p-1)(p-2)}{2 \cdot 3} x^3 + \cdots \ldots$$
$$= 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 +$$
$$\frac{p(p-1)(p-2)(p-3)}{4!} x^4 + \cdots + \frac{p(p-1)(p-2)(p-(n-1))}{n!} x^n$$

Since the initial problem y(0) = 1 has one solution the series converges for |x|<1

So this is a power solution,

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n$$

Which is binomial series.

Example 4: Solve the equation $y' = x - y,\ y\,(0) = 0$

Solution: Assume that y has a power series solution of the form

$$y = \sum_{n=0}^{\infty} \text{an } x^n$$

which converges for $|x| < R, R > 0$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Now $y' = x - y$

$$(a_1 + 2a_2x + 3a_3x^2 + \cdots) = x - (a_0 + a_1x + a_2x^2 + \cdots)$$

Equating the coefficients of x, x^2,

$$\begin{aligned} a_1 &= -a_0 = 0, \quad \text{Since} \quad y(0) = 0 \\ 2a_2 &= 1 - a_1 \\ &= 1 - 0 \end{aligned}$$

$$\Rightarrow a_2 = \frac{1}{2}$$

$$3a_3 = -a_2$$

$$a_3 = \frac{-a_2}{3} = -\frac{1}{2 \cdot 3}$$

$$4a_4 = -a_3$$

$$\Rightarrow a_4 = \frac{1}{2 \cdot 3 \cdot 4}$$

$$\therefore y = 0 + 0 + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \ldots \ldots$$

$$= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) + x - 1$$

$$= e^{-x} + x - 1$$

By direct method

$$y' = x - y$$

$$\frac{dy}{dx} = x - y \Rightarrow \frac{dy}{dx} + y = x \qquad \left(\frac{dy}{dx} + py = Q\ form\right)$$

here $P(x) = 1$, integrating factor

$$= e^{\int p(x) \cdot dx}$$

$$= e^x$$

$$\therefore ye^x = \int xe^x \cdot dx$$

$$ye^x = x \cdot e^x - \int e^x \cdot dx$$

$$= xe^x - e^x$$

$$ye^x = e^x(x - 1) + c$$

$$y = \frac{e^x(x-1) + c}{dx} = x - 1 + \frac{c}{e^x} = ce^{-x} + (x - 1)$$

$$\therefore y = (x - 1) + ce^{-x}$$

CHAPTER 2

SECOND ORDER LINEAR EQUATION, ORDINARY POINTS

Consider the general homogeneous second order linear equation,

$$y'' + P(x)y' + Q(x)y = 0 \qquad (1)$$

As we know, it is occasionally possible to solve such an equation in terms of familiar elementary functions. This is true, for instance, when $P(x)$ and $Q(x)$ are constants, and in a few other cases as well. For the most part, however, the equations of this type having the greatest significance in both pure and applied mathematics are beyond the reach of elementary methods, and can only be solved by means of power series.

P(x) and Q(x) are called coefficients of the equation. The behaviour of its solutions near a point x_0 depends on the behaviour of its coefficient functions $P(x)$ and $Q(x)$ near this point. we confine ourselves to the case in which $P(x)$ and $Q(x)$ are well behaved in the sense of being analytic at $x0$, which means that each has a power series expansion valid in some neighbourhood of this point. In this case $x0$ is called an *ordinary point* of equation (1). Any point that is not an ordinary point of (1) is called a *singular point.*

Consider the equation,

$$y'' + y = 0 \qquad (2)$$

the coefficient functions are $P(x) = 0$ and $Q(x) = 1$, These functions are analytic at all points, so we seek a solution of the form,

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \qquad (3)$$

Differentiating (3) we get,

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1} x^n + \cdots \qquad (4)$$

And

$$y'' = 2a_2 + 2\cdot 3a_3x + 3\cdot 4a_4x^2 + \cdots + (n+1)(n+2)a_{n+2}x^n + \cdots \quad (5)$$

If we substitute (5) and (3) into (2) and add the two series term by term, we get

$$y'' + y = \begin{matrix}(2a_2 + a_0) + (2\cdot 3a_3 + a_1)x + (3\cdot 4a_4 + a_2)x^2 + \\ (4\cdot 5a_5 + a_3)x^3 + \cdots + [(n+1)(n+2)a_{n+2} + a_n]x^n + \cdots\end{matrix} = 0$$

and equating to zero the coefficients of successive powers of *x* gives

$$2a_2 + a_0 = 0, \qquad 2\cdot 3a_3 + a_1 = 0, \qquad 3\cdot 4a_4 + a_2 = 0$$

$$4\cdot 5a_5 + a_3 = 0, \dots\dots, \qquad (n+1)(n+2)a_{n+2} + a_n = 0, \dots$$

By means of these equations we can express a_n in terms of a_0 or a_0, according as *n* is even or odd:

$$a_2 = -\frac{a_0}{2}, \qquad a_3 = -\frac{a_1}{2\cdot 3}, \qquad a_4 = -\frac{a_2}{3\cdot 4} = \frac{a_0}{2\cdot 3\cdot 4}$$

$$a_5 = -\frac{a_3}{4\cdot 5} = \frac{a_1}{2\cdot 3\cdot 4\cdot 5}, \cdots$$

With these coefficients, (3) becomes

$$y = a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{2\cdot 3}x^3 + \frac{a_0}{2\cdot 3\cdot 4}x^4 + \frac{a_1}{2\cdot 3\cdot 4\cdot 5}x^5 - \cdots$$

$$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \qquad (6)$$

$i.e, \qquad y = a_0\cos\ x + a_1\sin\ x$

Since each of the series in the parenthesis converges for all *x*. This implies the series (2) for all *x*.

Solve the legenders equation,

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

Solution

Consider $(1-x^2)y'' - 2xy' + p(p+1)y = 0$ as equation (1)

Assume that y has a power series solution of the form

$$y = \sum a_n x^n$$

Which converges $|x| < R, R > 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

put $n = n + 2$ (Since y'' is not x^n form)

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+2-1) a_{n+2} x^{n+2-2}$$

$$\therefore y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

Now (1) $\Rightarrow$ $\quad y'' - x^2 y'' - 2xy' + p(p+1)y = 0$

$$\Rightarrow \sum (n+1)(n+2) a_{n+2} x^n - \sum n(n-1) a_n x^n - \sum 2n a_n x^n + \sum p(p+1) a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [((n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + p(p+1) a_n) x^n] = 0$$

for n = 0,1,2,3…….

$$\Rightarrow (n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + p(p+1) a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{[n(n-1) + 2n - p(p+1)]}{(n+1)(n+2)} a_n$$

$$= \frac{(n^2 - n + 2n - p^2 - p) a_n}{(n+1)(n+2)}$$

$$= \frac{(n^2 + n - p^2 - p) a_n}{(n+1)(n+2)}$$

$$\therefore a_{n+2} = \frac{-(p-n)(p+n+1)}{(n+1)(n+2)} a_n , \qquad n = 0,1,2 \ldots$$

This is an Recursion formula

put $n = 0,$ $\quad a_2 = \dfrac{-p(p+1)}{1 \cdot 2} a_0$

$n = 1,$ $\quad a_3 = \dfrac{-(p-1)(p+2)}{2 \cdot 3} \cdot a_1$

$n = 2,$ $\quad a_4 = \dfrac{-(p-2)(p+3)}{3i4} a_2$

$$= \frac{p(p-2)(p+1)(p+3)}{4!} a_0$$

$n = 3,$ $\quad a_5 = \dfrac{-(p-3)[p+4)}{4 \cdot 5} a_3$

$$= \frac{(p-1)(p-3)(p+2)(p+4)}{5!} a_1$$

$n = 4,$ $\quad a_6 = \dfrac{-(p-4)(p+5)}{5 \cdot 6} a_4$

$$= \frac{-p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} a_0$$

$n = 5,$ $\quad a_7 = -\dfrac{(p-5)(p+6)}{6 \cdot 7} a_5$

$$= -\frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} a_1$$

$$\begin{aligned} y = a_0 \Big[1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p-2)(p+1)(p+3)}{4!} x^4 \\ - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} x^6 + \cdots \Big] \\ + a_1 \Big[x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!} x^5 \\ - \frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} x^7 + \cdots \Big] \end{aligned}$$

Find the general solution of $(1+x^2)y'' + 2xy' - 2y = 0$ in terms of power series in x. Can you express this solution by means of elementary functions?

Solution

Consider the equation $(1+x^2)y'' + 2xy' - 2y = 0$ as equation (1)

Assume that y has a power series solution of the form

$$y = \sum a_n x^n$$

Which converges $|x| < R, R > 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1+x^2)y'' = y'' + x^2 y''$$

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

Now $y'' = \displaystyle\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

put $n = n + 2$

$\Rightarrow$

$$\sum_{n=0}^{\infty} (n+2)(n+2-1)a_n + 2x^{n+2=2}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n$$

$$(1) \Rightarrow \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x_n$$
$$+ \sum_{n=1}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\Rightarrow \sum[((n+1)(n+2)a_{n+2} + n(n-1)a_n + 2na_n - 2a_n)x^n] = 0$$

$$\Rightarrow (n+1)(n+2)a_{n+2} + n(n-1)a_n + 2na_n - 2a_n = 0$$

$$a_{n+2} = \frac{[-n(n-1) - 2n + 2]}{(n+1)(n+2)} a_n$$
$$= \frac{(-n^2 + n - 2n + 2)}{(n+1)(n+2)} a_n$$

put $n = 0, \qquad a_2 = \frac{2}{1 \cdot 2} a_0 = \frac{2a_0}{2!} = a_0$

$n = 1, \qquad a_3 = \frac{(1 - 1 - 2 + 2)}{2 \cdot 3} a_1 = 0$

$n = 2, \qquad a_4 = \frac{2 - 4 - 4 + 2}{3 \cdot 4} a_2 \quad = \frac{-4}{3 \cdot 4} a_0 = \frac{-a_0}{3}$

$n = 3, \qquad a_5 = \frac{3 - 9 - 16 + 2}{4.5} a_3 \quad = 0$

$n = 4, \qquad a_6 = \frac{4 - 16 - 8 + 2}{5.6} a_4 \quad = \frac{-3}{5} a_4 \quad = \frac{3a_0}{3.5} = \frac{a_0}{5}$

$$\therefore y = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$= a_0 + a_1 x + a_0 x^2 - \frac{a_0}{3} x^4 + \frac{a_0}{5} x^6 \ldots.$$
$$= a_0 \left[1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \cdots\right] + a_1 x$$
$$= a_0 \left[1 + x\left(x - \frac{x^3}{3} + \frac{x^5}{5} \cdots\right)\right] + a_1 x$$
$$= a_0(1 + x\tan^{-1} x) + a_1 x$$

Consider the equation $y'' + xy' + y = 0$

(a) Find its general solution $y = \sum a_n x^n$ in the form $y = a_0 y_1(x) + a_1 y_2(x)$ where $y_1(x)$ and $y_2(x)$ are power series

(b) use the ratio test to verify that the two series $y_1(x)$ and $y_2(x)$ converges for all x.

Solution:

Given $$y'' + xy' + y = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

Assume that y has a power series solution the form $\sum a_n x^n$ which converges for $|x|$ $R > 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n$$

$$(1) \Rightarrow \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum \left[\left((n+1)(n+2) a_{n+2} + n a_n + a_n \right) x^n \right] = 0$$

$$\Rightarrow (n+1)(n+2) a_{n+2} + n a_n + a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{(-n-1) a_n}{(n+1)(n+2)} = \frac{-a_n}{n+2}$$

put $$n = 0, a_2 = -\frac{a_0}{2}$$

$$n = 1, a_3 = \frac{-2a_1}{2 \cdot 3} = \frac{-a_1}{3}$$

$$n = 2, \quad a_4 = \frac{-3a_2}{3 \cdot 4} = \frac{-a_2}{4} = \frac{a_0}{8}$$
$$n = 3, \quad a_5 = \frac{-4a_3}{4 \cdot 5} = \frac{a_1}{15}$$
$$n = 4, \quad a_6 = \frac{-5a_4}{5 \cdot 6} = \frac{-a_0}{48}$$

$\therefore$ we get $$y = a_0 + a_1 x + -\frac{a_0}{2}x^2 - \frac{a_1}{3}x^3 + \frac{a_0}{8}x^4 + \frac{a_1}{15}x^5 - \frac{a_0}{48}x^6 + \cdots$$

$$= a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots\right] + a_1\left[x - \frac{x^3}{3} + \frac{x^5}{3.5} + \cdots\right]$$

where $$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{\dot{x}^2}{2 \cdot 4 \cdot 6} +$$

$$y_2(x) = x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \cdots$$

(b) $$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots$$

$$R = \lim_{n \to \infty} \left|\frac{a_n}{a_{n+1}}\right|$$
$$= \lim_{n \to \infty} \left|\frac{(-1)^n}{2 \cdot 4 \cdot (2n)} \Big/ \frac{(-1)^{n+1}}{2 \cdot 4 \cdot\cdot (2n+2)}\right|$$
$$= \lim_{n \to \infty} \left|\frac{2(n+1)}{-1}\right|$$
$$= \lim_{n \to \infty} |-2n(1 + \frac{1}{n})| = \infty$$

$\therefore y_1(x)$ converges for all x

$$y_2(x) = x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} - \cdots$$

$$R = \lim_{n\to\infty} \left| \frac{(-1)^n}{3 \cdot 5 \cdots (2n+1)} \Big/ \frac{(-1)^{n+1}}{3 \cdot 5 \cdot \cdots (2n+3)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1) \cdot 3 \cdot 5 \cdots (2n+1)(2n+3)}{3 \cdot 5 \cdots \cdot (2n+1)} \right|$$

$$= \lim_{n\to\infty} |(-1)n(2+3/n)| = \infty$$

$\therefore y_2(x)$ converges for all x

REGULAR SINGULAR POINTS

A singular point x_0 of equation

$$y'' + P(x)y' + Q(x)y = 0$$

is said to be regular if the functions $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic, and irregular otherwise. Roughly speaking, this means that the singularity in $P(x)$ cannot be worse than $1/(x - x_0)$, and that in $Q(x)$ cannot be worse than $1/(x - x_0)^2$.

If we consider Legendre's equation in the form

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$$

it is clear that x = 1 and x = −1 are singular points. The first is regular because

$$(x-1)P(x) = \frac{2x}{x+1} \text{ and } (x-1)^2 Q(x) = -\frac{(x-1)p(p+1)}{x+1}$$

are analytic at x = 1, and the second is also regular for similar reasons.

Example: *Bessel's equation of order p*, where p is a nonnegative constant:

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

If this is written in the form

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0,$$

it is apparent that the origin is a regular singular point because $xP(x) = 1$ and $x^2 Q(x) = x^2 - p^2$ are analytic at x = 0.

CONCLUSION

The purpose of this project gives a simple account of series solution of first order equation, second order linear equation, ordinary points. The study of these topics given excellent introduction to the subject called ‘POWER SERIES’

we used application of power series extensively throughout this project. We take it for granted that most readers are reasonably well acquainted with these series from an earlier course in calculus. Nevertheless, for the benefit of those whose familiarity with this topic may have faded slightly, we presented a brief review of the main facts of power series.

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DEPARTMENT OF MATHEMATICS

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BONAFIDE CERTIFICATE

Certified that this project report on " INNER PRODUCT SPACES" is the bonafide work of ANASWARA SASIDHARAN who carried out the project work under my supervision.

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DECLARATION

I, ANASWARA SASIDHARAN hereby declare that the project work entitled 'INNER PRODUCT SPACES′ has been prepared by me and submitted to Kannur University in partial fulfilment of requirement for the award of Bachelor of Science is a record of original work done by me under the supervision of Mr. ANIL M V, Assistant Professor, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu.

I, also declare that this Project work has been submitted by me fully or partially for the award of any Degree, Diploma, Title or recognition before any authority.

Place : Angadikadavu

ANASWARA SASIDHARAN

Date :

DB18CMSR02

ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express mydeepest gratitude to people along the way.

No words can adequately express the sense of gratitude, still I try to express my heartfelt thanks through words. The outset, I am deeply indebted to my project supervisor Mr. ANIL M.V, Assistant Professor, Department of Mathematics, Don Bosco Arts and ScienceCollege, Angadikadavu, for the invaluable guidance, loving encouragement and meticulous care towards me throughout my career.I express my deep sense of gratitude to all the faculty members of the Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu.

I could not name many who sincerely supported and helped for the successful completion of this Project. It is my pleasure and duty to thank each and every one of them who walked with me.

ANASWARA SASIDHARAN

Contents

INTODUCTION

In linear algebra, an inner product space is a vector space with an additional structure called an inner product. This additional structure associates each pair of vectors in the space with a scalar quantity known as the inner product of the vectors. Inner products allow the rigorous introduction of intuitive geometrical notions such as the length of a vector or the angle between two vectors. They also provide the means of defining orthogonality between vectors (zero inner product). Inner product spaces generalize Euclidean spaces (in which the inner product is the dot product, also known as the scalar product) to vector spaces of any (possibly infinite) dimension and are studied in functional analysis. The first usage of the concept of a vector space with an inner product is due to Peano, in 1898.

An inner product naturally induces an associated norm, thus an inner product space is also a normed vector space. A complete space with an inner product is called a Hilbert space. An (incomplete) space with an inner product is called a pre-Hilbert space.

PRELIMINARIES

LINEAR SPACES

Definition 1: A linear (vector) space X over a field $\mathbf{F}$ is a set of elements together with a function, called addition, from $X \times X$ into X and a function called scalar multiplication, from $\mathbf{F} \times X$ into X which satisfy the following conditions for all $x, y, z \in X$ and $\alpha, \beta \in \mathbf{F}$;

i. $(x + y) + z = x + (y + z)$

ii. $x + y = y + x$

iii. There is an element 0 in X such that $x + 0 = x$ for all $x \in X$.

iv. For each $x \in X$ there is an element $-x \in X$ such that $x + (-x) = 0$.

v. $(x + y) = \alpha x + \alpha y$

vi. $(\alpha + \beta)x = \alpha x + \beta x$

vii. $\alpha(\beta x) = (\alpha\beta)x$

viii. $1 \cdot x = x$.

Properties i to iv imply that X is an abelian group under addition and v to vi relate the operation of scalar multiplication to addition X and to addition and multiplication in $\mathbf{F}$.

Examples:

(a) $V_n(\mathbf{R})$. The vectors are n-tuples of real numbers and the scalars are real

numbers with addition and scalar multiplication defined by

$$(\alpha_1,\cdots,\alpha_n)+(\beta_1,\cdots,\beta_n)=(\alpha_1+\beta_1,\cdots,\alpha_n+\beta_n) \tag{1}$$

$$\beta(\alpha_1,\cdots,\alpha_n)=(\beta\alpha_1,\cdots,\beta\alpha_n) \tag{2}$$

$V_n(\mathbf{R})$ is a linear space over **R**. Similarly, the set of all n-tuples of complex numbers with the above definition of addition and multiplication is a linear space over **C** and is denoted as $V_n(\mathbf{C})$.

(b) The set of all functions from a nonempty set X into a field **F** with addition and scalar multiplication defined by

$$[f+g](t)=f(t)+g(t) \quad \text{and} \quad [\alpha f](t)=\alpha f(t);\ f,g\in X,\ t\in T \tag{3}$$

is a linear space.

Let T = **N** the set of all positive integers and X is the set of all sequences of elements **F** with addition and scalar multiplication defined by

$$(\alpha_n+\beta_n)=\ (\alpha_n+\beta_n) \tag{4}$$

$$\beta\left(\alpha_n\right)=\left(\beta\alpha_n\right) \tag{5}$$

denoted as $V_\infty(\mathbf{F})$, form a linear space.

METRIC SPACES

Remember the distance function in the Euclidean space $\mathbf{R}^n$.

Let $x,y,z \in \mathbf{R}^n$, then

(1) $|x - y| \geq 0$; $|x - y| = 0$ if and only if $x = y$;

(2) $|x - y| = |y - x|$;

(3) $|x - y| \leq |x - z| + z - y|$.

Definition 2: A **metric** or **distance function** on a set X is a real valued function d defined on $X \times X$ which has the following properties: for all $x, y, z \in X$.

(1) $d(x,y) \geq 0$; $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq d(x, z) + d(z, y)$

A **metric space** (X, d) is a nonempty set X and a metric d defined on X.

Examples: In addition to the Euclidean spaces let us have the following examples. *Here all functions are assumed to be continuous*. Let L^p denotes a set of complex valued functions in $\mathbf{R}^n$ such that $|f|^p$ is integrable. Let us recall some results concerning such functions.

Höder's Inequality: If $p > 1$, $1/q = 1 - 1/p$

$$\int | fg | \leq [\int | f |^p]^{1/p} [\int | g |^q]^{1/q} .$$

Minkowski's Inequality: If $p \geq 1$,

$$[\int | f + g |^p]^{1/p} \leq [\int | f |^p]^{1/p} + [\int | g |^p]^{1/p}$$

If x_k and y_k for $k = 1, \ldots, m$ are complex numbers, let $f(t) = |x_k|$ and $g(t) = |y_k|$ for $t \in [k, k+1]$ and $f(t) = 0 = g(t)$ for $t \in [1, m+1]$. Then we obtain the summation form of the above inequalities from the integral form

Hölder's Inequality

$$\sum_{k=1}^{m} |x_k y_k| \le \left[\sum_{k=1}^{m} |x_k|^p\right]^{1/p} \left[\sum_{k=1}^{m} |y_k|^q\right]^{1/q}$$

Minkowski's Inequality:

$$\left[\sum_{k=1}^{m} |x_k + y_k|^p\right]^{1/p} \le \left[\sum_{k=1}^{m} |x_k|^p\right]^{1/p} + \left[\sum_{k=1}^{m} |y_k|^p\right]^{1/p}$$

NORMED LINEAR SPACES

Definition 3. A norm on X is a real valued function, whose value at x is denoted by $||x||$, satisfying the following conditions for all $x, y \in X$ and $\alpha \in \mathbf{F}$;

(1) $||x|| > 0$ if $x \neq 0$

(2) $||\alpha x|| = |\alpha| ||x||$

(3) $||x + y|| \le ||x|| + ||y||$.

A linear space X with a norm defined on it is called a **normed linear space**.

Example: l^p **space**. On the linear space $V_n(\mathbf{F})$, define

$$|| x || = [\sum_{k=1}^{n} | \alpha_i |^p]^{1/p}$$

where $p \ge 1$ is any real number and $x = (\alpha_1, \cdots, \alpha_n)$. This defines a norm (called p-norm) on $V_n(\mathbf{F})$. This space is called l^p space.

CHAPTER 1

INNER PRODUCT SPACES

INNER PRODUCTS

Let F be the field of real numbers or the field of complex numbers, and V a vector space over F an inner product on V is a function which assigns to each ordered' pair of vectors α, β in V a scalar $(\alpha|\beta)$ in F in such a way that for all α, β, γ in V and all scalars c.

(a) $(\alpha + \beta|\gamma) = (\alpha|\gamma) + (\beta|\gamma)$;

(b) $(c\alpha|\beta) = c(\alpha|\beta)$;

(c) $(\beta|\alpha) = (\overline{\alpha|\beta})$, the bar denoting complex conjugation

(d) $(\alpha|\alpha) > 0$ if $\alpha \neq 0$

It should be observed that conditions (a), (b) and (c) implies that

$(e) = (\alpha \mid c\beta + \gamma) = (\bar{c}(\alpha|\beta) + (\alpha|\gamma)$

One other point should be made. When F is the field R of real nunbers. The complex conjugates appearing in (c) and (e) are superflom. However, in the complex case they are necessary for the consistency of the conditions. Without these complex conjugates we would have the contradiction

$$(\alpha|\alpha) > 0 \text{ and } (i\alpha \mid i\alpha) = -1(\alpha|\alpha)$$

Example 1:

On F^n there is an inner product which we call the standard inner product. It is defined on $\alpha = (x_1, \cdots x_n)$ and $\beta = (y_1, \dots, y_n)$, by

$$(\alpha|\beta) = \sum_i x_i \overline{y_i}$$

When F is R this may be also written as

$$(\alpha|\beta) = \sum_i x_i y_i$$

In the real case, the standard inner product is often called the dot or scalar product and denoted by $\alpha \cdot \beta$.

INNER PRODUCTS SPACES

An inner product space is a real or complex vector space together with a specified inner product on that space.

- A finite-dimensional real inner product space is often called a Euclidean spare. A complex inner product spare often referred to as a unitary spare.
- Every inner product space is a normed linear space and every normed space is a metric space. Hence , every inner product space is a metric space.

Theorem

If V is an inner product space, then for any vector's α, β in V and any scalar c

(1) $||c\alpha|| = |c|||\alpha||$;

(ii) $\left||\alpha|\right| > 0$ for $\alpha \neq 0$

(iii) $|(\alpha \mid \beta)| \leqslant ||\alpha|| \; ||\beta||$

(iv) $\| \alpha + \beta|| \leqslant \| \alpha \| + \| \beta||$

Proof:

Statements (i) and (ii) follow almost immediately form the various definitions involved. The inequality in (iii) is clearly valid when $\alpha = 0.$ if $\alpha \neq 0$, put

$$\gamma = \beta - \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha$$

Then, $(\gamma \mid \alpha) = 0$ and

$$0 \leqslant \| \gamma \|^2 = \left(\beta - \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha \middle/ \beta - \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha\right)$$

$$= (\beta|\beta) - \frac{(\beta|\alpha)(\alpha|\beta)}{\| \alpha \|^2}$$

$$= \| \beta \|^2 - \frac{|(\alpha|\beta)|^2}{\| \alpha \|^2}$$

Hence,

$$|(\alpha \mid \beta)|^2 \leqslant \| \alpha \|^2 \| \beta \|^2$$

Now using (c) we find that

$$\| \alpha + \beta \|^2 = \| \alpha \|^2 + (\alpha \mid \beta) + (\beta \mid \alpha) + \| \beta \|^2$$

$$\begin{aligned} &= \| \alpha \|^2 + 2\mathrm{Re}\,(\alpha \mid \beta) + \| \beta \|^2 \\ &\leqslant \| \alpha \|^2 + 2 \| \alpha \| \| \beta \| + \| \beta \|^2 \\ &= (\| \alpha \| + \| \beta \|)^2 \end{aligned}$$

Thus,

$$\| \alpha + \beta \| \leqslant \| \alpha \| + \| \beta \|$$

the inequality (iii) is called the Cauchy -Schwarz inequality. It has a wide variety of application

the proof shows that if $\alpha$is non-zero then

$$\big((\alpha \mid \beta)\big) < \| \alpha \| \| \beta \|, \text{unless}$$

$$\beta = \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha$$

Then equality occurs in (iii) if and only if α and β are linearly independent.

CHAPTER 2

ORTHOGONAL SETS

Definition

Let α and β be the vectors in an inner product space V. Then α is orthogonal to β if (α | β) = 0. We simply say that and are orthogonal.

Definition

If S is a set of vectors in V, S is called an orthogonal set provided all set pairs of distinct vectors in S are orthogonal.

Definition

An orthogonal set is an orthogonal set S with the additional property that $\| \alpha \| = 1$ for every α in S.

- The zero vectors are orthogonal to every vector in V and is the only vector with this property.
- It is an appropriate to think of an orthonormal set as a set of mutually perpendicular vectors each having length l.

Example: the vector (x , y) is R^2 is orthogonal to $(-\text{y}, \text{x})$ with respect to the standard inner product, for,

$$((\text{x}, \text{y})|(-\text{y}, \text{x})) = -\text{xy} + \text{yx} = 0$$

- The standard basis of either R^n or C^n is an orthonormal set with respect to the standard inner product.

Theorem : An orthogonal set of nonzero vectors is linearly independent.

Proof:

Let S be a finite or infinite orthogonal set of nonzero vectors in a given inner product space suppose $\alpha_{1,}\alpha_2, \dots \alpha_n$ are distinct vectors in S and that β=$c_1\alpha_{1+} + \cdots c_n\alpha_n$

Then (β$|\alpha_k$)=($c_1\alpha_{1+} + \cdots c_n\alpha_n|\alpha_k$)

= $c_1(\alpha_1|\alpha_k)$ + $c_2(\ \alpha_2|\alpha_k)$+…+$c_n(\alpha_n|\alpha_k)$

= $c_k(\alpha_n|\alpha_k)$, since $\left(\alpha_i\,\middle|\alpha_j\right) = 0$,if i ≠ j and $\left(\alpha_i\,\middle|\alpha_j\right) = 1$,if i=j

Hence, c_k= (β $|\alpha_k)\ /(\alpha_k\ ,\alpha_k)$)

c_k=(β $|\alpha_k)/||\alpha_k||^2$,1≤ k ≤ m

Thus, when β=0 each c_k=0; so S is a linearly independent set.

Corollary:

If $\{\alpha_{1,}\alpha_2, \dots \alpha_m\}$ is an orthogonal set of nonzero vectors in a finite dimensional inner product space V, then m ≤ dimV.

That is number of mutually orthogonal vectors in V cannot exceed the dimensional V.

Corollary:

If a vector β is linear combination of an orthogonal of nonzero vectors $\alpha_{1,}\alpha_2, \dots \alpha_n$, then β is the particular linear combination

$$β =\sum_{k=1}^{m}\frac{(\beta\ |\alpha_k)}{||\alpha_k||^2}\alpha_k$$

Proof :

Since β is the linear combination of an orthogonal sequence of nonzero vectors $\alpha_{1,}\alpha_2, \dots \alpha_n$, we can write β =$c_1\alpha_1 + \cdots c_n\alpha_n$.

Where $c_k = \frac{(\beta|\alpha_k)}{||\alpha_k||^2}$, 1 ≤ k ≤ m (ref. by previous theorem)

Hence, β$= \sum_{k=1}^{m}\frac{(\beta|\alpha_k)}{||\alpha_k||^2}$

Theorem (Gram Schmidt Orthogonalization Process)

Let V be an inner product space and $\{\beta_{1,} \ldots, \beta_n\}$ be any linearly independent vectors in V. Then one may construct orthogonal vectors $\{\alpha_{1,}\alpha_2, \ldots \alpha_n\}$ in V, such that for each k = 1, 2, …n, the set $\{\alpha_{1,}\alpha_2, \ldots \alpha_k\}$is an orthogonal basis for the subspace of V spanned by $\beta_{1,} \ldots, \beta_n$.

Proof :

The vectors are obtained by means of a construction known as the Gram Schmidt orthogonalization process.

First let $\alpha_1 = \beta_1$ The other vectors are then given inductively as follows:

Suppose $\alpha_{1,}\alpha_2, \ldots \alpha_m$ $(1 \leq m \leq n)$ have been chosen so that for every k

$$\{\alpha_{1,}\alpha_2, \ldots \alpha_k\} \ (1 \leq k \leq m)$$

is an orthogonal basis for the space of v that is spanned by $\beta_{1,} \ldots, \beta_n$

To construct the next vector $\alpha_{m+1,}$ let

$$\alpha_{m+1,} = \beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}\alpha_k$$

Then $\alpha_{m+1} \neq 0$. For otherwise, $\beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}\alpha_k = 0$, implies,

$\beta_{m+1} = \beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}\alpha_k$,implies, β_{m+1} is a linear combination of $\alpha_{1,}\alpha_2, \ldots \alpha_m$ and hence a linear combination of $\beta_1, \beta_2, \ldots, \beta_m$, a contradiction.

Furthermore, if $1 \leq j \leq m$, then,

$$(\alpha_{m+1}|\ \alpha_j) = (\beta_{m+1}|\ \alpha_j) - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}(\alpha_k\ |\alpha_j)$$

$$= (\beta_{m+1}|\ \alpha_m) - (\beta_{m+1}|\ \alpha_j)$$, using the orthonormality of $\{\alpha_{1,}\alpha_2, \ldots \alpha_m\}$.

Therefore $\{\alpha_{1,}\alpha_2, \ldots, \alpha_{m+1}\}$ is an orthogonal set consisting of m+1 nonzero vectors in the subspace spanned by $\beta_{1,} \ldots, \beta_{m+1}$. Hence by an earlier Theorem , it is a basis for this subspace .Thus the vectors , $\alpha_{1,}\alpha_2, \ldots \alpha_n$ may be constructed using the formula

$$\alpha_{m+1} = \beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2} \alpha_k$$

In particular, when n=3 ,we have

$$\alpha_1 = \beta_1$$

$$\alpha_2 = \beta_2 - \frac{(\alpha_2|\beta_2)}{||\alpha_k||^2} \alpha 1$$

$$\alpha_3 = \beta_3 - \frac{(\beta_3|\alpha_1)}{||\alpha_1||^2} \alpha 1 - \frac{(\alpha_2|\beta_3)}{||\alpha_k||^2} \alpha_2$$

Corollary :

Every finite dimensional inner product space has an orthonormal basis.

Proof :

Let V be a finite dimensional inner product space and { $\beta_{1,} \dots, \beta_n$} a basis for V. Apply the gram Schmidt orthogonalization process to construct an orthogonal basis , simply replace each vector α_n by $\frac{\alpha_k}{||\alpha_k||}$.

Gram-Schmidt process can be used to test for linear dependence . For suppose $\beta_{1,} \dots, \beta_n$ are linearly independent vectors in an inner product space; to exclude a trivial case , assume that β≠0. Let m be largest integers for which $\beta_{1,} \dots, \beta_m$are independent. Then 1≤m<n.

Let $\alpha_1, \alpha_2, \dots \alpha_m$ be the vectors obtained by applying the orthogonalization process to $\beta_{1,} \dots, \beta_m$. Then the vector α_{m+1} given by $\alpha_{m+1} = \beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2} \alpha_k$ is necessarily 0.

For α_{m+1} is in the subspace spanned by $\alpha_1, \alpha_2, \dots \alpha_m$ and orthogonal to each of the vectors , hence it is 0 as β=$\sum_{k=1}^{m} \frac{(\beta|\alpha_k)}{||\alpha_k||^2} \alpha_k$. Conversely, if $\alpha_1, \alpha_2, \dots \alpha_m$ are different from 0 and α_{m+1} =0, then $\beta_{1,} \dots, \beta_{m+1}$ are linearly independent .

Definition:

A best approximation to β ∈ V by vectors in a subspace W of V is a vector α∈ W such that

$$\|\beta - \alpha\| \le \|\beta - \gamma\| \text{ for every vector } \gamma \in W .$$

Theorem

Let W be a subspace of an inner product space V and let $\beta \in V$.

1. The vector $\alpha \in W$ is a best approximation to $\beta \in V$ by vectors in W if and only if $\beta - \alpha$ is orthogonal to every vector in W .
2. If a best approximation to $\beta \in V$ by vectors in W exists, it is unique.
3. If W is finite-dimensional and $\{\alpha_{1,}\alpha_2, \ldots \alpha_n\}$ is any orthonormal basis for W , then the vector

$$\alpha = \sum_{k=1}^{n} \frac{(\beta \mid \alpha_k)}{\|\alpha_k\|^2} \alpha_k$$

is the (unique) best approximation to β by vectors in W.

Definition:

Let V be an inner product space and S be any set of vectors in V. The orthogonal complement of S is the set $S^{\perp}$ of all vectors in V which are orthogonal to every vector in S.

That is, $S^{\perp} = \{\alpha \in V : (\alpha \mid \beta) = 0, \forall \beta \in S\}$

Definition:

Whenever the vector α in the above theorem exists it is called the orthogonal projection of β on W. If every vector in V has an orthogonal projection on W, the mapping that assigns to each vector in V its orthogonal projection on W is called the orthogonal projection of V on W.

Corollary :

Let V be an inner product space and W a finite dimensional subspace and E be the orthogonal projection of V on W. Then the mapping

$$\beta \to \beta - E\beta$$

is the orthogonal projection of V on $W^{\perp}$.

Proof :

Let $\beta \in V$. Then $\beta - E\beta \in W^{\perp}$, and for any $\gamma \in W^{\perp}$, $\beta - \gamma = E\beta + (\beta - E\beta - \gamma)$

Since $E\beta \in W$ and $\beta - E\beta - \gamma \in W^{\perp}$,

It follows that

$$||\beta - \gamma||^2 = (E\beta + (\beta - E\beta - \gamma), E\beta + (\beta - E\beta - \gamma))$$

$$= ||E\beta||^2 + ||\beta - E\beta - \gamma||^2$$

$$\geq ||\beta - (\beta - E\beta)||^2$$

with strict inequality when $\gamma \neq \beta - E\beta$. Therefore, $\beta - E\beta$ is the best approximation to β by vectors in $W^{\perp}$.

Theorem

Let W be a finite dimensional subspace of an inner product space V and let E be the orthogonal projection of V on W. Then E is an idempotent linear transformation of V onto W , $W^{\perp}$ is the null space of E , and $V = W \oplus W^{\perp}$.

Proof

Let β be an arbitrary vector in V. Then $E\beta$ is the best approximation to β that lies in W . In particular, $E\beta = \beta$ when β is in W . Therefore, $E(E\beta) = E\beta$ for every β in V; that is, E is idempotent : $E^2 = E$. To prove that E is linear transformation, let α and β be any vectors in V and c an arbitrary scalar ,Then by theorem,

$\alpha - E\alpha$ and $\beta - E\beta$ are each orthogonal to every vector in W . Hence the vector

$c(\alpha - E\alpha) + (\beta - E\beta) = (c\alpha + \beta) - (cE\alpha + E\beta)$

Also belongs to $W^{\perp}$. Since $cE\alpha + E\beta$ is a vector in W , it follows from theorem that

$E(c\alpha + \beta) = cE\alpha + E\beta$.

Again let β be any vector in V. Then $E\beta$ is the unique vector in W such that $\beta - E\beta$ is in $W^{\perp}$. Thus $E\beta = 0$ when β is in $W^{\perp}$.

Conversely, β is in $W^{\perp}$ when $E\beta = 0$. Thus $W^{\perp}$ is the null space of E.

The equation ,

$$\beta = E\,\beta + \beta - E\beta$$

shows that V = $W + W^{\perp}$; moreover $W \cap W^{\perp} = \{0\}$; for if α is a vector in $W \cap W^{\perp}$,then $(\alpha \mid \alpha)$ =0. Therefore, α=0 and V is the direct sum of W and $W^{\perp}$.

Corollary :

Under the conditions of theorem, $I - E$ is the orthogonal projection of V on $W^{\perp}$.

It is an independent linear transformation of V onto $W^{\perp}$ with null space W .

Proof :

We have seen that the mapping $\beta \to \beta - E\,\beta$ is the orthogonal projection of V on $W^{\perp}$.

Since E is a linear transformation , this projection $W^{\perp}$ is the linear transformation $I - E$ from its geometric properties one sees that $I - E$ is an idempotent .Transformation of V onto W .

This also follows from the computation $(I - E)(I - E) = I - E - E + E^2$

$$= I - E$$

Moreover , $(I - E)\beta = 0$ If and only if $\beta = E\beta$, and this is the case if and only if β is in W .

Therefore W is the null space of $I - E$.

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DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS

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by

ANILA BIJU

DB18CMSR18

Under the guidance of

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March 2021

Examiners 1: Examiner 2:

CERTIFICATE

It is to certify that this project report '**DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS**' is the bonafide project of **ANILA BIJU** who carried out the project under my supervision.

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DECLARATION

I, **ANILA BIJU**, hereby declare that this project report entitled '**DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS**' is an original record of studies and bonafide project carried out by me during the period from November 2019 to March 2020, under the guidance of **Ms.Sneha P Sebastian**, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not been submitted by me elsewhere for the award of any degree, diploma, title or recognition, before.

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ANILA BIJU

CONTENTS

INTRODUCTION

In mathematics, a group is a set equipped with a binary operation that combines any two elements to form a third element in such a way that the three conditions called group axioms are satisfied , namely associativity , identity and invertability.

Let us take a moment to review our present stockpile of groups. Starting with finite groups, we have the cyclic group $\mathbb{Z}_n$,the symmetric group S_n , and the alternating group A_n for each positive integer n. We also have the dihedral group D_n and klein 4-group . Of course we know that subgroups of these groups exists. Turning to infinite groups , we have $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ under addition , and their non zero elements under multiplication we also have the group S_A of all permutation of an infinite set A , as well as various groups formed from matrices .

One purpose of this section is to show a way to use known groups as building blocks to form more groups. Given two groups G and H, it is possible to construct a new group from the cartesian product of G and H . Conversely , given a large group , it is sometimes possible to decompose the group ; that is , a group is sometimes isomorphic to the direct product of two smaller groups. Rather than studying a large group , it is often easier to study the component group of that group.

PRELIMINARY

Groups : A non empty set G together with an operation $*$ is said to be a group , denote by $(G,*)$, if it satisfy the following axioms.

- Closure property
- Associative property
- Existence of identity
- Existence of inverse

Abelian group

A group $(G,*)$ is said to be abelian if it satisfies commutative law .

Finite group

If the underlying set G of the group $(G,*)$ consist of finite number of elements , then the group is finite group .

Infinite group

A group that is not finite is an infinite group .

Order of a group : The number of elements in a finite group is called the order of the group , denoted by $O(G)$.

Example

Show that the set of integers $\mathbb{Z}$ is a group with respect to the operation of addition of integers.

$$\mathbb{Z} = \{\dots\dots\dots.-3,-2,-1,0,1,2,3,\dots\dots\dots\}$$

Since the addition of two integers gives an integer , it satisfy closure property .

If $a, b, c \epsilon \mathbb{Z}$ then the $(a+b)+c = a+(b+c)$, hence associativity holds .

There is a number $0 \epsilon \mathbb{Z}$ such that $0+a = a+0$, hence identity exists

If $a \epsilon \mathbb{Z}$ then there exists $-a \epsilon \mathbb{Z}$, such that $-a + a = 0 = a + -a$

Therefore inverse exist .

Therefore $\mathbb{Z}$ is a group under addition .

Subgroup

A subset H of G is said to be a subgroup of G if H itself is a group under the same operation in G.

There are two different types of group structure of order 4 .

$\mathbb{Z}_4 = \{ 0,1,2,3\}$

Klein 4 – group , $V = \{e, a, b, c\}$

Cyclic group

A group G is cyclic if there is some element 'a' in G that generate G. And the element 'a' is called generator of G.

Group Homomorphism

A function $\Psi: G \rightarrow G'$ is a group homomorphism (or simply homomorphism).

If $\Psi(ab) = \Psi(a)\, \Psi(b)$ hold for all $a , b \in G$, is called homomorphism property .

Isomorphism

A one to one and onto homomorphism $\Psi: G \rightarrow G'$ is called an isomorphism .

CHAPTER – 1

DIRECT PRODUCT OF GROUPS

Definition

The Cartesian product of sets $S, S_2, \dots \dots, S_n$ is the set of all ordered n-tuples $(a_1, a_2, \dots \dots, a_n)$, where $a_i \in S_i$ for $i = 1,2,3, \dots \dots, n$. The Cartesian product is denoted by either

$S_1 \times S_2 \times \dots \dots \times S_n$ or by $\Pi_{i=1}^{n} S_i$.

Let $G_1, G_2, \dots \dots, G_n$ be groups and let us use multiplicative notation for all the group operations.

If we consider G_i as a set , $i = 1,2, \dots \dots . n$. we have the products $G_1 \times G_2 \times \dots \dots, \times G_n$ we denote it by $\Pi_{i=1}^{n} G_i$. This product is called direct-product of groups. We can make $\Pi_{i=1}^{n} G_i$ into a group by means of a binary operation of multiplication by components.

Theorem

Let $G_1, G_2, \dots \dots, G_n$ be groups. For $(a_1, a_2, \dots \dots, a_n)$ and $(b_1, b_2, \dots \dots, b_n)$ in $\Pi_{i=1}^{n} G_i$ define ;

$$(a_1, a_2, \dots \dots, a_n)(b_1, b_2, \dots \dots, b_n) = (a_1 b_1, a_2 b_2, \dots \dots, a_n b_n)$$

Then $\Pi_{i=1}^{n} G_i$ is a group.

Proof

We have ,

$$\Pi_{i=1}^{n} G_i = \{(a_1, a_2, \dots \dots, a_n) : a_i \in G_i\}$$

(1) **Closure property**

Let $(a_1, a_2, \dots \dots, a_n), (b_1, b_2, \dots \dots, b_n) \in \Pi_{i=1}^{n} G_i$

And we have ,

$$(a_1, a_2, \dots \dots, a_n)(b_1, b_2, \dots \dots, b_n) = (a_1 b_1, a_2 b_2, \dots \dots, a_n b_n)$$

Here $a_i \in G_i$ and $b_i \in G_i$ for $i = 1,2, \dots \dots, n$

$\because$ G_i is a group , $a_i b_i \in G_i$ for $i = 1,2, \dots \dots, n$

$\Rightarrow (a_1 b_1, a_2 b_2, \dots \dots, a_n b_n) \in \Pi_{i=1}^{n} G_i$

i.e. $\Pi_{i=1}^{n} G_i$ is closed under the binary operation.

(2) **Associativity**

Let $(a_1, a_2, \dots \dots, a_n), (b_1, b_2, \dots \dots, b_n), (c_1, c_2, \dots \dots, c_n) \in \Pi_{i=1}^{n} G_i$

We have,

$$(a_1, a_2, \dots \dots, a_n)(b_1, b_2, \dots \dots, b_n)(c_1, c_2, \dots \dots, c_n)$$

$$= (a_1 b_1 c_1 , a_2 b_2 c_2, \dots \dots, a_n b_n c_n) \in \Pi_{i=1}^{n} G_i$$

$$[(a_1, a_2, \dots \dots, a_n)(b_1, b_2, \dots \dots, b_n)](c_1, c_2, \dots \dots, c_n)$$

$$\begin{aligned}
&= [a_1 b_1, a_2 b_2, \dots \dots, a_n b_n](c_1, c_2, \dots \dots, c_n) \\
&= [(a_1 b_1) c_1, (a_2 b_2) c_2, \dots \dots, (a_n b_n) c_n] \\
&= [a_1 (b_1 c_1), a_2 (b_2 c_2), \dots \dots, a_n (b_n c_n)] \\
&= (a_1, a_2, \dots \dots, a_n)[b_1 c_1, b_2 c_2, \dots \dots, b_n c_n] \\
&= (a_1, a_2, \dots \dots, a_n)[(b_1, b_2, \dots \dots, b_n)(c_1, c_2, \dots \dots, c_n)]
\end{aligned}$$

Hence associativity holds.

(3) **Existence of identity**

If e_i is the identity element in G_i.

Then,

$$(e_1, e_2, \dots \dots ., e_n) \in \Pi_{i=1}^{n} G_i$$

Also for ,

$$(a_1, a_2, \dots \dots ., a_n) \in \Pi_{i=1}^{n} G_i,$$

$$\begin{aligned}(a_1, a_2, \dots \dots ., a_n)(e_1, e_2, \dots \dots ., e_n) &= (a_1 e_1, a_2 e_2, \dots \dots ., a_n e_n) \\ &= (a_1, a_2, \dots \dots ., a_n)\end{aligned}$$

$\therefore$ $(e_1, e_2, \dots \dots ., e_n)$ is the identity element 'e' in $\Pi_{i=1}^{n} G_i$

(4) **Existence of inverse**

Let $(a_1, a_2, \dots \dots ., a_n) \in \Pi_{i=1}^{n} G_i$

Here $a_i \in G_i$ for $i = 1,2, \dots \dots ., n$.

Since G_i is a group ,

$\exists$ an inverse element a_i^{-1} in G_i : $a_i a_i^{-1} = e_i \qquad i = 1,2, \dots \dots ., n$

Clearly, $\qquad (a_1^{-1}, a_2^{-1}, \dots \dots ., a_n^{-1}) \in \Pi_{i=1}^{n} G_i$ &

$$(a_1, a_2, \dots \dots ., a_n)(a_1^{-1}, a_2^{-1}, \dots \dots ., a_n^{-1}) = (e_1, e_2, \dots \dots ., e_n)$$

Hence $\Pi_{i=1}^{n} G_i$ is a group.

Note

If the operation of each G_i is a commutative. We sometimes use additive notation in $\Pi_{i=1}^{n} G_i$ and refer to $\Pi_{i=1}^{n} G_i$ as the direct sum of the group G_i. The notation $\oplus_{i=1}^{n} G_i$, especially with abelian groups with operation $+$.

The direct sum of abelian groups $G_1, G_2, \dots \dots, G_n$ may be written $G_1 \oplus G_2 \oplus \dots \dots \oplus G_n$

- Direct product of abelian group is abelian

Example

Q. Check whether $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic or not.

$$\mathbb{Z}_2 = \{0,1\}$$

$$\mathbb{Z}_3 = \{0,1,2\}$$

$$\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$$

Consider,

$1(1,1) = (1,1)$

$2(1,1) = (1,1) + (1,1) = (0,2)$

$3(1,1) = (1,1) + (1,1) + (1,1) = (1,0)$

$4(1,1) = 3(1,1) + (1,1) = (1,0) + (1,1) = (0,1)$

$5(1,1) = 4(1,1) + (1,1) = (0,1) + (1,1) = (1,2)$

$6(1,1) = 5(1,1) + (1,1) = (1,2) + (1,1) = (0,0)$

$\therefore$ $(1,1)$ is a generator of $\mathbb{Z}_2 \times \mathbb{Z}_3$

$\therefore$ $\mathbb{Z}_2 \times \mathbb{Z}_3$ is a cyclic group generated by $(1,1)$.

Q. Check whether $\mathbb{Z}_3 \times \mathbb{Z}_3$ is cyclic or not.

$$\mathbb{Z}_3 = \{0,1,2\}$$

$$\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$$

$1(0,1) = (0,1)$

$2(0,1) = (0,2)$

$3(0,1) = (0,3) = (0,0)$ $\quad\therefore$ order $(0,1) = 3$

$1(0,2) = (0,2)$

$2(0,2) = (0,4) = (0,1)$

$3(0,2) = (0,6) = (0,0)$ $\quad\therefore$ order $(0,2) = 3$

Every element added to itself three times gives the identity. Thus no element can generate the group. Hence $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not cyclic.

similarly $\mathbb{Z}_m \times \mathbb{Z}_m$ is not cyclic for any m.

Theorem

The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime, that is, the gcd of m and n is 1.

Proof

Suppose $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and isomorphic to $\mathbb{Z}_{mn}$.

To show that m and n are relatively prime.

Suppose not, let d be the gcd of m and n.

So that $d > 1$

Consider $\frac{mn}{d}$, which is an integer since $d|m$ and $d|n$

Let (r, s) be an arbitrary element of $\mathbb{Z}_m \times \mathbb{Z}_n$, add (r, s) repeatedly $\frac{mn}{d}$ times

$$\left.(r, s) + (r, s) +, \dots \dots, +(r, s)\right\} \; \frac{mn}{d} times = (0,0)$$

$\therefore$ no element of $\mathbb{Z}_m \times \mathbb{Z}_n$ having order mn. $\therefore$ no element of $\mathbb{Z}_m \times \mathbb{Z}_n$ can generate $\mathbb{Z}_m \times \mathbb{Z}_n$ which is not possible. $\because$ $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic. Hence $gcd(m, n) = 1$.

i.e. m and n are relatively prime.

Conversely, suppose m and n are relatively prime, i.e. $\gcd(m, n) = 1$

To show that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic.

If $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic, then it is isomorphic to $\mathbb{Z}_{mn}$, $\because$ $\mathbb{Z}_m \times \mathbb{Z}_n$ has mn elements.

Consider the cyclic subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$ generated by the element $(1,1)$.The order of this cyclic subgroup is the smallest power of $(1,1)$,that gives the identity $(0,0)$. Here taking a power of $(1,1)$ in our additive notation will involve adding $(1,1)$ to itself repeatedly.

Consider $(1,1) + (1,1) +, \dots \dots, +(1,1)$

If we add first coordinates m times , we get zero.

$\therefore$ order of first coordinate $= m$.

Similarly , Order of second coordinate $= n$.

The two coordinates together become zero. If we add them $lcm(m, n)$ times.

$\because \gcd(m, n) = 1$, We get the $lcm = mn$.

i.e. $(1,1)$ generates a cyclic subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$ of order mn , which is the order of the whole group.

$$\Rightarrow \mathbb{Z}_m \times \mathbb{Z}_n = < (1,1) >$$

$\Rightarrow\ \mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic.

Corollary

The group $\prod_{i=1}^{n} \mathbb{Z}_{m_i}$ is cyclic and isomorphic to $\mathbb{Z}_{m_1 m_2 \ldots\ldots m_n}$ if and only if the numbers m_i for $i = 1,2, \ldots \ldots, n$ are such that the gcd of any two of them is 1.

Example

If n is written as a product of powers of distinct prime numbers , as in ,

$$n = (p_1)^{n_1} . (p_2)^{n_2} \ldots \ldots . (p_n)^{n_r}$$

Then $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{(p_1)^{n_1}} \times \mathbb{Z}_{(p_2)^{n_2}} \times \ldots \ldots . \times \mathbb{Z}_{(p_r)^{n_r}}$.

In particular , $\mathbb{Z}_{72}$ is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_9$.

Consider set of integers $\mathbb{Z}$, cyclic subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$, $n \in \mathbb{Z}$. Consider $2\mathbb{Z}$ and $3\mathbb{Z}$, then $< 2 > \cap < 3 > = < 6 >$

$\therefore$ if we take $r\mathbb{Z}$, $s\mathbb{Z}$ of $\mathbb{Z}$, then the $lcm(r, s)$ =generator of $< r > \cap < s >$

Using this we can define the lcm of the positive integers.

Definition

Let $r_1, r_2, \ldots \ldots, r_n$ be positive integers. Their least common multiple (abbreviated lcm) is the positive generator of the cyclic group of all common multiples of the r_i , that is the cyclic group of all integers divisible by each r_i for $i = 1,2, \ldots \ldots, n$.

Theorem

Let $(a_1, a_2, \dots \dots., a_n) \in \prod_{i=1}^{n} G_i$.

If a_i is of finite order r_i in G_i , then the order of $(a_1, a_2, \dots \dots., a_n)$ in $\prod_{i=1}^{n} G_i$ is equal to the least common multiple of all the r_i.

Proof

Given,

order of $a_1 = r_1 \Rightarrow a_1^{r_1} = e_1$ in G_1

order of $a_2 = r_2 \Rightarrow a_2^{r_2} = e_2$ in G_2

.

.

.

order of $a_n = r_n \Rightarrow a_n^{r_n} = e_n$ in G_n.

We have to find a power k for $(a_1, a_2, \dots \dots., a_n)$.

So that $(a_1, a_2, \dots \dots., a_n)^k = (e_1, e_2, \dots \dots., e_n)$.

The power must simultaneously be a multiple of r_1 , multiple of r_2 and so on. But k is the least positive integers having the above property.

$$\therefore k = lcm(r_1, r_2, \dots \dots., r_n).$$

Q. Find the order of $(8,4,10)$ in the group $\mathbb{Z}_{12} \times \mathbb{Z}_{60} \times \mathbb{Z}_{24}$.

$O(8) = 3$ in Z_{12}

$O(4) = 15$ in Z_{60}

$O(10) = 12$ in Z_{24}

$O(8,4,10) = lcm(3,15,12) = 60$

Q. Find a generator of $\mathbb{Z} \times \mathbb{Z}_2$

$$\mathbb{Z} \times \mathbb{Z}_2 = \{(n, 0), (n, 1): n \in \mathbb{Z}\}$$
$$(n, 0) = n(1,0)$$
$$(n, 1) = (n, 0) + (0,1) = n(1,0) + (0,1)$$

$\therefore$ $\mathbb{Z} \times \mathbb{Z}_2$ is generated by $\{(1,0), (0,1)\}$

In general , $\mathbb{Z} \times \mathbb{Z}_n$ is generated by ,

$$\{(1,0,0, \dots \dots ,0), (0,1,0, \dots \dots ,0), \dots \dots , (0,0, \dots \dots ,1)\}$$

Q. Find the order of (3,10,9) in $(\mathbb{Z}_4, \mathbb{Z}_{12}, \mathbb{Z}_{15})$

$O(3) = 4$ in $\mathbb{Z}_4$

$O(10) = 6$ in $\mathbb{Z}_{12}$

$O(9) = 5$ in $\mathbb{Z}_{15}$

$\therefore O(3,10,9) = lcm(4,6,5)$
$= 60$

$\therefore$ order of (3,10,9) in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$ is 60.

CHAPTER-2

FUNDAMENTAL THEOREM OF FINITELY GENERATED ABELIAN GROUPS

Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups in the form,

$$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \dots \dots . \times \mathbb{Z}_{(p_n)^{r_n}} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \dots \dots . \times \mathbb{Z}$$

Where the p_i are primes , not necessarily distinct and the r_i are positive integers.

Remark

- The direct product is unique except for possible rearrangement of the factors.
- The number of factors $\mathbb{Z}$ is unique and this number is called Betti number.

Example

Find all abelian groups , upto isomorphism of order

1)8 , 2)16 , 3)360

(1) Order 8

$$8 = 1 \times 8$$

$$8 = 2 \times 4 = 2 \times 2^2$$

$$8 = 2 \times 2 \times 2$$

3 non-isomorphic groups are $\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

(2) Order 16

$16 = 1 \times 16 = 1 \times 2^4$

$16 = 2 \times 8 \ \ = 2 \times 2^3$

$16 = 4 \times 4 \ \ = 2^2 \times 2^2$

$16 = 2 \times 2 \times 2 \times 2$

$16 = 2 \times 2 \times 2^2$

$\mathbb{Z}_{16}, \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$

(3) Order 360

$360 = 2^2 \cdot 3^2 \cdot 5$

Possibilities are,

1) $\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
2) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
3) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
4) $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
5) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
6) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

Definition

A group G is decomposable if it is isomorphic to a direct product of two proper non-trivial subgroups , otherwise G is indecomposable.

Example

$\mathbb{Z}_6$ is decomposable while $\mathbb{Z}_5$ is indecomposable.

$\mathbb{Z}_6$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$

$\mathbb{Z}_{mn}$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$, if m and n are prime.

Theorem

The finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.

Proof

Let G be a finite indecomposable abelian group $\because G$ is finitely generated , we can apply fundamental theorem of finitely generated abelian groups.

$\therefore G \cong \mathbb{Z}_{(p)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \ldots \ldots . \times \mathbb{Z}_{(p_n)^{r_n}}$

$\because G$ is indecomposable and $\mathbb{Z}_{(p_i)^{r_i}}$'s are proper subgroups we get in the above , there is only one factor say $\mathbb{Z}_{(p_i)^{r_i}}$ which is cyclic group with order a prime power.

Theorem

If m divides the order of a finite abelian group , then G has a subgroup of order m.

Proof

Given G is a finite abelian group.

$\therefore$ we can apply Fundamental Theorem ,

Hence,

$$G \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \dots \dots . \times \mathbb{Z}_{p_n^{r_n}}$$

Here all primes need not be distinct.

Then,

$$O(G) = p_1^{r_1} . p_2^{r_2} \dots \dots p_n^{r_n}$$

Let m is a $+ve$ integer which divides $O(G)$.

$0 \leq s_i \leq r_i$ By theorem , " let G be a cyclic group with n elements and generated by a. Let $b \in G$, $b = a^s$, then ' b' generates a cyclic subgroup H of G containing $\frac{n}{d}$ elements , where $d = \gcd(n, s)$."

$p_i^{r_i - s_i}$ generates a cyclic subgroup of $\mathbb{Z}_{p_i^{r_i}}$ having order $\frac{p_i^{r_i}}{gcd(p_i^{r_i}, p_i^{r_i - s_i})}$

$$= \frac{p_i^{r_i}}{p_i^{r_i - s_i}} = p_i^{s_i}$$

$$\therefore O(< p_i^{r_i - s_i} >) = p_i^{s_i}$$

i.e. $< p_1^{r_1 - s_1} >$ is a subgroup of $\mathbb{Z}_{p_1^{r_1}}$ having order $p_1^{s_1}$.

$< p_2^{r_2 - s_2} >$ is a subgroup of $\mathbb{Z}_{p_2^{r_2}}$ having order $p_2^{s_2}$.

…………………………………………………………

$< p_n^{r_n - s_n} >$ is a subgroup of $\mathbb{Z}_{p_n^{r_n}}$ having order $p_n^{s_n}$.

$\therefore < p_1^{r_1 - s_1} > \times < p_2^{r_2 - s_2} > \times \dots \dots . \times < p_n^{r_n - s_n} >$ is a subgroup of $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \dots \dots . \times \mathbb{Z}_{p_n^{r_n}}$ having order $p_1^{s_1} \cdot p_2^{s_2} \cdots\cdots p_n^{s_n} = m$.

Theorem

If m is a square free integer , that is m is not divisible by the square of any prime . Then every abelian group of order m is cyclic.

Proof

Let m be a square free integer , then $p^i \nmid m$ for every i greater than 1 for a prime p.

Given G is a finite abelian group having order m , by fundamental theorem , then

$$G \cong \mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \dots \dots . \times \mathbb{Z}_{(p_n)^{r_n}}$$

Then,

$$O(G) = p_1^{r_1} \cdot p_2^{r_2} \cdots\cdots p_n^{r_n}$$

$\because$ $O(G)$ is a square free integer , the only possibility

$$r_1 = r_2 = \dots \dots . . = r_n = 1$$

Then,

$$G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \dots . \times \mathbb{Z}_{p_n}$$

$$\cong \mathbb{Z}_{p_1, p_2, \dots\dots, p_n}$$, which is cyclic.

Example

15 is a square free integer. So an abelian group of order 15 is cyclic.

CONCLUSION

Direct product of groups is the product $G_1 \times G_2, \dots \dots . G_n$,where each G_i is a set. We have discussed about definition and some properties related to the direct product of groups. The fundamental theorem of finitely generated abelian group helped us to get a deeper understanding about the topic. The theorems gives us complete structural information about abelian group, in particular finite abelian group. We have also discussed some examples in order to develope more intrest in algebra.

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Project Report on

INNER PRODUCT SPACES

Dissertation submitted in the partial fulfilment

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Bachelor of Science in Mathematics of

Kannur University

Name : ANJALY BABY

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2.

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BONAFIDE CERTIFICATE

Certified that this project report on " INNER PRODUCT SPACES” is the bonafide work of ANJALY BABY who carried out the project work under my supervision.

Mrs. Riya Baby

Head of Department

Mr. Anil M V

Supervisor

DECLARATION

I, ANJALY BABY hereby declare that the project work entitled 'INNER PRODUCT SPACES′ has been prepared by me and submitted to Kannur University in partial fulfilment of requirement for the award of Bachelor of Science is a record of original work done by me under the supervision of Mr. ANIL M V, Assistant Professor, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu.

I, also declare that this Project work has been submitted by me fully or partially for the award of any Degree, Diploma, Title or recognition before any authority.

Place : Angadikadavu

ANJALY BABY

Date :

DB18CMSR19

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ANJALY BABY

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INTODUCTION

In linear algebra, an inner product space is a vector space with an additional structure called an inner product. This additional structure associates each pair of vectors in the space with a scalar quantity known as the inner product of the vectors. Inner products allow the rigorous introduction of intuitive geometrical notions such as the length of a vector or the angle between two vectors. They also provide the means of defining orthogonality between vectors (zero inner product). Inner product spaces generalize Euclidean spaces (in which the inner product is the dot product, also known as the scalar product) to vector spaces of any (possibly infinite) dimension and are studied in functional analysis. The first usage of the concept of a vector space with an inner product is due to Peano, in 1898.

An inner product naturally induces an associated norm, thus an inner product space is also a normed vector space. A complete space with an inner product is called a Hilbert space. An (incomplete) space with an inner product is called a pre-Hilbert space.

PRELIMINARIES

LINEAR SPACES

Definition 1: A linear (vector) space X over a field $\mathbf{F}$ is a set of elements together with a function, called addition, from $X \times X$ into X and a function called scalar multiplication, from $\mathbf{F} \times X$ into X which satisfy the following conditions for all $x, y, z \in X$ and $\alpha, \beta \in \mathbf{F}$;

i. $(x + y) + z = x + (y + z)$

ii. $x + y = y + x$

iii. There is an element 0 in X such that $x + 0 = x$ for all $x \in X$.

iv. For each $x \in X$ there is an element $-x \in X$ such that $x + (-x) = 0$.

v. $(x + y) = \alpha x + \alpha y$

vi. $(\alpha + \beta)x = \alpha x + \beta x$

vii. $\alpha(\beta x) = (\alpha\beta)x$

viii. $1 \cdot x = x$.

Properties i to iv imply that X is an abelian group under addition and v to vi relate the operation of scalar multiplication to addition X and to addition and multiplication in $\mathbf{F}$.

Examples:

(a) $V_n(\mathbf{R})$. The vectors are n-tuples of real numbers and the scalars are real

numbers with addition and scalar multiplication defined by

$$(\alpha_1,\cdots,\alpha_n)+(\beta_1,\cdots,\beta_n)=(\alpha_1+\beta_1,\cdots,\alpha_n+\beta_n) \tag{1}$$

$$\beta(\alpha_1,\cdots,\alpha_n)=(\beta\alpha_1,\cdots,\beta\alpha_n) \tag{2}$$

$V_n(\mathbf{R})$ is a linear space over **R**. Similarly, the set of all n-tuples of complex numbers with the above definition of addition and multiplication is a linear space over **C** and is denoted as $V_n(\mathbf{C})$.

(b) The set of all functions from a nonempty set X into a field **F** with addition and scalar multiplication defined by

$$[f+g](t)=f(t)+g(t) \quad \text{and} \quad [\alpha f](t)=\alpha f(t);\ f,g\in X,\ t\in T \tag{3}$$

is a linear space.

Let T = **N** the set of all positive integers and X is the set of all sequences of elements **F** with addition and scalar multiplication defined by

$$(\alpha_n+\beta_n)=\ (\alpha_n+\beta_n) \tag{4}$$

$$\beta\left(\alpha_n\right)=\left(\beta\alpha_n\right) \tag{5}$$

denoted as $V_\infty(\mathbf{F})$, form a linear space.

METRIC SPACES

Remember the distance function in the Euclidean space $\mathbf{R}^n$.

Let $x, y, z \in \mathbf{R}^n$, then

(1) $|x - y| \geq 0$; $|x - y| = 0$ if and only if $x = y$;

(2) $|x - y| = |y - x|$;

(3) $|x - y| \leq |x - z| + z - y|$.

Definition 2: A **metric** or **distance function** on a set X is a real valued function d defined on $X \times X$ which has the following properties: for all $x, y, z \in X$.

(1) $d(x, y) \geq 0$; $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq d(x, z) + d(z, y)$

A **metric space** (X, d) is a nonempty set X and a metric d defined on X.

Examples: In addition to the Euclidean spaces let us have the following examples. *Here all functions are assumed to be continuous*. Let L^p denotes a set of complex valued functions in $\mathbf{R}^n$ such that $|f|^p$ is integrable. Let us recall some results concerning such functions.

Höder's Inequality: If $p > 1$, $1/q = 1 - 1/p$

$$\int | fg | \leq [\int | f |^p]^{1/p} [\int | g |^q]^{1/q} .$$

Minkowski's Inequality: If $p \geq 1$,

$$[\int | f + g |^p]^{1/p} \leq [\int | f |^p]^{1/p} + [\int | g |^p]^{1/p}$$

If x_k and y_k for $k = 1, \ldots, m$ are complex numbers, let $f(t) = |x_k|$ and $g(t) = |y_k|$ for $t \in [k, k+1]$ and $f(t) = 0 = g(t)$ for $t \in [1, m+1]$. Then we obtain the summation form of the above inequalities from the integral form

Hölder's Inequality

$$\sum_{k=1}^{m} |x_k y_k| \le \left[\sum_{k=1}^{m} |x_k|^p\right]^{1/p} \left[\sum_{k=1}^{m} |y_k|^q\right]^{1/q}$$

Minkowski's Inequality:

$$\left[\sum_{k=1}^{m} |x_k + y_k|^p\right]^{1/p} \le \left[\sum_{k=1}^{m} |x_k|^p\right]^{1/p} + \left[\sum_{k=1}^{m} |y_k|^p\right]^{1/p}$$

NORMED LINEAR SPACES

Definition 3. A norm on X is a real valued function, whose value at x is denoted by $||x||$, satisfying the following conditions for all $x, y \in X$ and $\alpha \in \mathbf{F}$;

(1) $||x|| > 0$ if $x \neq 0$

(2) $||\alpha x|| = |\alpha| ||x||$

(3) $||x + y|| \le ||x|| + ||y||$.

A linear space X with a norm defined on it is called a **normed linear space**.

Example: l^p **space**. On the linear space $V_n(\mathbf{F})$, define

$$||x|| = [\sum_{k=1}^{n} |\alpha_i|^p]^{1/p}$$

where $p \ge 1$ is any real number and $x = (\alpha_1, \cdots, \alpha_n)$. This defines a norm (called p-norm) on $V_n(\mathbf{F})$. This space is called l^p space.

CHAPTER 1

INNER PRODUCT SPACES

INNER PRODUCTS

Let F be the field of real numbers or the field of complex numbers, and V a vector space over F an inner product on V is a function which assigns to each ordered' pair of vectors α, β in V a scalar $(\alpha|\beta)$ in F in such a way that for all α, β, γ in V and all scalars c.

(a) $(\alpha+\beta|\gamma) = (\alpha|\gamma) + (\beta|\gamma)$;

(b) $(\mathrm{c}\alpha|\beta) = c(\alpha|\beta)$;

(c) $(\beta|\alpha) = (\overline{\alpha|\beta})$, the bar denoting complex conjugation

(d) $(\alpha|\alpha) > 0$ if $\alpha \neq 0$

It should be observed that conditions (a), (b) and (c) implies that

$(e) = (\alpha \mid c\beta + \gamma) = (\bar{c}(\alpha|\beta) + (\alpha|\gamma)$

One other point should be made. When F is the field R of real nunbers. The complex conjugates appearing in (c) and (e) are superflom. However, in the complex case they are necessary for the consistency of the conditions. Without these complex conjugates we would have the contradiction

$$(\alpha|\alpha) > 0 \text{ and } (i\alpha \mid i\alpha) = -1(\alpha|\alpha)$$

Example 1:

On F^n there is an inner product which we call the standard inner product. It is defined on $\alpha = (x_1, \cdots x_n)$ and $\beta = (y_1, \dots, y_n)$, by

$$(\alpha|\beta) = \sum_i x_i \overline{y_i}$$

When F is R this may be also written as

$$(\alpha|\beta) = \sum_i x_i y_i$$

In the real case, the standard inner product is often called the dot or scalar product and denoted by $\alpha \cdot \beta$.

INNER PRODUCTS SPACES

An inner product space is a real or complex vector space together with a specified inner product on that space.

- A finite-dimensional real inner product space is often called a Euclidean spare. A complex inner product spare often referred to as a unitary spare.
- Every inner product space is a normed linear space and every normed space is a metric space. Hence , every inner product space is a metric space.

Theorem

If V is an inner product space, then for any vector's α, β in V and any scalar c

(1) $||c\alpha|| = |c|||\alpha||$;

(ii) $\left||\alpha|\right| > 0$ for $\alpha \neq 0$

(iii) $|(\alpha \mid \beta)| \leqslant ||\alpha|| \, ||\beta||$

(iv) $\| \alpha + \beta|| \leqslant \| \alpha \| + \| \beta||$

Proof:

Statements (i) and (ii) follow almost immediately form the various definitions involved. The inequality in (iii) is clearly valid when $\alpha = 0.$ if $\alpha \neq 0$, put

$$\gamma = \beta - \frac{(\beta|\alpha)}{\| \alpha \|^2} \alpha$$

Then, $(\gamma \mid \alpha) = 0$ and

$$0 \leqslant \| \gamma \|^2 = \left(\beta - \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha \middle/ \beta - \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha\right)$$

$$= (\beta|\beta) - \frac{(\beta|\alpha)(\alpha|\beta)}{\| \alpha \|^2}$$

$$= \| \beta \|^2 - \frac{|(\alpha|\beta)|^2}{\| \alpha \|^2}$$

Hence,

$$|(\alpha \mid \beta)|^2 \leqslant \| \alpha \|^2 \| \beta \|^2$$

Now using (c) we find that

$$\| \alpha + \beta \|^2 = \| \alpha \|^2 + (\alpha \mid \beta) + (\beta \mid \alpha) + \| \beta \|^2$$

$$\begin{aligned} &= \| \alpha \|^2 + 2\mathrm{Re}\,(\alpha \mid \beta) + \| \beta \|^2 \\ &\leqslant \| \alpha \|^2 + 2 \| \alpha \| \| \beta \| + \| \beta \|^2 \\ &= (\| \alpha \| + \| \beta \|)^2 \end{aligned}$$

Thus,

$$\| \alpha + \beta \| \leqslant \| \alpha \| + \| \beta \|$$

the inequality (iii) is called the Cauchy -Schwarz inequality. It has a wide variety of application

the proof shows that if $\alpha$is non-zero then

$$\big((\alpha \mid \beta)\big) < \| \alpha \| \| \beta \|, \text{unless}$$

$$\beta = \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha$$

Then equality occurs in (iii) if and only if α and β are linearly independent.

CHAPTER 2

ORTHOGONAL SETS

Definition

Let α and β be the vectors in an inner product space V. Then α is orthogonal to β if (α | β) = 0. We simply say that and are orthogonal.

Definition

If S is a set of vectors in V, S is called an orthogonal set provided all set pairs of distinct vectors in S are orthogonal.

Definition

An orthogonal set is an orthogonal set S with the additional property that $\| \alpha \| = 1$ for every α in S.

- The zero vectors are orthogonal to every vector in V and is the only vector with this property.
- It is an appropriate to think of an orthonormal set as a set of mutually perpendicular vectors each having length l.

Example: the vector (x , y) is R^2 is orthogonal to $(-y, x)$ with respect to the standard inner product, for,

$$((x, y)|(-y, x)) = -xy + yx = 0$$

- The standard basis of either R^n or C^n is an orthonormal set with respect to the standard inner product.

Theorem : An orthogonal set of nonzero vectors is linearly independent.

Proof:

Let S be a finite or infinite orthogonal set of nonzero vectors in a given inner product space suppose $\alpha_{1,}\alpha_2, \dots \alpha_n$ are distinct vectors in S and that β=$c_1\alpha_{1+} + \cdots c_n\alpha_n$

Then (β|α_k)=($c_1\alpha_{1+} + \cdots c_n\alpha_n|\alpha_k$)

$= c_1(\alpha_1|\alpha_k) + c_2(\alpha_2|\alpha_k)+\dots+c_n(\alpha_n|\alpha_k)$

$= c_k(\alpha_n|\alpha_k)$, since $(\alpha_i\,|\alpha_j) = 0$,if i ≠ j and $(\alpha_i\,|\alpha_j) = 1$,if i=j

Hence, c_k= (β $|\alpha_k)\,/(\alpha_k\,,\alpha_k)$)

c_k=(β $|\alpha_k)/||\alpha_k||^2$,1≤ k ≤ m

Thus, when β=0 each c_k=0; so S is a linearly independent set.

Corollary:

If $\{\alpha_{1,}\alpha_2, \dots \alpha_m\}$ is an orthogonal set of nonzero vectors in a finite dimensional inner product space V, then m ≤ dimV.

That is number of mutually orthogonal vectors in V cannot exceed the dimensional V.

Corollary:

If a vector β is linear combination of an orthogonal of nonzero vectors $\alpha_{1,}\alpha_2, \dots \alpha_n$, then β is the particular linear combination

$$\beta = \sum_{k=1}^{m} \frac{(\beta\,|\alpha_k)}{||\alpha_k||^2}\alpha_k$$

Proof :

Since β is the linear combination of an orthogonal sequence of nonzero vectors $\alpha_{1,}\alpha_2, \dots \alpha_n$, we can write β =$c_1\alpha_1 + \cdots c_n\alpha_n$.

Where $c_k = \frac{(\beta|\alpha_k)}{||\alpha_k||^2}$, 1 ≤ k ≤ m (ref. by previous theorem)

Hence, β= $\sum_{k=1}^{m} \frac{(\beta|\alpha_k)}{||\alpha_k||^2}$

Theorem (Gram Schmidt Orthogonalization Process)

Let V be an inner product space and $\{\beta_{1,} \dots, \beta_n\}$ be any linearly independent vectors in V. Then one may construct orthogonal vectors $\{\alpha_{1,}\alpha_2, \dots \alpha_n\}$ in V, such that for each k = 1, 2, …n, the set $\{\alpha_{1,}\alpha_2, \dots \alpha_k\}$is an orthogonal basis for the subspace of V spanned by $\beta_{1,} \dots, \beta_n$.

Proof :

The vectors are obtained by means of a construction known as the Gram Schmidt orthogonalization process.

First let $\alpha_1 = \beta_1$ The other vectors are then given inductively as follows:

Suppose $\alpha_{1,}\alpha_2, \dots \alpha_m$ (1 ≤ m ≤ n) have been chosen so that for every k

$$\{\alpha_{1,}\alpha_2, \dots \alpha_k\} \ (1 \le k \le m)$$

is an orthogonal basis for the space of v that is spanned by $\beta_{1,} \dots, \beta_n$

To construct the next vector $\alpha_{m+1,}$ let

$$\alpha_{m+1,} = \beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2} \alpha_k$$

Then $\alpha_{m+1} \neq 0$. For otherwise, $\beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2} \alpha_k = 0$, implies,

$\beta_{m+1} = \beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2} \alpha_k$,implies, β_{m+1} is a linear combination of $\alpha_{1,}\alpha_2, \dots \alpha_m$ and hence a linear combination of $\beta_1, \beta_2, \dots, \beta_m$, a contradiction.

Furthermore, if 1≤j≤m, then,

$$(\alpha_{m+1}|\ \alpha_j) = (\beta_{m+1}|\ \alpha_j) - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2} (\alpha_k\ |\alpha_j)$$

$= (\beta_{m+1}|\ \alpha_m) - (\beta_{m+1}|\ \alpha_j)$, using the orthonormality of $\{\alpha_{1,}\alpha_2, \dots \alpha_m\}$.

Therefore $\{\alpha_{1,}\alpha_2, \dots, \alpha_{m+1}\}$ is an orthogonal set consisting of m+1 nonzero vectors in the subspace spanned by $\beta_{1,} \dots, \beta_{m+1}$. Hence by an earlier Theorem , it is a basis for this subspace .Thus the vectors , $\alpha_{1,}\alpha_2, \dots \alpha_n$ may be constructed using the formula

$$\alpha_{m+1}= \beta_{m+1}-\sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}\alpha_k$$

In particular, when n=3 ,we have

$$\alpha_1=\beta_1$$

$$\alpha_2=\beta_2- \frac{(\alpha_2|\beta_2)}{||\alpha_k||^2}\alpha 1$$

$$\alpha_3=\beta_3 - \frac{(\beta_3|\alpha_1)}{||\alpha_1||^2}\alpha 1 - \frac{(\alpha_2|\beta_3)}{||\alpha_k||^2}\alpha_2$$

Corollary :

Every finite dimensional inner product space has an orthonormal basis.

Proof :

Let V be a finite dimensional inner product space and $\{ \beta_{1,} \dots, \beta_n\}$ a basis for V. Apply the gram Schmidt orthogonalization process to construct an orthogonal basis , simply replace each vector α_n by $\frac{\alpha_k}{||\alpha_k||}$.

Gram-Schmidt process can be used to test for linear dependence . For suppose $\beta_{1,} \dots, \beta_n$ are linearly independent vectors in an inner product space; to exclude a trivial case , assume that β≠0. Let m be largest integers for which $\beta_{1,} \dots, \beta_m$are independent. Then 1≤m<n.

Let $\alpha_1, \alpha_2, \dots \alpha_m$ be the vectors obtained by applying the orthogonalization process to $\beta_{1,} \dots, \beta_m$. Then the vector α_{m+1} given by $\alpha_{m+1}= \beta_{m+1}-\sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}\alpha_k$ is necessarily 0.

For α_{m+1} is in the subspace spanned by $\alpha_1, \alpha_2, \dots \alpha_m$ and orthogonal to each of the vectors , hence it is 0 as $\beta=\sum_{k=1}^{m} \frac{(\beta|\alpha_k)}{||\alpha_k||^2}\alpha_k$. Conversely, if $\alpha_1, \alpha_2, \dots \alpha_m$ are different from 0 and α_{m+1} =0, then $\beta_{1,} \dots, \beta_{m+1}$ are linearly independent .

Definition:

A best approximation to β ∈ V by vectors in a subspace W of V is a vector $\alpha \in W$ such that

$$\|\beta-\alpha\| \le \|\beta-\gamma\| \text{ for every vector } \gamma \in W \text{ .}$$

Theorem

Let W be a subspace of an inner product space V and let $\beta \in V$.

1. The vector $\alpha \in W$ is a best approximation to $\beta \in V$ by vectors in W if and only if $\beta - \alpha$ is orthogonal to every vector in W .
2. If a best approximation to $\beta \in V$ by vectors in W exists, it is unique.
3. If W is finite-dimensional and $\{\alpha_{1,}\alpha_2, \dots \alpha_n\}$ is any orthonormal basis for W ,

   then the vector

$$\alpha = \sum_{k=1}^{n} \frac{(\beta \mid \alpha_k)}{\|\alpha_k\|^2} \alpha_k$$

   is the (unique) best approximation to β by vectors in W.

Definition:

Let V be an inner product space and S be any set of vectors in V. The orthogonal complement of S is the set $S^{\perp}$ of all vectors in V which are orthogonal to every vector in S.

That is, $S^{\perp} = \{\alpha \in V : (\alpha \mid \beta) = 0, \forall \beta \in S\}$

Definition:

Whenever the vector α in the above theorem exists it is called the orthogonal projection of β on W. If every vector in V has an orthogonal projection on W, the mapping that assigns to each vector in V its orthogonal projection on W is called the orthogonal projection of V on W.

Corollary :

Let V be an inner product space and W a finite dimensional subspace and E be the orthogonal projection of V on W. Then the mapping

$$\beta \rightarrow \beta - E\beta$$

is the orthogonal projection of V on $W^{\perp}$.

Proof :

Let $\beta \in V$. Then $\beta - E\beta \in W^{\perp}$, and for any $\gamma \in W^{\perp}$, $\beta - \gamma = E\beta + (\beta - E\beta - \gamma)$

Since $E\beta \in W$ and $\beta - E\beta - \gamma \in W^{\perp}$,

It follows that

$$||\beta - \gamma||^2 = (E\beta + (\beta - E\beta - \gamma), E\beta + (\beta - E\beta - \gamma))$$

$$= ||E\beta||^2 + ||\beta - E\beta - \gamma||^2$$

$$\geq ||\beta - (\beta - E\beta)||^2$$

with strict inequality when $\gamma \neq \beta - E\beta$. Therefore, $\beta - E\beta$ is the best approximation to β by vectors in $W^{\perp}$.

Theorem

Let W be a finite dimensional subspace of an inner product space V and let E be the orthogonal projection of V on W. Then E is an idempotent linear transformation of V onto W , $W^{\perp}$ is the null space of E , and $V = W \oplus W^{\perp}$.

Proof

Let β be an arbitrary vector in V. Then $E\beta$ is the best approximation to β that lies in W. In particular, $E\beta = \beta$ when β is in W. Therefore, $E(E\beta) = E\beta$ for every β in V; that is, E is idempotent : $E^2 = E$. To prove that E is linear transformation, let α and β be any vectors in V and c an arbitrary scalar ,Then by theorem,

$\alpha - E\alpha$ and $\beta - E\beta$ are each orthogonal to every vector in W. Hence the vector

$c(\alpha - E\alpha) + (\beta - E\beta) = (c\alpha + \beta) - (cE\alpha + E\beta)$

Also belongs to $W^{\perp}$. Since $cE\alpha + E\beta$ is a vector in W, it follows from theorem that

$E(c\alpha + \beta) = cE\alpha + E\beta$.

Again let β be any vector in V. Then $E\beta$ is the unique vector in W such that $\beta - E\beta$ is in $W^{\perp}$.

Thus $E\beta = 0$ when β is in $W^{\perp}$.

Conversely, β is in $W^{\perp}$ when $E\beta = 0$. Thus $W^{\perp}$ is the null space of E.

The equation ,

$$\beta = E\beta + \beta - E\beta$$

shows that V = $W + W^{\perp}$; moreover $W \cap W^{\perp} = \{0\}$; for if α is a vector in $W \cap W^{\perp}$,then

$(\alpha \mid \alpha)$=0. Therefore, α=0 and V is the direct sum of W and $W^{\perp}$.

Corollary :

Under the conditions of theorem, $I - E$ is the orthogonal projection of V on $W^{\perp}$.

It is an independent linear transformation of V onto $W^{\perp}$ with null space W.

Proof :

We have seen that the mapping $\beta \rightarrow \beta - E\beta$ is the orthogonal projection of V on $W^{\perp}$.

Since E is a linear transformation , this projection $W^{\perp}$ is the linear transformation $I - E$ from its geometric properties one sees that $I - E$ is an idempotent .Transformation of V onto W.

This also follows from the computation $(I - E)(I - E) = I - E - E + E^2$

$$= I - E$$

Moreover , $(I - E)\beta = 0$ If and only if $\beta = E\beta$, and this is the case if and only if β is in W .

Therefore W is the null space of $I - E$.

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CERTIFICATE

It is to certify that this project report '**DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS**' is the bonafide project of **ARJUN MOHANAN P V** who carried out the project under my supervision.

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DECLARATION

I, **ARJUN MOHANAN P V**, hereby declare that this project report entitled '**DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS**' is an original record of studies and bonafide project carried out by me during the period from November 2019 to March 2020, under the guidance of **Ms.Sneha P Sebastian**, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not been submitted by me elsewhere for the award of any degree, diploma, title or recognition, before.

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CONTENTS

INTRODUCTION

In mathematics, a group is a set equipped with a binary operation that combines any two elements to form a third element in such a way that the three conditions called group axioms are satisfied , namely associativity , identity and invertability.

Let us take a moment to review our present stockpile of groups. Starting with finite groups, we have the cyclic group $\mathbb{Z}_n$,the symmetric group S_n , and the alternating group A_n for each positive integer n. We also have the dihedral group D_n and klein 4-group . Of course we know that subgroups of these groups exists. Turning to infinite groups , we have $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ under addition , and their non zero elements under multiplication we also have the group S_A of all permutation of an infinite set A , as well as various groups formed from matrices .

One purpose of this section is to show a way to use known groups as building blocks to form more groups. Given two groups G and H, it is possible to construct a new group from the cartesian product of G and H . Conversely , given a large group , it is sometimes possible to decompose the group ; that is , a group is sometimes isomorphic to the direct product of two smaller groups. Rather than studying a large group , it is often easier to study the component group of that group.

PRELIMINARY

Groups : A non empty set G together with an operation $*$ is said to be a group , denote by $(G\ ,*)$, if it satisfy the following axioms.

- Closure property
- Associative property
- Existence of identity
- Existence of inverse

Abelian group

A group $(G\ ,*)$ is said to be abelian if it satisfies commutative law .

Finite group

If the underlying set G of the group $(G\ ,*)$ consist of finite number of elements , then the group is finite group .

Infinite group

A group that is not finite is an infinite group .

Order of a group : The number of elements in a finite group is called the order of the group , denoted by $O(G)$.

Example

Show that the set of integers $\mathbb{Z}$ is a group with respect to the operation of addition of integers.

$$\mathbb{Z} = \{\ldots\ldots\ldots. -3, -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\}$$

Since the addition of two integers gives an integer , it satisfy closure property .

If $a, b, c \epsilon \mathbb{Z}$ then the $(a+b)+c = a+(b+c)$, hence associativity holds .

There is a number $0 \epsilon \mathbb{Z}$ such that $0+a = a+0$, hence identity exists

If $a \epsilon \mathbb{Z}$ then there exists $-a \epsilon \mathbb{Z}$, such that $-a + a = 0 = a + -a$

Therefore inverse exist .

Therefore $\mathbb{Z}$ is a group under addition .

Subgroup

A subset H of G is said to be a subgroup of G if H itself is a group under the same operation in G.

There are two different types of group structure of order 4 .

$\mathbb{Z}_4 = \{ 0,1,2,3\}$

Klein 4 – group , $V = \{e, a, b, c\}$

Cyclic group

A group G is cyclic if there is some element 'a' in G that generate G. And the element 'a' is called generator of G.

Group Homomorphism

A function $\Psi: G \rightarrow G'$ is a group homomorphism (or simply homomorphism).

If $\Psi(ab) = \Psi(a)\, \Psi(b)$ hold for all $a , b \in G$, is called homomorphism property .

Isomorphism

A one to one and onto homomorphism $\Psi: G \rightarrow G'$ is called an isomorphism .

CHAPTER – 1

DIRECT PRODUCT OF GROUPS

Definition

The Cartesian product of sets $S, S_2, \dots \dots, S_n$ is the set of all ordered n-tuples $(a_1, a_2, \dots \dots, a_n)$, where $a_i \in S_i$ for $i = 1,2,3, \dots \dots, n$. The Cartesian product is denoted by either

$S_1 \times S_2 \times \dots \dots \times S_n$ or by $\Pi_{i=1}^{n} S_i$.

Let $G_1, G_2, \dots \dots, G_n$ be groups and let us use multiplicative notation for all the group operations.

If we consider G_i as a set , $i = 1,2, \dots \dots n$. we have the products $G_1 \times G_2 \times \dots \dots, \times G_n$ we denote it by $\Pi_{i=1}^{n} G_i$. This product is called direct-product of groups. We can make $\Pi_{i=1}^{n} G_i$ into a group by means of a binary operation of multiplication by components.

Theorem

Let $G_1, G_2, \dots \dots, G_n$ be groups. For $(a_1, a_2, \dots \dots, a_n)$ and $(b_1, b_2, \dots \dots, b_n)$ in $\Pi_{i=1}^{n} G_i$ define ;

$$(a_1, a_2, \dots \dots, a_n)(b_1, b_2, \dots \dots, b_n) = (a_1 b_1, a_2 b_2, \dots \dots, a_n b_n)$$

Then $\Pi_{i=1}^{n} G_i$ is a group.

Proof

We have ,

$$\Pi_{i=1}^{n} G_i = \{(a_1, a_2, \dots \dots, a_n) : a_i \in G_i\}$$

(1) **Closure property**

Let $(a_1, a_2, \dots \dots, a_n), (b_1, b_2, \dots \dots, b_n) \in \Pi_{i=1}^{n} G_i$

And we have ,

$$(a_1, a_2, \dots \dots, a_n)(b_1, b_2, \dots \dots, b_n) = (a_1 b_1, a_2 b_2, \dots \dots, a_n b_n)$$

Here $a_i \in G_i$ and $b_i \in G_i$ for $i = 1,2, \dots \dots, n$

$\because$ G_i is a group , $a_i b_i \in G_i$ for $i = 1,2, \dots \dots, n$

$\Rightarrow (a_1 b_1, a_2 b_2, \dots \dots, a_n b_n) \in \Pi_{i=1}^{n} G_i$

i.e. $\Pi_{i=1}^{n} G_i$ is closed under the binary operation.

(2) **Associativity**

Let $(a_1, a_2, \dots \dots, a_n), (b_1, b_2, \dots \dots, b_n), (c_1, c_2, \dots \dots, c_n) \in \Pi_{i=1}^{n} G_i$

We have,

$$(a_1, a_2, \dots \dots, a_n)(b_1, b_2, \dots \dots, b_n)(c_1, c_2, \dots \dots, c_n)$$

$$= (a_1 b_1 c_1 , a_2 b_2 c_2, \dots \dots, a_n b_n c_n) \in \Pi_{i=1}^{n} G_i$$

$$[(a_1, a_2, \dots \dots, a_n)(b_1, b_2, \dots \dots, b_n)](c_1, c_2, \dots \dots, c_n)$$

$$\begin{aligned}
&= [a_1 b_1, a_2 b_2, \dots \dots, a_n b_n](c_1, c_2, \dots \dots, c_n) \\
&= [(a_1 b_1) c_1, (a_2 b_2) c_2, \dots \dots, (a_n b_n) c_n] \\
&= [a_1 (b_1 c_1), a_2 (b_2 c_2), \dots \dots, a_n (b_n c_n)] \\
&= (a_1, a_2, \dots \dots, a_n)[b_1 c_1, b_2 c_2, \dots \dots, b_n c_n] \\
&= (a_1, a_2, \dots \dots, a_n)[(b_1, b_2, \dots \dots, b_n)(c_1, c_2, \dots \dots, c_n)]
\end{aligned}$$

Hence associativity holds.

(3) **Existence of identity**

If e_i is the identity element in G_i.

Then,

$$(e_1, e_2, \dots \dots ., e_n) \in \Pi_{i=1}^{n} G_i$$

Also for ,

$$(a_1, a_2, \dots \dots ., a_n) \in \Pi_{i=1}^{n} G_i,$$

$$\begin{aligned}(a_1, a_2, \dots \dots ., a_n)(e_1, e_2, \dots \dots ., e_n) &= (a_1 e_1, a_2 e_2, \dots \dots ., a_n e_n)\\ &= (a_1, a_2, \dots \dots ., a_n)\end{aligned}$$

$\therefore$ $(e_1, e_2, \dots \dots ., e_n)$ is the identity element 'e' in $\Pi_{i=1}^{n} G_i$

(4) **Existence of inverse**

Let $(a_1, a_2, \dots \dots ., a_n) \in \Pi_{i=1}^{n} G_i$

Here $a_i \in G_i$ for $i = 1,2, \dots \dots ., n$.

Since G_i is a group ,

$\exists$ an inverse element a_i^{-1} in G_i : $a_i a_i^{-1} = e_i$ $\qquad i = 1,2, \dots \dots ., n$

Clearly, $\qquad (a_1^{-1}, a_2^{-1}, \dots \dots ., a_n^{-1}) \in \Pi_{i=1}^{n} G_i$ &

$$(a_1, a_2, \dots \dots ., a_n)(a_1^{-1}, a_2^{-1}, \dots \dots ., a_n^{-1}) = (e_1, e_2, \dots \dots ., e_n)$$

Hence $\Pi_{i=1}^{n} G_i$ is a group.

Note

If the operation of each G_i is a commutative. We sometimes use additive notation in $\Pi_{i=1}^{n} G_i$ and refer to $\Pi_{i=1}^{n} G_i$ as the direct sum of the group G_i. The notation $\oplus_{i=1}^{n} G_i$, especially with abelian groups with operation $+$.

The direct sum of abelian groups $G_1, G_2, \dots \dots, G_n$ may be written $G_1 \oplus G_2 \oplus \dots \dots \oplus G_n$

- Direct product of abelian group is abelian

Example

Q. Check whether $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic or not.

$$\mathbb{Z}_2 = \{0,1\}$$

$$\mathbb{Z}_3 = \{0,1,2\}$$

$\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$

Consider,

$1(1,1) = (1,1)$

$2(1,1) = (1,1) + (1,1) = (0,2)$

$3(1,1) = (1,1) + (1,1) + (1,1) = (1,0)$

$4(1,1) = 3(1,1) + (1,1) = (1,0) + (1,1) = (0,1)$

$5(1,1) = 4(1,1) + (1,1) = (0,1) + (1,1) = (1,2)$

$6(1,1) = 5(1,1) + (1,1) = (1,2) + (1,1) = (0,0)$

$\therefore$ $(1,1)$ is a generator of $\mathbb{Z}_2 \times \mathbb{Z}_3$

$\therefore$ $\mathbb{Z}_2 \times \mathbb{Z}_3$ is a cyclic group generated by $(1,1)$.

Q. Check whether $\mathbb{Z}_3 \times \mathbb{Z}_3$ is cyclic or not.

$$\mathbb{Z}_3 = \{0,1,2\}$$

$$\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$$

$1(0,1) = (0,1)$

$2(0,1) = (0,2)$

$3(0,1) = (0,3) = (0,0)$ $\quad\quad \therefore$ order $(0,1) = 3$

$1(0,2) = (0,2)$

$2(0,2) = (0,4) = (0,1)$

$3(0,2) = (0,6) = (0,0)$ $\quad\quad \therefore$ order $(0,2) = 3$

Every element added to itself three times gives the identity. Thus no element can generate the group. Hence $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not cyclic.

similarly $\mathbb{Z}_m \times \mathbb{Z}_m$ is not cyclic for any m.

Theorem

The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime, that is, the gcd of m and n is 1.

Proof

Suppose $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and isomorphic to $\mathbb{Z}_{mn}$.

To show that m and n are relatively prime.

Suppose not, let d be the gcd of m and n.

So that $d > 1$

Consider $\frac{mn}{d}$, which is an integer since $d|m$ and $d|n$

Let (r,s) be an arbitrary element of $\mathbb{Z}_m \times \mathbb{Z}_n$, add (r,s) repeatedly $\frac{mn}{d}$ times

$$\left.(r,s) + (r,s)+, \dots \dots, +(r,s)\right\} \ \frac{mn}{d} times = (0,0)$$

$\therefore$ no element of $\mathbb{Z}_m \times \mathbb{Z}_n$ having order mn. $\therefore$ no element of $\mathbb{Z}_m \times \mathbb{Z}_n$ can generate $\mathbb{Z}_m \times \mathbb{Z}_n$ which is not possible. $\because$ $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic. Hence $gcd(m,n) = 1$.

i.e. m and n are relatively prime.

Conversely, suppose m and n are relatively prime, i.e. $\gcd(m,n) = 1$

To show that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic.

If $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic, then it is isomorphic to $\mathbb{Z}_{mn}$, $\because$ $\mathbb{Z}_m \times \mathbb{Z}_n$ has mn elements.

Consider the cyclic subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$ generated by the element $(1,1)$.The order of this cyclic subgroup is the smallest power of $(1,1)$,that gives the identity $(0,0)$. Here taking a power of $(1,1)$ in our additive notation will involve adding $(1,1)$ to itself repeatedly.

Consider $(1,1) + (1,1)+, \dots \dots, +(1,1)$

If we add first coordinates m times , we get zero.

$\therefore$ order of first coordinate $= m$.

Similarly , Order of second coordinate $= n$.

The two coordinates together become zero. If we add them $lcm(m,n)$ times.

$\because$ $\gcd(m,n) = 1$, We get the $lcm = mn$.

i.e. $(1,1)$ generates a cyclic subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$ of order mn , which is the order of the whole group.

$$\Rightarrow \mathbb{Z}_m \times \mathbb{Z}_n = < (1,1) >$$

$\Rightarrow\ \mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic.

Corollary

The group $\prod_{i=1}^{n} \mathbb{Z}_{m_i}$ is cyclic and isomorphic to $\mathbb{Z}_{m_1 m_2 \ldots\ldots m_n}$ if and only if the numbers m_i for $i = 1,2, \ldots \ldots, n$ are such that the gcd of any two of them is 1.

Example

If n is written as a product of powers of distinct prime numbers , as in ,

$$n = (p_1)^{n_1}.(p_2)^{n_2} \ldots \ldots.(p_n)^{n_r}$$

Then $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{(p_1)^{n_1}} \times \mathbb{Z}_{(p_2)^{n_2}} \times \ldots \ldots \times \mathbb{Z}_{(p_r)^{n_r}}$.

In particular , $\mathbb{Z}_{72}$ is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_9$.

Consider set of integers $\mathbb{Z}$, cyclic subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$, $n \in \mathbb{Z}$. Consider $2\mathbb{Z}$ and $3\mathbb{Z}$, then $< 2 > \cap < 3 > = < 6 >$

$\therefore$ if we take $r\mathbb{Z}$, $s\mathbb{Z}$ of $\mathbb{Z}$, then the $lcm(r, s)$ =generator of $< r > \cap < s >$

Using this we can define the lcm of the positive integers.

Definition

Let $r_1, r_2, \ldots \ldots, r_n$ be positive integers. Their least common multiple (abbreviated lcm) is the positive generator of the cyclic group of all common multiples of the r_i , that is the cyclic group of all integers divisible by each r_i for $i = 1,2, \ldots \ldots, n$.

Theorem

Let $(a_1, a_2, \dots \dots ., a_n) \in \prod_{i=1}^{n} G_i$.

If a_i is of finite order r_i in G_i , then the order of $(a_1, a_2, \dots \dots ., a_n)$ in $\prod_{i=1}^{n} G_i$ is equal to the least common multiple of all the r_i.

Proof

Given,

order of $a_1 = r_1 \Rightarrow a_1^{r_1} = e_1$ in G_1

order of $a_2 = r_2 \Rightarrow a_2^{r_2} = e_2$ in G_2

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order of $a_n = r_n \Rightarrow a_n^{r_n} = e_n$ in G_n.

We have to find a power k for $(a_1, a_2, \dots \dots ., a_n)$.

So that $(a_1, a_2, \dots \dots ., a_n)^k = (e_1, e_2, \dots \dots ., e_n)$.

The power must simultaneously be a multiple of r_1 , multiple of r_2 and so on. But k is the least positive integers having the above property.

$$\therefore k = lcm(r_1, r_2, \dots \dots ., r_n).$$

Q. Find the order of $(8,4,10)$ in the group $\mathbb{Z}_{12} \times \mathbb{Z}_{60} \times \mathbb{Z}_{24}$.

$$O(8) = 3 \text{ in } Z_{12}$$

$O(4) = 15$ in Z_{60}

$O(10) = 12$ in Z_{24}

$O(8,4,10) = lcm(3,15,12) = 60$

Q. Find a generator of $\mathbb{Z} \times \mathbb{Z}_2$

$$\mathbb{Z} \times \mathbb{Z}_2 = \{(n, 0), (n, 1): n \in \mathbb{Z}\}$$
$$(n, 0) = n(1,0)$$
$$(n, 1) = (n, 0) + (0,1) = n(1,0) + (0,1)$$

$\therefore$ $\mathbb{Z} \times \mathbb{Z}_2$ is generated by $\{(1,0), (0,1)\}$

In general , $\mathbb{Z} \times \mathbb{Z}_n$ is generated by ,

$$\{(1,0,0, \dots \dots . ,0), (0,1,0, \dots \dots . ,0), \dots \dots . , (0,0, \dots \dots . ,1)\}$$

Q. Find the order of (3,10,9) in $(\mathbb{Z}_4, \mathbb{Z}_{12}, \mathbb{Z}_{15})$

$O(3) = 4$ in $\mathbb{Z}_4$

$O(10) = 6$ in $\mathbb{Z}_{12}$

$O(9) = 5$ in $\mathbb{Z}_{15}$

$\therefore O(3,10,9) = lcm(4,6,5)$
$= 60$

$\therefore$ order of (3,10,9) in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$ is 60.

CHAPTER-2

FUNDAMENTAL THEOREM OF FINITELY GENERATED ABELIAN GROUPS

Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups in the form,

$$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \dots \dots . \times \mathbb{Z}_{(p_n)^{r_n}} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \dots \dots . \times \mathbb{Z}$$

Where the p_i are primes , not necessarily distinct and the r_i are positive integers.

Remark

- The direct product is unique except for possible rearrangement of the factors.
- The number of factors $\mathbb{Z}$ is unique and this number is called Betti number.

Example

Find all abelian groups , upto isomorphism of order

1)8 , 2)16 , 3)360

(1) Order 8

$$8 = 1 \times 8$$

$$8 = 2 \times 4 = 2 \times 2^2$$

$$8 = 2 \times 2 \times 2$$

3 non-isomorphic groups are $\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

(2) Order 16

$16 = 1 \times 16 = 1 \times 2^4$

16 = $2 \times 8 \quad = 2 \times 2^3$

16 = $4 \times 4 \quad = 2^2 \times 2^2$

$16 = 2 \times 2 \times 2 \times 2$

$16 = 2 \times 2 \times 2^2$

$\mathbb{Z}_{16}, \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$

(3) Order 360

$360 = 2^2 \cdot 3^2 \cdot 5$

Possibilities are,

1) $\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
2) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
3) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
4) $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
5) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
6) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

Definition

A group G is decomposable if it is isomorphic to a direct product of two proper non-trivial subgroups , otherwise G is indecomposable.

Example

$\mathbb{Z}_6$ is decomposable while $\mathbb{Z}_5$ is indecomposable.

$\mathbb{Z}_6$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$

$\mathbb{Z}_{mn}$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$, if m and n are prime.

Theorem

The finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.

Proof

Let G be a finite indecomposable abelian group $\because G$ is finitely generated , we can apply fundamental theorem of finitely generated abelian groups.

$\therefore G \cong \mathbb{Z}_{(p)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \dots \dots . \times \mathbb{Z}_{(p_n)^{r_n}}$

$\because G$ is indecomposable and $\mathbb{Z}_{(p_i)^{r_i}}$'s are proper subgroups we get in the above , there is only one factor say $\mathbb{Z}_{(p_i)^{r_i}}$ which is cyclic group with order a prime power.

Theorem

If m divides the order of a finite abelian group , then G has a subgroup of order m.

Proof

Given G is a finite abelian group.

$\therefore$ we can apply Fundamental Theorem ,

Hence,

$$G \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \dots \dots . \times \mathbb{Z}_{p_n^{r_n}}$$

Here all primes need not be distinct.

Then,

$$O(G) = p_1^{r_1} . p_2^{r_2} \dots \dots p_n^{r_n}$$

Let m is a $+ve$ integer which divides $O(G)$.

$0 \le s_i \le r_i$ By theorem , " let G be a cyclic group with n elements and generated by a. Let $b \in G$, $b = a^s$, then ' b' generates a cyclic subgroup H of G containing $\frac{n}{d}$ elements , where $d = \gcd(n, s)$."

$p_i^{r_i - s_i}$ generates a cyclic subgroup of $\mathbb{Z}_{p_i^{r_i}}$ having order $\frac{p_i^{r_i}}{gcd(p_i^{r_i}, p_i^{r_i - s_i})}$

$$= \frac{p_i^{r_i}}{p_i^{r_i - s_i}} = p_i^{s_i}$$

$$\therefore O\left(< p_i^{r_i - s_i} >\right) = p_i^{s_i}$$

i.e. $< p_1^{r_1 - s_1} >$ is a subgroup of $\mathbb{Z}_{p_1^{r_1}}$ having order $p_1^{s_1}$.

$< p_2^{r_2 - s_2} >$ is a subgroup of $\mathbb{Z}_{p_2^{r_2}}$ having order $p_2^{s_2}$.

………………………………………………………………

$< p_n^{r_n - s_n} >$ is a subgroup of $\mathbb{Z}_{p_n^{r_n}}$ having order $p_n^{s_n}$.

$\therefore < p_1^{r_1 - s_1} > \times < p_2^{r_2 - s_2} > \times \dots \dots . \times < p_n^{r_n - s_n} >$ is a subgroup of $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \dots \dots . \times \mathbb{Z}_{p_n^{r_n}}$ having order $p_1^{s_1} \cdot p_2^{s_2} \cdots\cdots p_n^{s_n} = m$.

Theorem

If m is a square free integer , that is m is not divisible by the square of any prime . Then every abelian group of order m is cyclic.

Proof

Let m be a square free integer , then $p^i \nmid m$ for every i greater than 1 for a prime p.

Given G is a finite abelian group having order m , by fundamental theorem , then

$$G \cong \mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \dots \dots . \times \mathbb{Z}_{(p_n)^{r_n}}$$

Then,

$$O(G) = p_1^{r_1} \cdot p_2^{r_2} \cdots\cdots p_n^{r_n}$$

$\because$ $O(G)$ is a square free integer , the only possibility

$$r_1 = r_2 = \dots \dots .. = r_n = 1$$

Then,

$$G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \dots . \times \mathbb{Z}_{p_n}$$

$\cong \mathbb{Z}_{p_1, p_2, \dots\dots, p_n}$, which is cyclic.

Example

15 is a square free integer. So an abelian group of order 15 is cyclic.

CONCLUSION

Direct product of groups is the product $G_1 \times G_2, \dots \dots . G_n$,where each G_i is a set. We have discussed about definition and some properties related to the direct product of groups. The fundamental theorem of finitely generated abelian group helped us to get a deeper understanding about the topic. The theorems gives us complete structural information about abelian group, in particular finite abelian group. We have also discussed some examples in order to develope more intrest in algebra.

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NORMED LINEAR SPACES

Project report submitted to
The Kannur University
for the award of the degree
of

Bachelor of Science

by

ARUN K A

DB18CMSR27

Under the guidance of

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CERTIFICATE

It is to certify that this project report **‘Normed Linear Spaces’** is the bonafide project of **Arun K A** carried out the project work under my supervision.

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DECLARATION

I **Arun K A** hereby declare that the project **'Normed Linear Space'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Athulya P , Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

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CONTENTS

INTRODUCTION

This chapter gives an introduction to the theory of normed linear spaces. A skeptical reader may wonder why this topic in pure mathematics is useful in applied mathematics. The reason is quite simple: Many problems of applied mathematics can be formulated as a search for a certain function, such as the function that solves a given differential equation. Usually the function sought must belong to a definite family of acceptable functions that share some useful properties. For example, perhaps it must possess two continuous derivatives. The families that arise naturally in formulating problems are often linear spaces. This means that any linear combination of functions in the family will be another member of the family. It is common, in addition, that there is an appropriate means of measuring the “distance” between two functions in the family. This concept comes into play when the exact solution to a problem is inaccessible, while approximate solutions can be computed. We often measure how far apart the exact and approximate solutions are by using a norm. In this process we are led to a normed linear space, presumably one appropriate to the problem at hand. Some normed linear spaces occur over and over again in applied mathematics, and these, at least, should be familiar to the practitioner. Examples are the space of continuous functions on a given domain and the space of functions whose squares have a finite integral on a given domain.

PRELIMINARIES

1) LINEAR SPACES

We introduce an algebraic structure on a set $X$and study functions on X which are well behaved with respect to this structure. From now onwards , $\boldsymbol{K}$ will denote either $\boldsymbol{R}$, the set of all real numbers or $\boldsymbol{C}$**,** the set of all complex numbers. For $k \in \boldsymbol{C}$, *Re k* and *Im k* will denote the real and imaginary part of k.

A linear space(or a vector space) over $\boldsymbol{K}$ is a non-empty set X along with a function $+\ : X \times X \to X$, called addition and a function $\cdot$: $\boldsymbol{K} \times X \to X$ called scalar multiplication, such that for all $x\ ,\ y\ ,\ z\ \in\ X$and $k\ ,\ l \in \boldsymbol{K}$, we have

$$x + y = y + x$$

$$x + (y + z) = (x + y) + z$$

$$\exists 0 \in X \ such\ that\ x + 0 = x,$$

$$\exists - x \in X\ such\ that\ x + (- x) = 0\ ,$$

$$k \cdot (x + y) = k \cdot x + k \cdot y,$$

$$(k + l)\cdot x = k \cdot x + l \cdot x,$$

$$(kl)\cdot x = k \cdot (l \cdot x),$$

$$1\cdot x = x.$$

We shall write kx in place of $k \cdot x$. We shall also adopt the following notations. For $x, y \in X, k \in \boldsymbol{K}$ and subsets $E, F$of X ,

$$x + F = \{x + y: y \in F\},$$

$$E + F = \{x + y: x \in E, y \in F\},`$$

$$kE = \{kx: x \in E\}.$$

2) BASIS

A nonempty subset E of X is said to be a subspace of X if $kx + ly \in E$ whenever $x, y \in E$ and $k, l \in \boldsymbol{K}$. If $\emptyset \neq E \subset X$, then the smallest subspace of X containing E is

$$spanE = \left\{k_1x_1 + \cdots + k_nx_n : x_1, ..., x_n \in E,\ k_1, ..., k_n \in K\right\}$$

It is called the span of E. If span $E = X$, we say that E spans X. A subset E of X is said to be linearly independent if for all $x_1, ..., x_n \in E$ and $k_1, ..., k_n \in \boldsymbol{K}$, the equation $k_1x_1 + \cdots + k_nx_n = 0$implies that $k_1 = \cdots = k_n = 0$. It is called linearly dependent if it is not linearly independent, that is, if there exist $x_1, ..., x_n \in E$ and $k_1, ..., k_n \in \boldsymbol{K}$ such that $k_1x_1 + \cdots + k_nx_n = 0$, where at least one $k_j$is nonzero.

A subset E of X is called a Hamel basis or simply basis for X if *span of* $E = X$ and E is linearly independent .

3) DIMENSION

If a linear space X has a basis consisting of a finite number of elements , then X is called finite dimensional and the number of elements in a basis for X is called the dimension of X, denoted as $dimX$. Every basis for a finite dimensional linear space has the same (finite) number of elements and hence the dimension is well-defined. The space $\{0\}$ is said to have zero dimension. Note that it has no basis !

If a linear space contains an infinite linearly independent subset, then it is said to be infinite dimensional.

4)METRIC SPACE

We introduce a distance structure on a set X and study functions on X which are well-behaved with respect to this structure.

A metric d on a nonempty set X is a function $d: X \times X \rightarrow \boldsymbol{R}$
such that for all $x, y, z \in X$

$$d(x, y) \geq 0 \text{ and } d(x, y) = 0 \text{ iff } x=y$$

$$d(y, x) = d(x, y)$$

$$d(x, y) \leq d(x, z) + d(z, y) .$$

The last condition is known as the triangle inequality. A metric space is a nonempty set X along with a metric on it.

5)CONTINUOUS FUNCTIONS

Roughly speaking, a function from a metric space to a metric space is continuous if it sends 'nearby' points to 'nearby' points. If X and Y are metric spaces with metrics d and e respectively, then a function $F: X \to Y$ is said to be continuous at $x_0 \in X$ if for every $\epsilon \rangle 0$, there is some $\delta > 0$ (possibly depending on ϵ and x_0) such that $e\left(F(x), F\left(x_0\right)\right) < \epsilon$ for all $x \in X$ satisfying $d\left(x, x_0\right) < \delta$. Further, F is said to be continuous on X if it is continuous at every point of X. It is easy to see that F is continuous on X if and only if the set $F^{-1}(E)$ is open in X whenever the set E is open in$Y$. Also , this happens iff $F(x_n) \to F(x)$ in Y whenever $x_n \to x$ in X.

6) UNIFORM CONTINUITY

We note that a continuous function $F: T \to S$ is, in fact, uniformly continuous, that is, for every $\epsilon > 0$, there exists some $\delta > 0$ such that $e(F(t), F(u)) < \epsilon$ whenever $d\left(t, u\ \right) < \delta$. This can be seen as follows. Let $t \in T$. By the continuity of F at $t \in T$, there is some δ_t, such that $e(F(t), F(u)) < \frac{\varepsilon}{2}$ whenever $d(t, u) < \delta_t$.

7) FIELD

A ring is a set R together with two binary operations + and ·(which we call addition and multiplication) such that the following axioms are satisfied .

- R is an abelian group with respect to addition
- Multiplication is associative
- $\forall a , b, c \in R$the left distributive law $a(b + c) = (a \cdot b) + (a \cdot c)$ and the right distributive law $(a + b)c = (a \cdot c) + (b \cdot c)$, hold .

A field is a commutative division ring

CHAPTER 1

NORMED LINEAR SPACE

Let X be a linear space over $\boldsymbol{K}$. A norm on X is the function $|| \ ||$ from X to $\boldsymbol{R}$ such that $\forall$ $x,y \in X$ *and* $k \in \boldsymbol{K}$,

$$||x|| \geq 0 \text{ and } ||x|| = 0 \text{ if and only if } x = 0 \ ,$$

$$||x + y|| \leq ||x|| + ||y|| \ ,$$

$$||kx|| = |k| \ ||x|| \ .$$

A norm is the formalization and generalization to real vector spaces of the intuitive notion of " length" in the real world .

A normed space is a linear space with norm on it .

For x and y in X , let

$$d(x,y) = ||x - y||$$

Then d is a metric on X so that (X,d) is a metric space , thus every normed space is a metric space

➢ <u>*Every normed linear space is a metric space . But converse may not be true .*</u>

Example :

$$d(x,y) = \frac{|x-y|}{1+|x-y|} \ , \ \forall x \ , \ y \in X$$

$$\Rightarrow \ ||x - y|| = \frac{|x-y|}{1+|x-y|}$$

$$\Rightarrow \ ||z|| = \frac{|z|}{1+|z|} \ , \ z = x - y \in X$$

$$||\alpha z|| = \frac{|\alpha z|}{1+|\alpha z|}$$

$$= \frac{|\alpha|\,|z|}{1+|\alpha|\,|z|}$$

$$= |\alpha| \left(\frac{|z|}{1+|\alpha|\,|z|} \right)$$

$$\neq |\alpha|\ ||z|| .$$

➢ *Result*

Let X be a normed linear space . Then ,

$$\big|\ ||x|| - ||y||\ \big| \leq ||x - y|| ,\ \forall x , y \in X$$

Proof :

$$||x|| = ||\ (x - y) + y|| \leq ||x - y|| + ||y||$$

$$\Rightarrow ||x|| - ||y|| \leq ||x - y|| \rightarrow (1)$$

$x \leftrightarrow y$

$$||y|| - ||x|| \leq ||y - x||$$

$$\Rightarrow -\ (\ ||x|| - ||y||\ ) \leq\ ||x - y|| \rightarrow (2)$$

From (1) and (2)

$$\big|\,||x|| - ||y||\,\big| \leq\ ||x - y||$$

➢ *Norm is a continuous function*

Let $x_n \rightarrow x$, as $n \rightarrow \infty$

$\Rightarrow x_n - x \to 0$ *as* $n \to \infty$

$\Rightarrow ||x_n - x|| \to 0$ *as* $n \to \infty$

$\big|\, ||x_n|| - ||x|| \,\big| \leq ||x_n - x|| \to 0$ *as* $n \to \infty$

$\Rightarrow ||x_n|| - ||x|| \to 0$, *as* $n \to \infty$

$\Rightarrow ||x||$ is continuous

- *Norm is a uniformly continuous function*

We have , $||\ || : X \to \boldsymbol{R}$. Let $x, y \in X$ and $\varepsilon > 0$

Then $\quad ||x|| = ||x - y + y||$

$\leq ||x - y|| + ||y||$

$\Rightarrow ||x|| - ||y|| \leq ||x - y|| \to (1)$

Interchanging x and y ,

$||y|| - ||x|| \leq ||y - x||$

$\Rightarrow -(\, ||x|| - ||y||\,) \leq ||x - y||$

$\Rightarrow \quad ||x|| - ||y|| \geq - ||x - y|| \to (2)$

Combining (1) and (2)

$- ||x - y|| \leq ||x|| - ||y|| \leq ||x - y||$

That is ,

$\big|\, ||x|| - ||y|| \,\big| \leq ||x - y||$

Take $\delta = \varepsilon$, then whenever $||x - y|| < \delta$, $\big|\, ||x|| - ||y|| \,\big| < \varepsilon$

Therefore $|| \ ||$ is a uniformly continuous function .

➢ *Continuity of addition and scalar multiplication*

To show that $+ : X \times X \to X$ and $\cdot : \boldsymbol{K} \times X \to X$ are continuous functions.

Let $(x,y) \in X \times X$. To show that + is continuous at (x ,y) , that is , to show that for each $(x,y) \in X \times X$ *if* $x_n \to x$ and $y_n \to y$ in X , then

$$+(x_n , y_n) \to +(x , y) ;$$

That is ,

$$x_n + y_n \to x + y .$$

Consider

$$||(x_n + y_n) - (x + y)|| = ||x_n - x + y_n - y||$$

$$\leq ||x_n - x|| + ||y_n - y||$$

Given $x_n \to x$ *and* $y_n \to y$ *, for each* $\epsilon > 0$, $\exists N_1 \ni$

$$||x_n - x|| < \frac{\varepsilon}{2} \forall n \geq N_1 , \quad \textit{and} \ \exists N_2 \ni$$

$$||y_n - y|| < \frac{\varepsilon}{2} \quad \forall n \geq N_2$$

\

Take $N = max \{ N_1, N_2\}$

Then $||x_n - x|| < \frac{\varepsilon}{2}$ and $||y_n - y|| < \frac{\varepsilon}{2} \ \forall n \geq N$

Therefore $||(x_n + y_n) - (x + y)|| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \forall n \geq N$

That is , $x_n + y_n \to x + y$

Now to show that $\cdot : \boldsymbol{K} \times X \to X$ is continuous

Let $(k , x) \in \boldsymbol{K} \times X$

To show that if $k_n \to k$ and $x_n \to x$, then $k_n x_n \to kx$

Since $k_n \to k$, $\forall\, \varepsilon > 0\ \exists\ N_1 \ni |k_n - k| < \frac{\varepsilon}{2}\quad \forall\, n \geq N_1$

Since $x_n \to x$, $\forall\, \varepsilon > 0\ \exists\ N_2 \ni ||x_n - x|| < \frac{\varepsilon}{2}\quad \forall\, n \geq N_2$

Consider $||k_n x_n - kx|| = ||k_n x_n - kx + x_n k - x_n k||$

$$= ||x_n (k_n - k) + k(x_n - x)||$$

$$\leq ||x_n(k_n - k)|| + ||k(x_n - x)||$$

$$= ||x_n||\, |k_n - k| + |k|\, ||x_n - x||$$

$$\leq ||x_n||\frac{\varepsilon}{2} + |k|\frac{\varepsilon}{2}$$

$\therefore\quad k_n x_n \to kx$

➢ <u>*Examples of normed space*</u>

1) Spaces $\boldsymbol{K}^n$ ($\boldsymbol{K} = \boldsymbol{R}$ *or* $\boldsymbol{C}$)

For $n = 1$, the absolute value of function | | is a norm on $\boldsymbol{K}$, since $\forall k \in \boldsymbol{K}$

We have ,

$||k|| = ||k \cdot 1|| = |k|\, ||1||$, by definition .

But $||1||$ is a positive scalar .

$\therefore$ $||k||$ is a positive scalar multiple of the absolute value function .

$\therefore$ any norm on $\boldsymbol{K}$ is a positive scalar multiple of the absolute value function

For $n > 1$, let $p \geq 1$ be a real number

$$\boldsymbol{K}^n = \{ (x(1) , x(2) , \ldots , x(n)) : x(i) \in \boldsymbol{K} , i = 1 , 2 , \ldots , n \}$$

For $x \in \boldsymbol{K}^n$, that is , $x = (x(1) , x(2) , \ldots , x(n))$, define

$$||x||_p = (|x(1)|^p + \ldots + |x(n)|^p)^{1/p}$$

Then $|| \; ||_p$ is a norm on $\boldsymbol{K}^n$

When $p = 1$,

Then , $||x||_1 = |x(1)| + |x(2)| + \ldots + |x(n)|$

Since $|x(i)| \geq 0 \;\; \forall \, i = 1 , 2 , \ldots , n$, $||x||_1 \geq 0$

And $||x||_1 = 0 \Leftrightarrow |x(1)| + \ldots + |x(n)| = 0$

$$\Leftrightarrow |x(i)| = 0 \quad \forall \, i$$

$$\Leftrightarrow x(i) = 0 \; \forall \, i$$

$$\Leftrightarrow x = (x(1) , \ldots , x(n)) = 0$$

Now $||kx||_1 = |kx(1)| + |kx(2)| + \ldots + |kx(n)|$

$$= |k| \, |x(1)| + \ldots + |k| \, |x(n)|$$

$$= |k| \, (|x(1)| + \ldots + |x(n)|)$$

$$= |k| \, ||x||_1$$

$$||x + y||_1 = |(x + y)(1)| + \ldots + |(x + y)(n)|$$

$$= |x(1) + y(1)| + \ldots + |x(n) + y(n)|$$

$$\leq |x(1)| + |y(1)| + \ldots + |x(n)| + |y(n)|$$

$$= |x(1)| + \ldots + |x(n)| + |y(1)| + \ldots + |y(n)|$$

$$= ||x||_1 + ||y||_1$$

Consider $1<p<\infty$

Now , $||x||_p = (|x(1)|^p +. . . + |x(n)|^p)^{1/p}$

Since $|x(i)|^p \geq 0 \ \forall i$, we have $||x||_p \geq 0$

And $||x||_p = 0 \Leftrightarrow (|x(1)|^p +. . . + |x(n)|^p)^{1/p} = 0$

$\Leftrightarrow |x(i)|^p = 0 \ \forall i$

$\Leftrightarrow |x(i)| = 0 \ \forall i$

$\Leftrightarrow x(i) = 0 \ \forall i$

$\Leftrightarrow x = (x(1) , . . . , x(n)) = 0$.

Now

$$||kx||_p = (|kx(1)|^p +. . . + |kx(n)|^p)^{1/p}$$

$$= (|k|^p |x(1)|^p +. . . + |k|^p |x(n)|^p)^{1/p}$$

$$= |k| (|x(1)|^p +. . . + |x(n)|^p)^{1/p}$$

$$= |k| \, ||x||_p .$$

$$||x + y||_p = (|x(1) + y(1)|^p +. . . + |x(n) + y(n)|^p)^{1/p}$$

We have by Minkowski's inequality ,

$$\left(\sum_{i=1}^{n} |x(i) + y(i)|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y(i)|^p\right)^{1/p}$$

Then

$$||x+y||_p \leq \left(|x(1)|^p + \ldots + |x(n)|^p\right)^{1/p} + \left(|y(1)|^p + \ldots + |y(n)|^p\right)^{1/p}$$

$$= ||x||_p + ||y||_p$$

Then , for $1 \leq p < \infty$, $|| \; ||_p$ is a norm on $\boldsymbol{K^n}$

When $p = \infty$, define $||x||_\infty = max\{|x(1)|, |x(2)|, \ldots, |x(n)|\}$

Then it is a norm on $\boldsymbol{K^n}$

$||x||_p \geq 0$ since each values $|x(i)| \geq 0$

So that

$$max\{|x(i)|, i=1, \ldots, n\} \geq 0$$

$$||x||_\infty = 0 \Leftrightarrow max\{|x(i)| : i = 1, \ldots, n\} = 0$$

$$\Leftrightarrow |x(i)| = 0 \ \ \forall i$$

$$\Leftrightarrow x(i) = 0 , \forall i$$

$$\Leftrightarrow x = 0$$

$$||kx||_\infty = max\{|kx(1)|, \ldots, |kx(n)|\}$$

$$= max\{|k| \, |x(1)|, \ldots, |k| \, |x(n)|\}$$

$$= |k| \, max\{|x(1)|, \ldots, |x(n)|\}$$

$$= |k| \, ||x||_\infty$$

$$||x+y||_\infty = max\{|x(1) + y(1)|, \ldots, |x(n) + y(n)|\}$$

$$\leq max\{|x(1)| + |y(1)|, \ldots, |x(n)| + |y(n)|\}$$

$\leq max \{ |x(1)| , \ldots , |x(n)| \} + max \{ |y(1)| , \ldots , |y(n)| \}$

$= ||x||_\infty + ||y||_\infty$

2) Sequence space

Let $1 \leq p < \infty$, $l^p = \{ x = (x(1) , x(2) , \ldots) ; \ x(i) \in \boldsymbol{K} \ and \ \sum_{j=1}^{\infty} |x(j)|^p < \infty \}$, that is , l^p is the space of p-summable scalar sequences in $\boldsymbol{K}$. For $x = (x(1) , x(2) , \ldots) \in l^p$,

let $||x||_p = (|x(1)|^p + |x(2)|^p + \ldots)^{1/p}$. Then it is a norm on l^p.

That is , $|| \, ||_p$ is a function from l^p to $\boldsymbol{R}$.

If $p = 1$, then l^1 is a linear space and $||x||_1 = (|x(1)| + |x(2)| + \ldots)$ is a norm on l^1

Let $p = \infty$. Then l^∞ is the linear space of all bounded scalar sequences . And ,

$$||x||_\infty = sup \{ |x(j)| : j = 1, 2, 3, \ldots \}$$

Then $|| \, ||_\infty$ is a norm on l^∞

CHAPTER 2

THEOREMS ON NORMED SPACES

a) Let Y be a subspace of a normed space X, then Y and its closure $\overline{Y}$ are normed spaces with the induced norm.

b) Let Y be a closed subspace of a normed space X, for $x+Y$ in the quotient space X/Y, let $|||x+Y||| = inf\{\ ||x+y|| : y \in Y\}$. Then $|||\ \ |||$ is a norm on X/Y, called the quotient norm.

A sequence $(x_n + Y)$ converges to $x + Y$ in X/Y iff there is a sequence (y_n) in Y, $(x_n + y_n)$ converges to x in X.

c) Let $||\ \ ||_p$ be a norm on the linear space X_p, $j = 1,2,\ldots$. Fix p such that $1 \le p \le \infty$

For $x = (x(1), x(2), \ldots, x(m))$ that is the product space $X = X_1 \times X_2 \times \ldots \times X_m$,

Let $||x||_p = \left(||x(1)||_1^p + ||x(2)||_2^p + \ldots + ||x(m)||_m^p \right)^{1/p}$, if $1 \le p < \infty$

$||x||_p = max\{\ ||x(1)||_1, \ldots, ||x(m)||_m\}$, if $p = \infty$.

Then $||\ \ ||_p$ is a norm on X.

A sequence (x_n) converges to x in X $\Leftrightarrow$ $(x_n(j))$ converges to $x(j)$ in X_j $\forall\, j=1,2,\ldots,m.$

Proof:

a) Since X is a normed space, there is a norm on X to Y. Since Y is a subspace of X,

$||\ \ ||_y : Y \to \boldsymbol{R}$ is a function. To show that $||\ \ ||_y$ is a norm on Y.

For $y \in Y$, $||y||_Y = ||y||$, then

$$||y||_Y \ge 0 \quad (\because ||y|| \ge 0) \quad and \quad ||y||_Y = 0 \Leftrightarrow y = 0$$

$$||ky||_Y = ||ky|| = |k|\ ||y|| = |k|\ ||y||_y .$$

Let $y_1, y_2 \in Y$. Then ,

$$||y_1 + y_2||_y = ||y_1 + y_2|| \le ||y_1|| + ||y_2|| = ||y_1||_y + ||y_2||_y$$

Now the continuity of addition and scalar multiplication shows that $\overline{Y}$ is a subspace of X, since if $x_n \to x$ and $y_n \to y$, $x_n, y_n \in \overline{Y}$, then

$x_n + y_n \to x + y$ (by continuity of addition) and

$kx_n \to kx$ (by continuity of scalar X^n) .

Since $\overline{Y}$ is closed , x + y ∈ $\overline{Y}$ and kx ∈ $\overline{Y}$. Therefore $\overline{Y} \leq$ X.

∴ norm on X induces a norm on Y and $\overline{Y}$

b) X/Y , the quotient space equals $X/Y=\{x + Y : x \in X\}$.

$$|||x + y||| = inf\{ ||x + y|| : y \in Y\}$$

Claim: ||| ||| is a norm on X/Y , called quotient norm

- Let $x \in X$,

$$|||x + Y||| = inf\{ ||x + y|| : y \in Y\} \geq 0.$$

∴ $|||x + Y||| \geq 0$.

If $|||x + y||| = 0$ (0 in X/Y is Y) , then there is a sequence (y_n) in Y ∋

$$||x + y_n|| \to 0$$

$$\Rightarrow \quad x + y_n \to 0$$

$$\Rightarrow \quad y_n \to -x$$

Since $y_n \in Y$ and Y is closed

$-x \in Y \quad \Leftrightarrow x \in Y$ (∵ Y is a subspace)

$\Leftrightarrow x + Y = Y$, zero in X/Y.

- For $k \in \boldsymbol{K}$,

$$|||k(x + Y)||| = |||kx + Y|||$$

$$= inf\{ ||k(x + y)|| : y \in Y\}$$

$$= inf\{ |k|\ ||x + y|| : y \in Y\}$$

$$= |k|\ inf\{ ||x + y|| : y \in Y\}$$

$$= |k|\ |||x + Y||| .$$

- Let x_1 , $x_2 \in X$. Then

$|||x_1 + Y||| = inf\{ ||x_1 + y|| : y \in Y\}$. Then ∃ $y_1 \in Y$ ∋

$$|||x_1 + Y||| + \frac{\varepsilon}{2} > ||x_1 + y_1|| , \ and$$

$$|||x_2 + Y||| = \inf\{\, ||x_2 + y|| : y \in Y\}\ ,\ Then\ \exists y_2 \in Y \ni$$

$$|||x_2 + Y||| + \tfrac{\varepsilon}{2} > ||x_2 + y_2||\,.$$

$$||x_1 + y_1 + x_2 + y_2|| \le ||x_1 + y_1|| + ||x_2 + y_2||$$

$$\le |||x_1 + Y||| + \tfrac{\varepsilon}{2} + |||x_2 + Y||| + \tfrac{\varepsilon}{2}$$

Let $y = y_1 + y_2 \in Y$. Then ,

$$||(x_1+x_2) + y|| \le |||x_1 + Y||| + |||x_2 + Y||| + \varepsilon \quad \text{—}(1)$$

Now , $|||(x_1 + Y) + (x_2 + Y)||| = |||x_1 + x_2 + Y|||$

$$= \inf\{\, ||x_1 + x_2 + y|| : y \in Y\,\}$$

$$< ||x_1 + x_2 + y||$$

$$\le |||x_1 + Y||| + |||x_2 + Y||| + \varepsilon \qquad \text{(by (1))}$$

since ε is arbitrary , we have

$$|||(x_1 + Y) + (x_2 + Y)||| \le |||x_1 + Y||| + |||x_2 + Y|||$$

$\therefore\ |||\quad|||$ is a norm on X/Y.

Let $(x_n + Y)$ be a sequence in X/Y . Assume that (y_n) is a sequence in $Y \ni (x_n + y_n)$ converges to x in X.

That is , $(x_n - x + y_n)$ converges to 0 . —(1)

Claim: $(x_n + Y)$ converges to $x + Y$.

Consider

$$|||x_n + Y - (x+Y)||| = |||(x_n - x) + Y|||$$

$$= \inf\{\, ||x_n - x + y_n|| : y \in Y\,\}$$

$$\le ||x_n - x + y_n|| \quad \forall y_n \in Y\,.$$

Then by (1) , $x_n + Y$ converges to $x + Y$ in X/Y.

Conversely assume that the sequence $(x_n + Y) \to x + Y$ in X/Y.

Consider $|||x_n + Y - (x + Y)||| = |||x_n - x + Y|||$

$$= \inf\{\, ||x_n - x + y|| : y \in Y\,\}$$

Then we can choose $y_n \in Y \ni$

$$||x_n - x + y_n|| < |||(x_n - x) + Y||| + \frac{1}{n}, \quad n=1,2,3,....$$

Since $x_n+Y \rightarrow x+Y$, we get

$(x_n - x + y_n)$ converges to zero as $n \rightarrow\infty$

That is, $(x_n + y_n)$ converges to x in X as $n \rightarrow\infty$

c) Consider $1\leq p < \infty$

Given that

$$||x||_p = (||x(1)||_1^p + ||x(2)||_2^p + ... + ||x(m)||_m^p)^{1/p}$$

Clearly, $||x||_p \geq 0$.

Since each $||x(i)||_i^p \geq 0$.

$$||x||_p = 0 \Leftrightarrow |x(j)|_j^p = 0 \quad \forall\, j = 1, . . . ,m$$

$$\Leftrightarrow x(j) = 0 \quad \forall\, j.$$

$$\Leftrightarrow x = (x(1), . . . ,x(m)) = 0$$

$$||kx||_p = \left(||kx(1)||_1^p + . . . + ||kx(m)||_m^p \right)^{1/p}$$

$$= \left(|k|^p ||x(1)||_1^p + . . . + |k|^p ||x(m)||_m^p \right)^{1/p}$$

$$= |k| \left(||x(1)||_1^p + . . . + ||x(m)||_m^p \right)^{1/p}$$

$$= |k|\, ||x||_p \quad , k\in \boldsymbol{K} \text{ and } x\in X$$

Now, $||x + y||_p = \left(||x(1) + y(1)||_1^p + . . . + ||x(m) + y(m)||_m^p \right)^{1/p}$

(by Minkowski's inequality)

$$\leq \left(\left(||x(1)||_1 + ||y(1)||_1\right)^p + . . . + \left(||x(m)||_m + ||y(m)||_m\right)^p\right)^{1/p}$$

$$\leq \left(\sum_{j=1}^{m} ||x(j)||_j^p\right)^{1/p} + \left(\sum_{j=1}^{m} ||y(j)||_j^p\right)^{1/p} \qquad \text{(Minkowski's inequality)}$$

$$= \left(||x(1)||_1^p + \ldots + ||x(m)||_m^p \right)^{1/p}$$

$$= ||x||_p + ||y||_p$$

Now suppose $p = \infty$

$$||x||_\infty = max\{ ||x(1)||_1, \ldots, ||x(m)||_m \}$$

$||x||_\infty \geq 0$ Since $||x(j)|| \geq 0$, $\forall j$

$$||x||_\infty = 0 \quad \Leftrightarrow \quad ||x(m)|| = 0 \quad \forall m$$

$$\Leftrightarrow \quad x(m) = 0 \quad \forall m$$

$$\Leftrightarrow x = 0$$

$$||kx||_\infty = max\{ ||kx(1)||_1, \ldots, ||kx(m)||_m \}$$

$$= |k| \, max\{ ||x(1)||_1, \ldots, ||x(m)||_m \}$$

$$= |k| \, ||x||_\infty$$

$$||x + y||_\infty = max\{ ||x(1) + y(1)||_1, \ldots, ||x(m) + y(m)||_m \}$$

$$\leq max\{ ||x(1)||_1 + ||y(1)||_1, \ldots, ||x(m)||_m + ||y(m)||_m \}$$

$$= max\{ ||x(1)||_1, \ldots, ||x(m)||_m \} + max\{ ||y(1)||_1, \ldots, ||y(m)||_m \}$$

$$= ||x||_\infty + ||y||_\infty$$

We now consider ,

$$||x_n - x(1)||_p = \left(||x_n(1) - x(1)||_1^p + \ldots + ||x_n(m) - x(m)||_m^p \right)^{1/p}$$

Then

$$x_n \to x \text{ in } X \quad \Leftrightarrow \quad ||x_n - x||_p \to 0$$

$$\Leftrightarrow \quad ||x_n(j) - x(j)||_j^p \to 0$$

$$\Leftrightarrow \quad x_n(j) - x(j) \to 0$$

$$\Leftrightarrow \quad x_n(j) \to x(j) \text{ in } X \, \forall j .$$

RIESZ LEMMA

Let X be a normed space . Y be a closed subspace of X and $X \neq Y$. Let r be a real number such that $0 < r < 1$. Then there exist some $x_r \in X$ such that $||x_r|| = 1$ and

$r < dist\,(x_r\,,\,Y) \leq 1$

Proof :

We have ,

$$dist\,(x\,,\,Y) = inf\,\{\,d(x\,,y) : y \in Y\}$$

$$= inf\,\{\,||x - y|| : y \in Y\}$$

Since $Y \neq X$, consider $x \in X \ni x \notin Y$.

If $dist(x\,,\,Y) = 0$, then $||x - y|| = 0 \Rightarrow x \in \overline{Y} = Y$ ($\because$ Y is closed)

Therefore ,

$$dist\,(x\,,\,Y) \neq 0$$

That is ,

$$dist\,(x\,,\,Y) > 0$$

Since $0 < r < 1$, $\frac{1}{r} > 1$

$$\Rightarrow \frac{dist\,(x\,,Y)}{r} > dist\,(x\,,\,Y)$$

That is , $\frac{dist\,(x\,,Y)}{r}$ is not a lower bound of $\{\,||x - y|| : y \in Y\,\}$

Then $\exists$ $y_0 \in Y \ni ||x - y_0|| < \dfrac{dist(x\,,Y)}{r}$ $\rightarrow$(1)

Let $x_r = \dfrac{x - y_0}{||x - y_0||}$. Then $x_r \in X$

($\because y_0 \in Y\,,\, x \notin Y \Rightarrow x - y_0 \in X$ and $||x - y_0|| \neq 0$)

Then $||x_r|| = \left|\left|\dfrac{x - y_0}{||x - y_0||}\right|\right| = \dfrac{||x - y_0||}{||x - y_0||} = 1$

Now to prove $r < dist(x_r, Y) \leq 1$

We have $dist(x_r , Y) = inf\{ ||x_r - y|| : y \in Y \}$

$\leq ||x_r - y|| \quad \forall y \in Y$

In particular, $0 \in Y$, so that $dist(x_r , Y) \leq ||x_r - 0|| = 1$

That is ,

$$dist\ (x_r , Y) \leq 1$$

Now ,

$$dist\ (x_r , Y) = dist\left(\frac{x - y_0}{||x - y_0||}, Y\right)$$

$$= \frac{1}{||x - y_0||} dist\ (x - y_0 , Y)$$

$$= \frac{1}{||x - y_0||} inf\{ ||x - y_0 - y|| : y \in Y\}$$

$$= \frac{1}{||x - y_0||} inf\{||x - (y_0 + y)|| : y_0 + y \in Y\}$$

$$= \frac{1}{||x - y_0||} dist\ (x , Y)$$

$$> \frac{r}{dist\ (x , Y)} dist\ (x , Y) \quad by\ (1)$$

$$\Rightarrow dist\ (x_r , Y) > r$$

That is ,

$$r < dist\ (x_r , Y) \leq 1$$

CONCLUSION

This project discusses the concept of normed linear space that is fundamental to functional analysis . A normed linear space is a vector space over a real or complex numbers ,on which the norm is defined . A norm is a formalization and generalization to real vector spaces of the intuitive notion of “length” in real world

In this project , the concept of a norm on a linear space is introduced and thus illustrated . It mostly includes the properties of normed linear spaces and different proofs related to the topic.

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DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS

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of

Bachelor of Science

by

ASWATHI BABU P B

DB18CMSR20

Under the guidance of

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March 2021

Examiners 1: Examiner 2:

CERTIFICATE

It is to certify that this project report '**DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS**' is the bonafide project of **ASWATHI BABU P B** who carried out the project under my supervision.

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DECLARATION

I, **ASWATHI BABU P B**, hereby declare that this project report entitled '**DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS**' is an original record of studies and bonafide project carried out by me during the period from November 2019 to March 2020, under the guidance of **Ms.Sneha P Sebastian**, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not been submitted by me elsewhere for the award of any degree, diploma, title or recognition, before.

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ASWATHI BABU P B

CONTENTS

INTRODUCTION

In mathematics, a group is a set equipped with a binary operation that combines any two elements to form a third element in such a way that the three conditions called group axioms are satisfied , namely associativity , identity and invertability.

Let us take a moment to review our present stockpile of groups. Starting with finite groups, we have the cyclic group $\mathbb{Z}_n$,the symmetric group S_n , and the alternating group A_n for each positive integer n. We also have the dihedral group D_n and klein 4-group . Of course we know that subgroups of these groups exists. Turning to infinite groups , we have $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ under addition , and their non zero elements under multiplication we also have the group S_A of all permutation of an infinite set A , as well as various groups formed from matrices .

One purpose of this section is to show a way to use known groups as building blocks to form more groups. Given two groups G and H, it is possible to construct a new group from the cartesian product of G and H . Conversely , given a large group , it is sometimes possible to decompose the group ; that is , a group is sometimes isomorphic to the direct product of two smaller groups. Rather than studying a large group , it is often easier to study the component group of that group.

PRELIMINARY

Groups : A non empty set G together with an operation $*$ is said to be a group , denote by $(G\ ,*)$, if it satisfy the following axioms.

- Closure property
- Associative property
- Existence of identity
- Existence of inverse

Abelian group

A group $(G\ ,*)$ is said to be abelian if it satisfies commutative law .

Finite group

If the underlying set G of the group $(G\ ,*)$ consist of finite number of elements , then the group is finite group .

Infinite group

A group that is not finite is an infinite group .

Order of a group : The number of elements in a finite group is called the order of the group , denoted by $O(G)$.

Example

Show that the set of integers $\mathbb{Z}$ is a group with respect to the operation of addition of integers.

$$\mathbb{Z} = \{\dots\dots\dots. -3, -2, -1, 0, 1, 2, 3, \dots\dots\dots\}$$

Since the addition of two integers gives an integer , it satisfy closure property .

If $a, b, c \in \mathbb{Z}$ then the $(a+b)+c = a+(b+c)$, hence associativity holds .

There is a number $0 \in \mathbb{Z}$ such that $0+a = a+0$, hence identity exists

If $a \in \mathbb{Z}$ then there exists $-a \in \mathbb{Z}$, such that $-a + a = 0 = a + -a$

Therefore inverse exist .

Therefore $\mathbb{Z}$ is a group under addition .

Subgroup

A subset H of G is said to be a subgroup of G if H itself is a group under the same operation in G.

There are two different types of group structure of order 4 .

$\mathbb{Z}_4 = \{0,1,2,3\}$

Klein 4 – group , $V = \{e, a, b, c\}$

Cyclic group

A group G is cyclic if there is some element 'a' in G that generate G. And the element 'a' is called generator of G.

Group Homomorphism

A function $\Psi: G \rightarrow G'$ is a group homomorphism (or simply homomorphism).

If $\Psi(ab) = \Psi(a)\,\Psi(b)$ hold for all $a, b \in G$, is called homomorphism property .

Isomorphism

A one to one and onto homomorphism $\Psi: G \rightarrow G'$ is called an isomorphism .

CHAPTER – 1

DIRECT PRODUCT OF GROUPS

Definition

The Cartesian product of sets $S, S_2, \ldots \ldots, S_n$ is the set of all ordered n-tuples $(a_1, a_2, \ldots \ldots, a_n)$, where $a_i \in S_i$ for $i = 1,2,3, \ldots \ldots, n$. The Cartesian product is denoted by either

$S_1 \times S_2 \times \ldots \ldots \times S_n$ or by $\Pi_{i=1}^{n} S_i$.

Let $G_1, G_2, \ldots \ldots, G_n$ be groups and let us use multiplicative notation for all the group operations.

If we consider G_i as a set , $i = 1,2, \ldots \ldots n$. we have the products $G_1 \times G_2 \times \ldots \ldots, \times G_n$ we denote it by $\Pi_{i=1}^{n} G_i$. This product is called direct-product of groups. We can make $\Pi_{i=1}^{n} G_i$ into a group by means of a binary operation of multiplication by components.

Theorem

Let $G_1, G_2, \ldots \ldots, G_n$ be groups. For $(a_1, a_2, \ldots \ldots, a_n)$ and $(b_1, b_2, \ldots \ldots, b_n)$ in $\Pi_{i=1}^{n} G_i$ define ;

$$(a_1, a_2, \ldots \ldots, a_n)(b_1, b_2, \ldots \ldots, b_n) = (a_1 b_1, a_2 b_2, \ldots \ldots, a_n b_n)$$

Then $\Pi_{i=1}^{n} G_i$ is a group.

Proof

We have ,

$$\Pi_{i=1}^{n} G_i = \{(a_1, a_2, \dots \dots, a_n) : a_i \in G_i\}$$

(1) **Closure property**

Let $(a_1, a_2, \dots \dots, a_n), (b_1, b_2, \dots \dots, b_n) \in \Pi_{i=1}^{n} G_i$

And we have ,

$$(a_1, a_2, \dots \dots, a_n)(b_1, b_2, \dots \dots, b_n) = (a_1 b_1, a_2 b_2, \dots \dots, a_n b_n)$$

Here $a_i \in G_i$ and $b_i \in G_i$ for $i = 1,2, \dots \dots, n$

$\because$ G_i is a group , $a_i b_i \in G_i$ for $i = 1,2, \dots \dots, n$

$\Rightarrow (a_1 b_1, a_2 b_2, \dots \dots, a_n b_n) \in \Pi_{i=1}^{n} G_i$

i.e. $\Pi_{i=1}^{n} G_i$ is closed under the binary operation.

(2) **Associativity**

Let $(a_1, a_2, \dots \dots, a_n), (b_1, b_2, \dots \dots, b_n), (c_1, c_2, \dots \dots, c_n) \in \Pi_{i=1}^{n} G_i$

We have,

$$(a_1, a_2, \dots \dots, a_n)(b_1, b_2, \dots \dots, b_n)(c_1, c_2, \dots \dots, c_n)$$

$$= (a_1 b_1 c_1 , a_2 b_2 c_2, \dots \dots, a_n b_n c_n) \in \Pi_{i=1}^{n} G_i$$

$$[(a_1, a_2, \dots \dots, a_n)(b_1, b_2, \dots \dots, b_n)](c_1, c_2, \dots \dots, c_n)$$

$$\begin{aligned}
&= [a_1 b_1, a_2 b_2, \dots \dots, a_n b_n](c_1, c_2, \dots \dots, c_n) \\
&= [(a_1 b_1) c_1, (a_2 b_2) c_2, \dots \dots, (a_n b_n) c_n] \\
&= [a_1 (b_1 c_1), a_2 (b_2 c_2), \dots \dots, a_n (b_n c_n)] \\
&= (a_1, a_2, \dots \dots, a_n)[b_1 c_1, b_2 c_2, \dots \dots, b_n c_n] \\
&= (a_1, a_2, \dots \dots, a_n)[(b_1, b_2, \dots \dots, b_n)(c_1, c_2, \dots \dots, c_n)]
\end{aligned}$$

Hence associativity holds.

(3) **Existence of identity**

If e_i is the identity element in G_i.

Then,

$$(e_1, e_2, \dots \dots ., e_n) \in \Pi_{i=1}^{n} G_i$$

Also for ,

$$(a_1, a_2, \dots \dots ., a_n) \in \Pi_{i=1}^{n} G_i,$$

$$\begin{aligned}(a_1, a_2, \dots \dots ., a_n)(e_1, e_2, \dots \dots ., e_n) &= (a_1 e_1, a_2 e_2, \dots \dots ., a_n e_n)\\ &= (a_1, a_2, \dots \dots ., a_n)\end{aligned}$$

$\therefore$ $(e_1, e_2, \dots \dots ., e_n)$ is the identity element 'e' in $\Pi_{i=1}^{n} G_i$

(4) **Existence of inverse**

Let $(a_1, a_2, \dots \dots ., a_n) \in \Pi_{i=1}^{n} G_i$

Here $a_i \in G_i$ for $i = 1,2, \dots \dots ., n$.

Since G_i is a group ,

$\exists$ an inverse element a_i^{-1} in G_i : $a_i a_i^{-1} = e_i \qquad i = 1,2, \dots \dots ., n$

Clearly, $\qquad (a_1^{-1}, a_2^{-1}, \dots \dots ., a_n^{-1}) \in \Pi_{i=1}^{n} G_i$ &

$$(a_1, a_2, \dots \dots ., a_n)(a_1^{-1}, a_2^{-1}, \dots \dots ., a_n^{-1}) = (e_1, e_2, \dots \dots ., e_n)$$

Hence $\Pi_{i=1}^{n} G_i$ is a group.

Note

If the operation of each G_i is a commutative. We sometimes use additive notation in $\Pi_{i=1}^{n} G_i$ and refer to $\Pi_{i=1}^{n} G_i$ as the direct sum of the group G_i. The notation $\oplus_{i=1}^{n} G_i$, especially with abelian groups with operation $+$.

The direct sum of abelian groups $G_1, G_2, \dots \dots, G_n$ may be written $G_1 \oplus G_2 \oplus \dots \dots \oplus G_n$

- Direct product of abelian group is abelian

Example

Q. Check whether $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic or not.

$$\mathbb{Z}_2 = \{0,1\}$$

$$\mathbb{Z}_3 = \{0,1,2\}$$

$$\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$$

Consider,

$1(1,1) = (1,1)$

$2(1,1) = (1,1) + (1,1) = (0,2)$

$3(1,1) = (1,1) + (1,1) + (1,1) = (1,0)$

$4(1,1) = 3(1,1) + (1,1) = (1,0) + (1,1) = (0,1)$

$5(1,1) = 4(1,1) + (1,1) = (0,1) + (1,1) = (1,2)$

$6(1,1) = 5(1,1) + (1,1) = (1,2) + (1,1) = (0,0)$

$\therefore$ $(1,1)$ is a generator of $\mathbb{Z}_2 \times \mathbb{Z}_3$

$\therefore$ $\mathbb{Z}_2 \times \mathbb{Z}_3$ is a cyclic group generated by $(1,1)$.

Q. Check whether $\mathbb{Z}_3 \times \mathbb{Z}_3$ is cyclic or not.

$$\mathbb{Z}_3 = \{0,1,2\}$$

$$\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$$

$1(0,1) = (0,1)$

$2(0,1) = (0,2)$

$3(0,1) = (0,3) = (0,0)$ $\quad\quad \therefore$ order $(0,1) = 3$

$1(0,2) = (0,2)$

$2(0,2) = (0,4) = (0,1)$

$3(0,2) = (0,6) = (0,0)$ $\quad\quad \therefore$ order $(0,2) = 3$

Every element added to itself three times gives the identity. Thus no element can generate the group. Hence $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not cyclic.

similarly $\mathbb{Z}_m \times \mathbb{Z}_m$ is not cyclic for any m.

Theorem

The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime, that is, the gcd of m and n is 1.

Proof

Suppose $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and isomorphic to $\mathbb{Z}_{mn}$.

To show that m and n are relatively prime.

Suppose not, let d be the gcd of m and n.

So that $d > 1$

Consider $\frac{mn}{d}$, which is an integer since $d|m$ and $d|n$

Let (r, s) be an arbitrary element of $\mathbb{Z}_m \times \mathbb{Z}_n$, add (r, s) repeatedly $\frac{mn}{d}$ times

$$\left.(r, s) + (r, s) +, \ldots \ldots., +(r, s)\right\} \frac{mn}{d} times = (0,0)$$

$\therefore$ no element of $\mathbb{Z}_m \times \mathbb{Z}_n$ having order mn. $\therefore$ no element of $\mathbb{Z}_m \times \mathbb{Z}_n$ can generate $\mathbb{Z}_m \times \mathbb{Z}_n$ which is not possible. $\because$ $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic. Hence $gcd(m, n) = 1$.

i.e. m and n are relatively prime.

Conversely, suppose m and n are relatively prime, i.e. $\gcd(m, n) = 1$

To show that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic.

If $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic, then it is isomorphic to $\mathbb{Z}_{mn}$, $\because$ $\mathbb{Z}_m \times \mathbb{Z}_n$ has mn elements.

Consider the cyclic subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$ generated by the element $(1,1)$.The order of this cyclic subgroup is the smallest power of $(1,1)$,that gives the identity $(0,0)$. Here taking a power of $(1,1)$ in our additive notation will involve adding $(1,1)$ to itself repeatedly.

Consider $(1,1) + (1,1) +, \ldots \ldots., +(1,1)$

If we add first coordinates m times , we get zero.

$\therefore$ order of first coordinate $= m$.

Similarly , Order of second coordinate $= n$.

The two coordinates together become zero. If we add them $lcm(m, n)$ times.

$\because \gcd(m, n) = 1$, We get the $lcm = mn$.

i.e. $(1,1)$ generates a cyclic subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$ of order mn , which is the order of the whole group.

$$\Rightarrow \mathbb{Z}_m \times \mathbb{Z}_n = < (1,1) >$$

$\Rightarrow\ \mathbb{Z}_m \times\ \mathbb{Z}_n$ is cyclic.

Corollary

The group $\prod_{i=1}^{n} \mathbb{Z}_{m_i}$ is cyclic and isomorphic to $\mathbb{Z}_{m_1 m_2 \ldots\ldots m_n}$ if and only if the numbers m_i for $i = 1,2, \ldots \ldots, n$ are such that the gcd of any two of them is 1.

Example

If n is written as a product of powers of distinct prime numbers , as in ,

$$n = (p_1)^{n_1}.(p_2)^{n_2} \ldots \ldots .(p_n)^{n_r}$$

Then $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{(p_1)^{n_1}} \times \mathbb{Z}_{(p_2)^{n_2}} \times \ldots \ldots . \times \mathbb{Z}_{(p_r)^{n_r}}$.

In particular , $\mathbb{Z}_{72}$ is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_9$.

Consider set of integers $\mathbb{Z}$, cyclic subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}, n \in \mathbb{Z}$. Consider $2\mathbb{Z}$ and $3\mathbb{Z}$, then $< 2 > \cap < 3 > = < 6 >$

$\therefore$ if we take $r\mathbb{Z}, s\mathbb{Z}$ of $\mathbb{Z}$, then the $lcm(r, s)$ =generator of $< r > \cap < s >$

Using this we can define the lcm of the positive integers.

Definition

Let $r_1, r_2, \ldots \ldots, r_n$ be positive integers. Their least common multiple (abbreviated lcm) is the positive generator of the cyclic group of all common multiples of the r_i , that is the cyclic group of all integers divisible by each r_i for $i = 1,2, \ldots \ldots, n$.

Theorem

Let $(a_1, a_2, \dots \dots, a_n) \in \prod_{i=1}^{n} G_i$.

If a_i is of finite order r_i in G_i , then the order of $(a_1, a_2, \dots \dots, a_n)$ in $\prod_{i=1}^{n} G_i$ is equal to the least common multiple of all the r_i.

Proof

Given,

order of $a_1 = r_1 \Rightarrow a_1^{r_1} = e_1$ in G_1

order of $a_2 = r_2 \Rightarrow a_2^{r_2} = e_2$ in G_2

.

.

.

order of $a_n = r_n \Rightarrow a_n^{r_n} = e_n$ in G_n.

We have to find a power k for $(a_1, a_2, \dots \dots, a_n)$.

So that $(a_1, a_2, \dots \dots, a_n)^k = (e_1, e_2, \dots \dots, e_n)$.

The power must simultaneously be a multiple of r_1 , multiple of r_2 and so on. But k is the least positive integers having the above property.

$$\therefore k = lcm(r_1, r_2, \dots \dots, r_n).$$

Q. Find the order of $(8,4,10)$ in the group $\mathbb{Z}_{12} \times \mathbb{Z}_{60} \times \mathbb{Z}_{24}$.

$O(8) = 3$ in Z_{12}

$O(4) = 15$ in Z_{60}

$O(10) = 12$ in Z_{24}

$O(8,4,10) = lcm(3,15,12) = 60$

Q. Find a generator of $\mathbb{Z} \times \mathbb{Z}_2$

$$\mathbb{Z} \times \mathbb{Z}_2 = \{(n, 0), (n, 1): n \in \mathbb{Z}\}$$
$$(n, 0) = n(1,0)$$
$$(n, 1) = (n, 0) + (0,1) = n(1,0) + (0,1)$$

$\therefore$ $\mathbb{Z} \times \mathbb{Z}_2$ is generated by $\{(1,0), (0,1)\}$

In general , $\mathbb{Z} \times \mathbb{Z}_n$ is generated by ,

$$\{(1,0,0, \dots \dots .,0), (0,1,0, \dots \dots .,0), \dots \dots ., (0,0, \dots \dots .,1)\}$$

Q. Find the order of (3,10,9) in $(\mathbb{Z}_4, \mathbb{Z}_{12}, \mathbb{Z}_{15})$

$O(3) = 4$ in $\mathbb{Z}_4$

$O(10) = 6$ in $\mathbb{Z}_{12}$

$O(9) = 5$ in $\mathbb{Z}_{15}$

$\therefore O(3,10,9) = lcm(4,6,5)$
$= 60$

$\therefore$ order of (3,10,9) in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$ is 60.

CHAPTER-2

FUNDAMENTAL THEOREM OF FINITELY GENERATED ABELIAN GROUPS

Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups in the form,

$$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \dots \dots . \times \mathbb{Z}_{(p_n)^{r_n}} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \dots \dots . \times \mathbb{Z}$$

Where the p_i are primes , not necessarily distinct and the r_i are positive integers.

Remark

- The direct product is unique except for possible rearrangement of the factors.
- The number of factors $\mathbb{Z}$ is unique and this number is called Betti number.

Example

Find all abelian groups , upto isomorphism of order

1)8 , 2)16 , 3)360

(1) Order 8

$$8 = 1 \times 8$$

$$8 = 2 \times 4 = 2 \times 2^2$$

$$8 = 2 \times 2 \times 2$$

3 non-isomorphic groups are $\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

(2) Order 16

$16 = 1 \times 16 = 1 \times 2^4$

$16 = 2 \times 8 \;\; = 2 \times 2^3$

$16 = 4 \times 4 \;\; = 2^2 \times 2^2$

$16 = 2 \times 2 \times 2 \times 2$

$16 = 2 \times 2 \times 2^2$

$\mathbb{Z}_{16}, \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$

(3) Order 360

$360 = 2^2 \cdot 3^2 \cdot 5$

Possibilities are,

1) $\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
2) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
3) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
4) $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
5) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
6) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

Definition

A group G is decomposable if it is isomorphic to a direct product of two proper non-trivial subgroups , otherwise G is indecomposable.

Example

$\mathbb{Z}_6$ is decomposable while $\mathbb{Z}_5$ is indecomposable.

$\mathbb{Z}_6$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$

$\mathbb{Z}_{mn}$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$, if m and n are prime.

Theorem

The finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.

Proof

Let G be a finite indecomposable abelian group $\because G$ is finitely generated , we can apply fundamental theorem of finitely generated abelian groups.

$\therefore G \cong \mathbb{Z}_{(p)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \dots \dots . \times \mathbb{Z}_{(p_n)^{r_n}}$

$\because G$ is indecomposable and $\mathbb{Z}_{(p_i)^{r_i}}$'s are proper subgroups we get in the above , there is only one factor say $\mathbb{Z}_{(p_i)^{r_i}}$ which is cyclic group with order a prime power.

Theorem

If m divides the order of a finite abelian group , then G has a subgroup of order m.

Proof

Given G is a finite abelian group.

$\therefore$ we can apply Fundamental Theorem ,

Hence,

$$G \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \dots \dots . \times \mathbb{Z}_{p_n^{r_n}}$$

Here all primes need not be distinct.

Then,

$$O(G) = p_1^{r_1} . p_2^{r_2} \dots \dots p_n^{r_n}$$

Let m is a $+ve$ integer which divides $O(G)$.

$0 \leq s_i \leq r_i$ By theorem , " let G be a cyclic group with n elements and generated by a. Let $b \in G$, $b = a^s$, then ' b' generates a cyclic subgroup H of G containing $\frac{n}{d}$ elements , where $d = \gcd(n, s)$."

$p_i^{r_i - s_i}$ generates a cyclic subgroup of $\mathbb{Z}_{p_i^{r_i}}$ having order $\frac{p_i^{r_i}}{gcd(p_i^{r_i}, p_i^{r_i - s_i})}$

$$= \frac{p_i^{r_i}}{p_i^{r_i - s_i}} = p_i^{s_i}$$

$$\therefore O(< p_i^{r_i - s_i} >) = p_i^{s_i}$$

i.e. $< p_1^{r_1 - s_1} >$ is a subgroup of $\mathbb{Z}_{p_1^{r_1}}$ having order $p_1^{s_1}$.

$< p_2^{r_2 - s_2} >$ is a subgroup of $\mathbb{Z}_{p_2^{r_2}}$ having order $p_2^{s_2}$.

………………………………………………………………

$< p_n^{r_n - s_n} >$ is a subgroup of $\mathbb{Z}_{p_n^{r_n}}$ having order $p_n^{s_n}$.

$\therefore < p_1^{r_1 - s_1} > \times < p_2^{r_2 - s_2} > \times \dots \dots . \times < p_n^{r_n - s_n} >$ is a subgroup of $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \dots \dots . \times \mathbb{Z}_{p_n^{r_n}}$ having order $p_1^{s_1} \cdot p_2^{s_2} \cdots\cdots p_n^{s_n} = m$.

Theorem

If m is a square free integer , that is m is not divisible by the square of any prime . Then every abelian group of order m is cyclic.

Proof

Let m be a square free integer , then $p^i \nmid m$ for every i greater than 1 for a prime p.

Given G is a finite abelian group having order m , by fundamental theorem , then

$$G \cong \mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \dots \dots . \times \mathbb{Z}_{(p_n)^{r_n}}$$

Then,

$$O(G) = p_1^{r_1} \cdot p_2^{r_2} \cdots\cdots p_n^{r_n}$$

$\because O(G)$ is a square free integer , the only possibility

$$r_1 = r_2 = \dots \dots .. = r_n = 1$$

Then,

$$G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \dots . \times \mathbb{Z}_{p_n}$$

$$\cong \mathbb{Z}_{p_1, p_2, \dots\dots, p_n} \text{ , which is cyclic.}$$

Example

15 is a square free integer. So an abelian group of order 15 is cyclic.

CONCLUSION

Direct product of groups is the product $G_1 \times G_2, \dots \dots . G_n$,where each G_i is a set. We have discussed about definition and some properties related to the direct product of groups. The fundamental theorem of finitely generated abelian group helped us to get a deeper understanding about the topic. The theorems gives us complete structural information about abelian group, in particular finite abelian group. We have also discussed some examples in order to develope more intrest in algebra.

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GRAPH COLORING

Project report submitted to

The Kannur University

for the award of the degree

of

Bachelor of Science

by

CHRISTY JOY

DB18CMSR21

Under the guidance of

MRS. Riya Baby

Department of Mathematics
Don Bosco Arts and Science College
Angadikadavu

March 2021

Examiners 1: Examiner 2:

CERTIFICATE

It is to certify that this project report '**GRAPH COLORING**' is the bona fide project of CHRISTY JOY who carried out the project under my supervision.

Mrs. Riya Baby

Supervisor, HOD

Department of Mathematics

Don Bosco Arts and Science College

Angadikadavu

DECLARATION

I, CHRISTY JOY, hereby declare that this project report entitled “**GRAPH COLORING**” is an original record of studies and bona fide project carried out by me during the period from November 2019 to March 2020, under the guidance of **Mrs. Riya Baby**, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not been submitted by me elsewhere for the award of any degree, diploma, title or recognition, before.

CHRISTY JOY

Department of Mathematics

Don Bosco Arts and Science College

Angadikadavu

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I sincerely express my deep sense of gratitude to all who have been of great help to me during the course of my dissertation. First and foremost I thank the almighty, for his blessing and protection during the period of this work. I express my thanks to Dr. Fr. Francies Karakkatt, Principal, for support in the completion of this dissertation. I express my gratitude to Mrs. Riya Baby, my project supervisor, for the constant encouragement, valuable guidance and timely corrections, which made this work a success.

I am also indebted to all my classmates and friends who supported me throughout the study. I would like to express my thanks to my parents and dear ones for their constant encouragement and support. I also thank all those who helped me directly or indirectly to complete this project.

CHRISTY JOY

CONTENTS

INTRODUCTION

A proper coloring of a graph is an assignment of colors to the vertices of the graph so that no two adjacent vertices have the same color.

Usually we drop the word "proper" unless other types of coloring are also under discussion. Of course, the "colors" don't have to be actual colors ; may can be any distinct labels - integers ,for examples , if a graph is not connected , each connected component can be colored independently; except where otherwise noted , we assume graphs are connected. We also assume graphs are simple in this section. Graph coloring has many applications in addition to its intrinsic interest.

In the same way the most important concept of graph coloring is utilized in resource allocation, scheduling. Also, paths, walks and circuits in graph theory are used in tremendous applications say travelling salesman problem, database design concepts, resource networking.

This project deals with coloring which is one of the most important topics in graph theory. In this project there are three chapters. First chapter is coloring . The second chapter is chromatic number. The last chapter deals with application of graph coloring.

BASIC CONCEPTS

1. GRAPH

A graph is an ordered triplet. G=(V(G), E(G), I(G)); V(G) is a non empty set, E(G) is a set disjoint from V(G) and I(G) is an incidence map that associates each element of E(G) and unrecorded pair of element of V(G). The elements of V(G) are called vertices (or nodes or points) of G and the elements of E(G) are Called edges or lines of G.

2. MULTIPLE EDGE / PARALLEL EDGE

A set of 2 or more edges of a graph G is called a multiple edge or parallel edge if they have the same end vertices.

3. LOOP

An edge for which the 2 end vertices are same is called a loop.

4. SIMPLE GRAPH

A graph is simple if it has no loop and no multiple edges.

5. DEGREE

Let G be a graph and v € V the number of edge incident at V in G is called the degree or vacancy of the vertex v in G.

CHAPTER - 1

COLORING

Graph coloring is nothing but a simple way of labeling graph components such as vertices , edges and regions under some constraints. In a graph, no two adjacent vertices, adjacent edges , or adjacent regions are colored with minimum number of colors .This number is called the chromatic number and the graph is called properly colored graph.

In graph theory coloring is a special case of graph labeling; it is an assignment of labels traditionally called “colors” to elements of a graph subject to certain constraints. In it simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color, it is called vertex coloring. Similarly, edge coloring assigns a color to each edge so that no two adjacent edges share the common color.

While graph coloring , the constraints that are set on the graph are colors , order of coloring , the way of assigning color , etc. A coloring is given to a vertex or a particular region . Thus, the vertices or regions having same colors form independent sets.

VERTEX COLORING

Vertex coloring is an assignment of colors to the vertices of a graph ‘G’ such that no two adjacent vertices have the same color .Simply put , no two vertices of an edge should be of the same color.

The most common type of vertex coloring seeks to minimize the number of colors for a given graph . Such a coloring is known as a minimum vertex coloring , and the minimum number of colors which with the vertices of a graph may be colored is called the chromatic number .

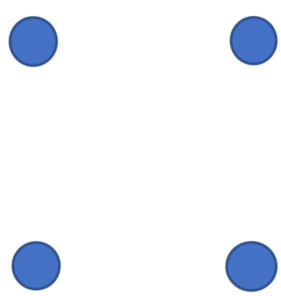
CHROMATIC NUMBER:

The minimum number of colors required for vertex coloring of graph ‘G’ is called as the chromatic number of G , denoted by X (G) .

X(G) = 1 iff ‘G’ is a null graph. If ‘G’ is not a null graph , then X(G) ≥ 2.

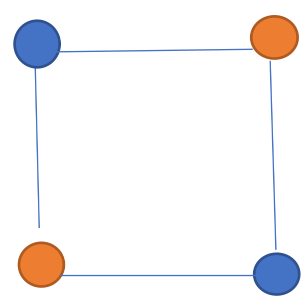
EXAMPLES;

1.



Null Graph (X (G) = 1)

2.

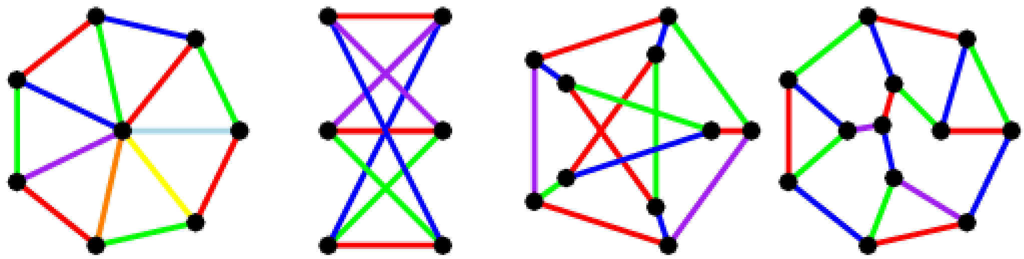


Not Null Graph (X (G) = 2)

EDGE COLORING

An edge coloring of a graph G is a coloring of the edges of G such that adjacent edges (or the edges bounding different regions) receive different colors. An edge coloring containing the smallest possible number of colors for a given graph is known as a minimum edge coloring.

The edge chromatic number gives the minimum number of colours with which graph's edges can be colored.



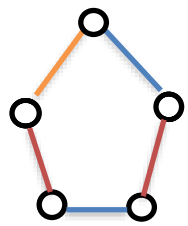
CHROMATIC INDEX

The minimum number of colors required for proper edge coloring of graph is called chromatic index.

A complete graph is the one in which each vertex is directly connected with all other vertices with an edge. If the number of vertices of a complete graph is n, then the chromatic index for an odd number of vertices will be n and the chromatic index for even number of vertices will be n-1.

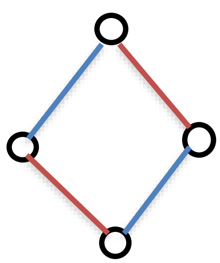
EXAMPLES;

1.



The given graph will require 3 unique colors so that no two incident edges have the Same color. So its chromatic index will be 3.

2.



The given graph will require 2 unique colors so that no two incident edges have the same color. So its chromatic index will be 2.

CHAPTER 2

Chromatic Number

The chromatic number of a graph is the smallest number of colors needed to color the vertices of so that no two adjacent vertices share the same color. That is the smallest value of possible to obtain a k-coloring.

- Graph Coloring is a process of assigning colors to the vertices of a graph.
- It ensures that no two adjacent vertices of the graph are colored with the same color.
- Chromatic Number is the minimum number of colors required to properly color any graph.

Graph Coloring Algorithm

- There exists no efficient algorithm for coloring a graph with minimum number of colors.

However, a following greedy algorithm is known for finding the chromatic number of any given graph.

Greedy Algorithm

<u>Step-01:</u>

Color first vertex with the first color.

Step-02:

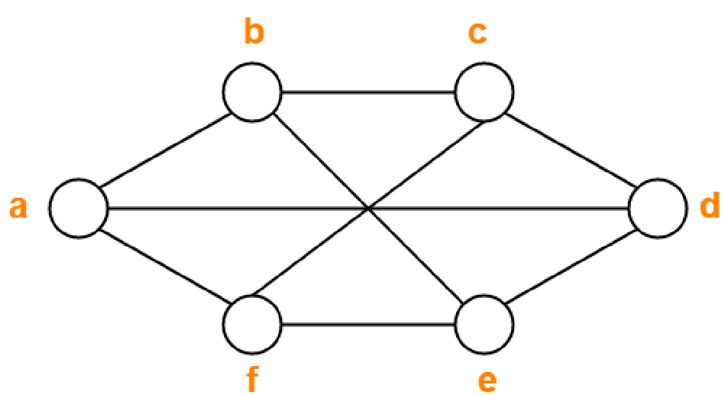
Now, consider the remaining (V-1) vertices one by one and do the following-

- Color the currently picked vertex with the lowest numbered color if it has not been used to color any of its adjacent vertices.
- If it has been used, then choose the next least numbered color.
- If all the previously used colors have been used, then assign a new color to the currently picked vertex.

Problems Based On Finding Chromatic Number of a Graph

Problem-01:

Find chromatic number of the following graph-



Solution-

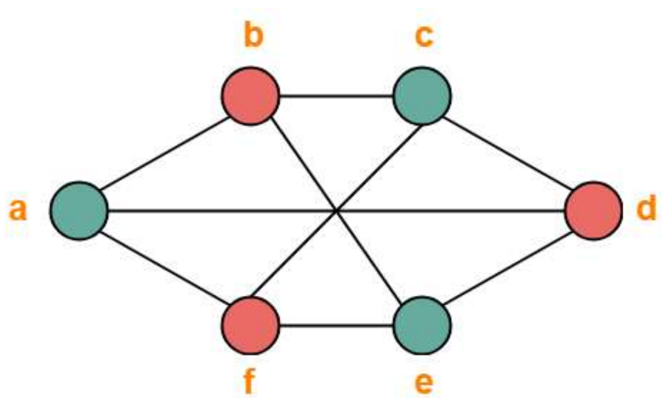
Applying Greedy Algorithm, we have

Vertex	a	b	C	d	e	f
Color	C1	C2	C1	C2	C1	C2

From here,

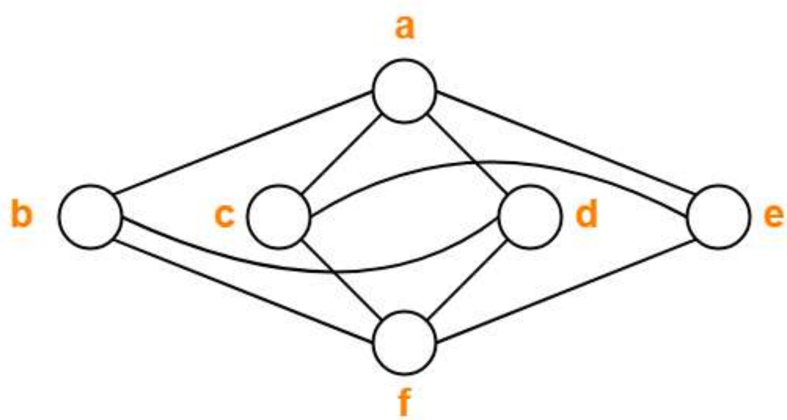
- Minimum numbers of colors used to color the given graph are 2.
- Therefore, Chromatic Number of the given graph = 2.

The given graph may be properly colored using 2 colors as shown below-



Problem-02:

Find chromatic number of the following graph-



Solution-

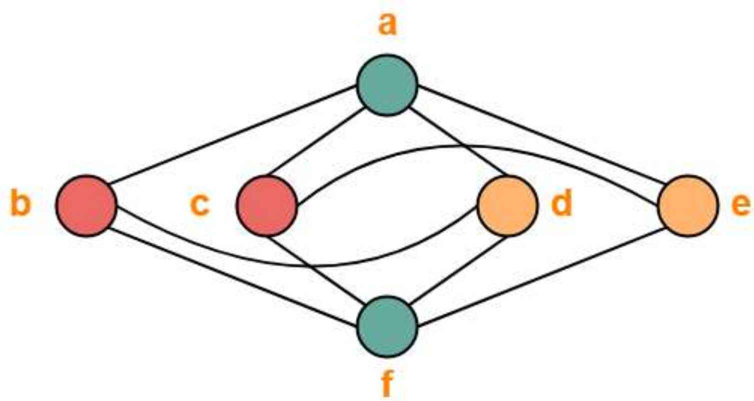
Applying Greedy Algorithm, we have-

Vertex	a	b	C	d	e	f
Color	C1	C2	C2	C3	C3	C1

From here,

- Minimum numbers of colors used to color the given graph are 3.
- Therefore, Chromatic Number of the given graph = 3.

The given graph may be properly colored using 3 colors as shown below-



Chromatic Number of Graphs

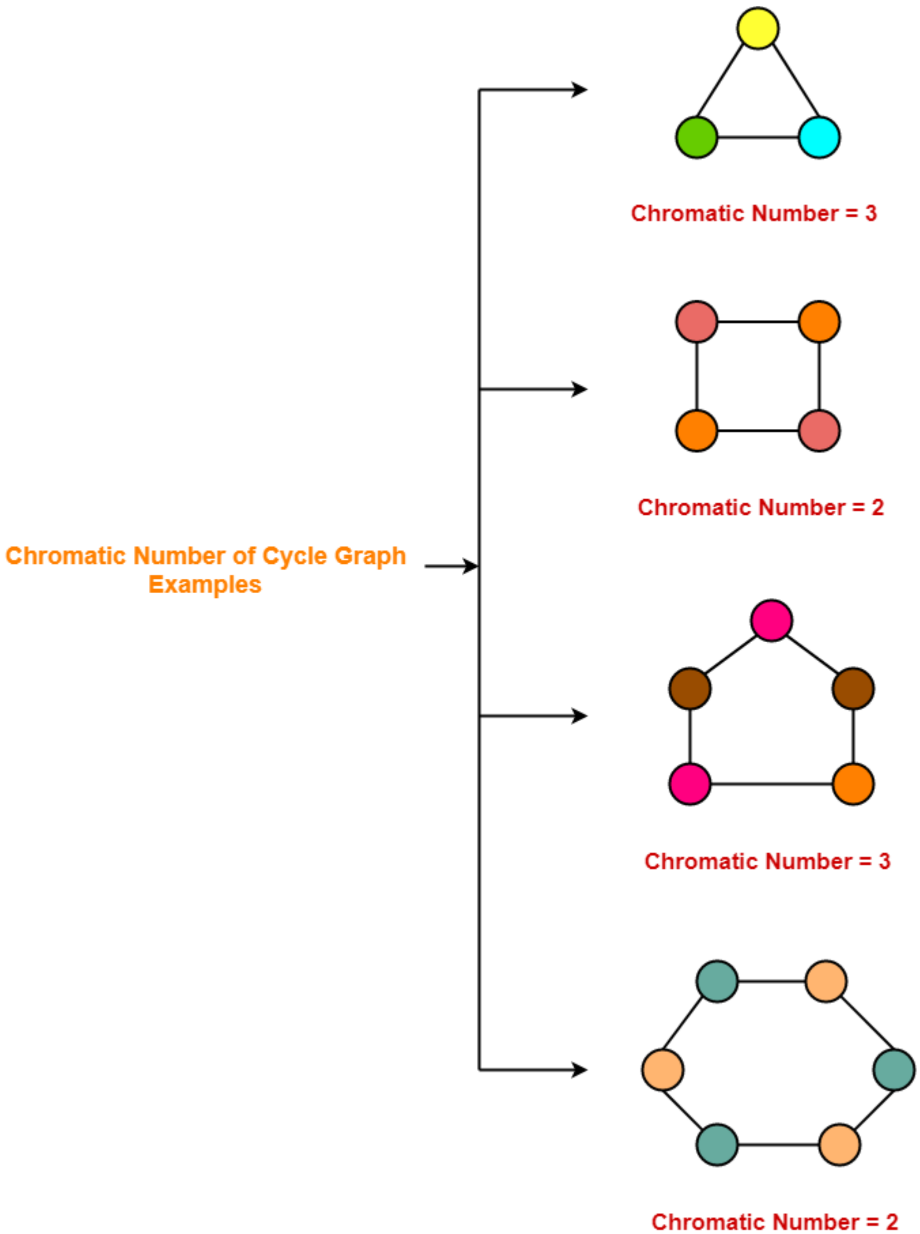
Chromatic Number of some common types of graphs are as follows-

1. Cycle Graph-

- A simple graph of ‘n’ vertices (n>=3) and ‘n’ edges forming a cycle of length ‘n’ is called as a cycle graph.
- In a cycle graph, all the vertices are of degree 2.

Chromatic Number
• If number of vertices in cycle graph is even, then its chromatic number = 2. • If number of vertices in cycle graph is odd, then its chromatic number = 3.

Examples-



2. Planar Graphs-

A planar graph is a graph that can be embedded in the plane, that is it can be drawn on the plane in such a way that its edges intersect only at their endpoint. In other words, it can be drawn in such a way that no edges cross each other.

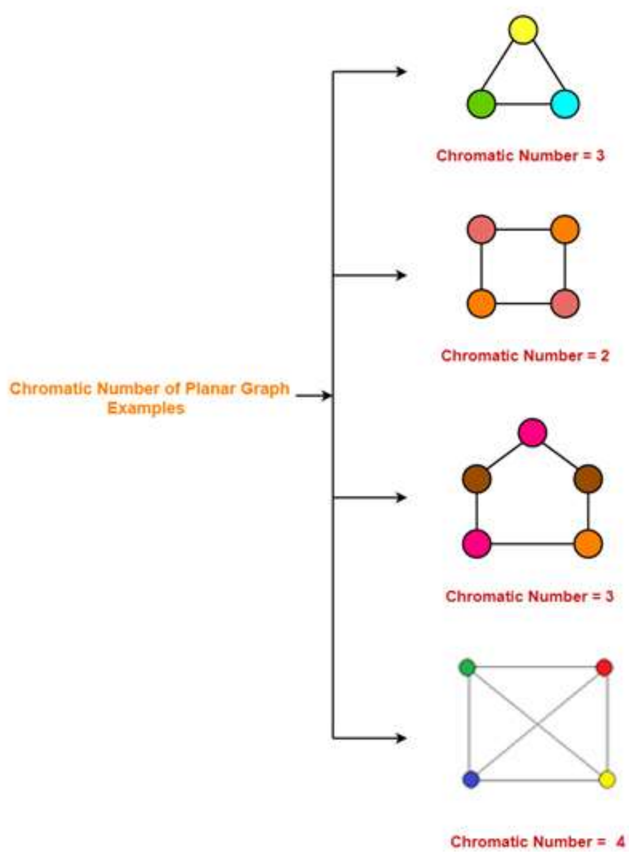
A **Planar Graph** is a graph that can be drawn in a plane such that none of its edges cross each other.

Chromatic Number Chromatic Number of any Planar Graph is less than or equal to 4

Examples-

+

- All the above cycle graphs are also planar graphs.
- Chromatic number of each graph is less than or equal to 4.

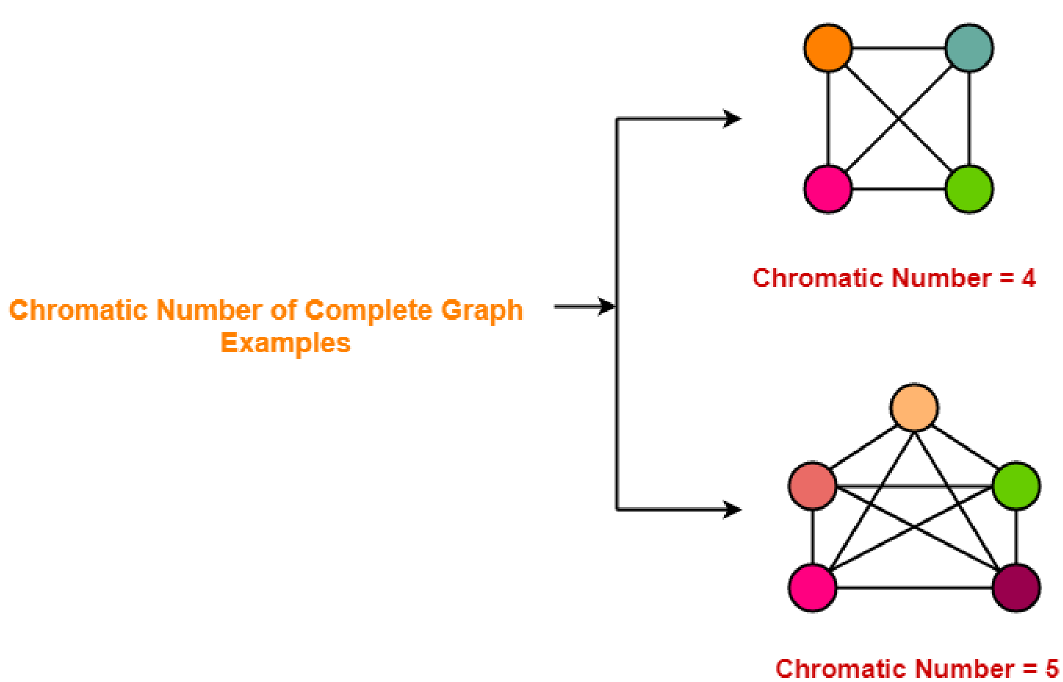


3. Complete Graphs-

- A complete graph is a graph in which every two distinct vertices are joined by exactly one edge.
- In a complete graph, each vertex is connected with every other vertex.
- So to properly it, as many different colors are needed as there are number of vertices in the given graph.

Chromatic Number Chromatic Number of any Complete Graph = Number of vertices in that Complete Graph

Examples-



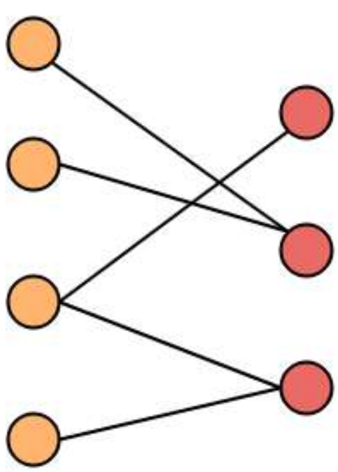
4. Bipartite Graphs-

A bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to one in V. Vertex sets U and V are usually called the parts of the graph.

- A **Bipartite Graph** consists of two sets of vertices X and Y.
- The edges only join vertices in X to vertices in Y, not vertices within a set.

Chromatic Number Chromatic Number of any Bipartite Graph = 2

Example-



Chromatic Number = 2

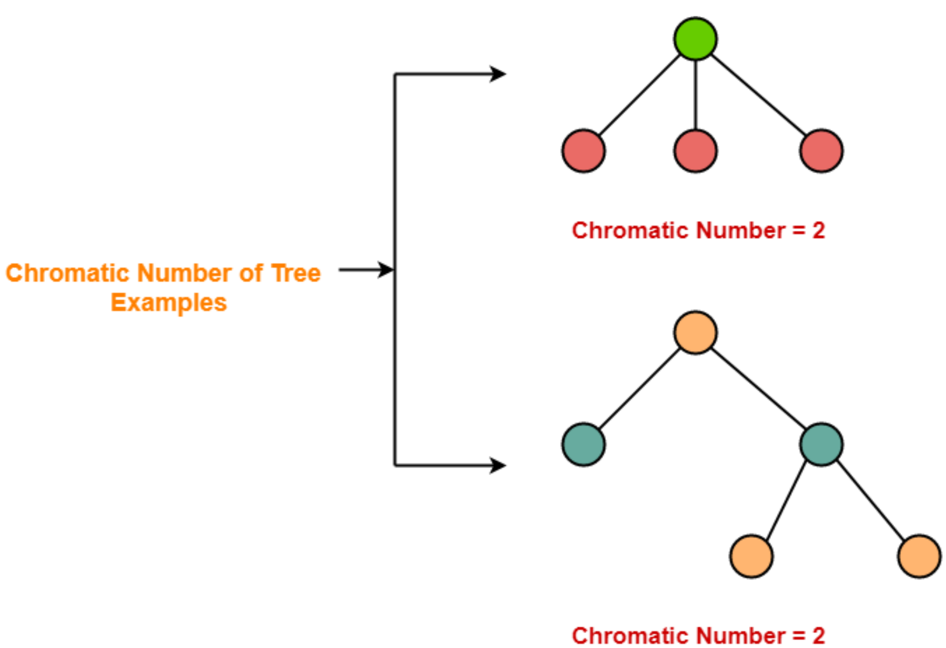
5. Trees-

A tree is an undirected graph in which any two vertices are connected by exactly one path, or equivalently a connected acyclic undirected graph.

- A **Tree** is a special type of connected graph in which there are no circuits.
- Every tree is a bipartite graph.
- So, chromatic number of a tree with any number of vertices = 2.

Chromatic Number
Chromatic Number of any tree = 2

Examples-



CHAPTER-3

APPLICATIONS OF GRAPH COLORING

1) Making Schedule or Time Table:

Suppose we want to make an exam schedule for a university. We have list different subjects and students enrolled in every subject. Many subjects would have common students (of same batch, some backlog students, etc). How do we schedule the exam so that no two exams with a common student are scheduled at same time? How many minimum time slots are needed to schedule all exams? This problem can be represented as a graph where every vertex is a subject and an edge between two vertices mean there is a common student. So this is a graph coloring problem where minimum number of time slots is equal to the chromatic number of the graph.

2) Mobile Radio Frequency Assignment:

When frequencies are assigned to towers, frequencies assigned to all towers at the same location must be different. How to assign frequencies with this constraint? What is the minimum number of frequencies needed? This problem is also an instance of graph coloring problem where every tower represents a vertex and an edge between two towers represents that they are in range of each other.

3) Register Allocation:

In compiler optimization, register allocation is the process of assigning a large number of target program variables onto a small number of CPU registers. This problem is also a graph coloring problem.

4) Sudoku:

Sudoku is also a variation of Graph coloring problem where every cell represents a vertex. There is an edge between two vertices if they are in same row or same column or same block.

5) Map Coloring:

Geographical maps of countries or states where no two adjacent cities cannot be assigned same color. Four colors are sufficient to color any map.

6) Bipartite Graphs:

We can check if a graph is bipartite or not by coloring the graph using two colors. If a given graph is 2-colorable, then it is Bipartite, otherwise not. See this for more details.

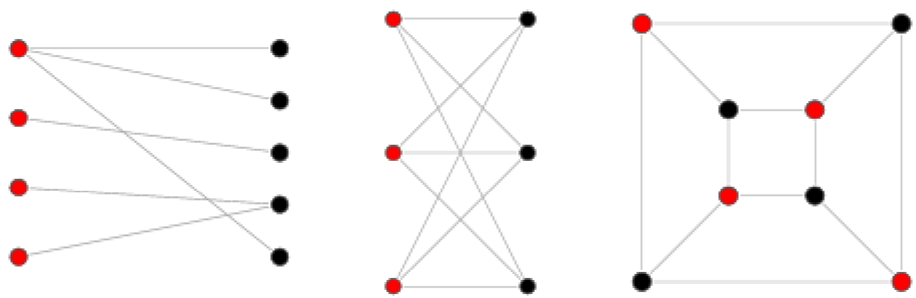
Explanation;

Algorithm:

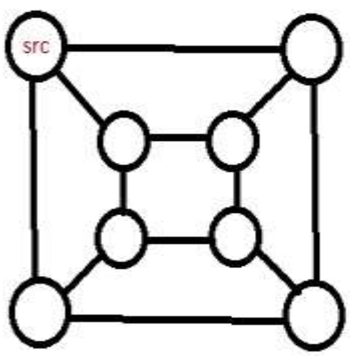
A bipartite graph is possible if it is possible to assign a color to each vertex such that no two neighbour vertices are assigned the same color. Only two colors can be used in this process.

Steps:

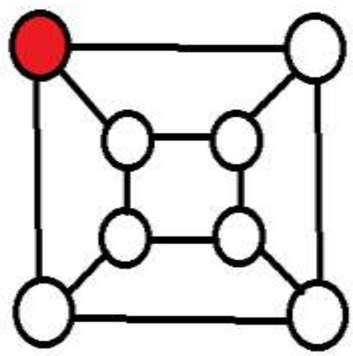
1. Assign a color (say red) to the source vertex.
2. Assign all the neighbours of the above vertex another color (say blue).
3. Taking one neighbour at a time, assign all the neighbour's neighbours the color red.
4. Continue in this manner till all the vertices have been assigned a color.
5. If at any stage, we find a neighbour which has been assigned the same color as that of the current vertex, stop the process. The graph cannot be colored using two colors. Thus the graph is not bipartite.



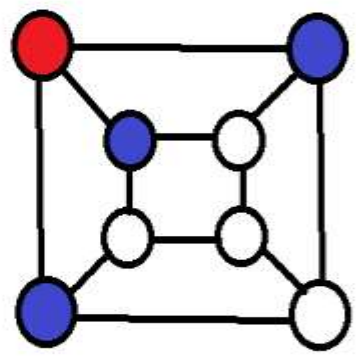
Example:



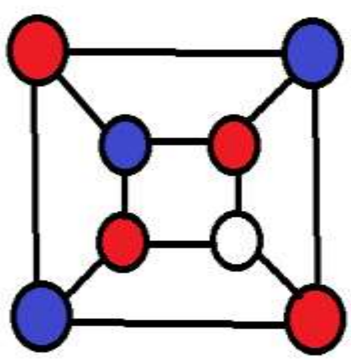
given a graph with source vertex



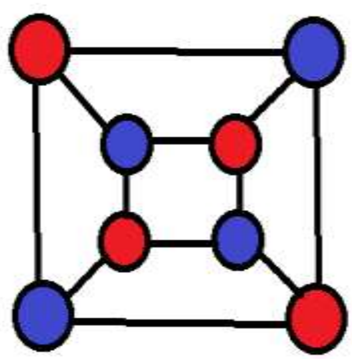
colour src vertex, say red



assign another colour to the neighbours, say blue



assign the neighbours of the vertices of the previous step the colour red



repeat till all vertices are coloured, or a conflicting colour assignment occurs.

set U: red colour
set V: blue colour

CONCLUSION

This project aims to provide a solid background in the basic topics of graph coloring. Graph coloring problem is to assign colors to certain elements of a graph subject to certain constraints. The nature of coloring problem depends on the number of colors but not on what they are.

The study of this topic gives excellent introduction to the subject called “Graph Coloring”.

This project includes two important topics such as vertex coloring and edge coloring and came to know about different ways and importance of coloring.

Graph coloring enjoys many practical applications as well as theoretical challenges. Besides the applications, different limitations can also be set on the graph or on the away a color is assigned or even on the color itself. It has been reached popularity with the general public in the form of the popular number puzzle Sudoku and it is also use in the making of time management which is an important application of coloring. So graph coloring is still a very active field of research.

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POWER SERIES SOLUTIONS AND SPECIAL FUNCTIONS

Project report submitted to
The Kannur University
for the award of the degree
of

Bachelor of Science

by

CYRIL SAJI

DB18CMSR28

Under the guidance of

Ms. Athulya P

Department of Mathematics
Don Bosco Arts and Science College
Angadikadavu

CERTIFICATE

Certified that this project **'Power Series'** is a bona fide project of **CYRIL SAJI** carried out the project work under my supervision.

Mrs. Riya Baby **Ms. Athulya P**

Head of Department **Supervisor**

Department of Mathematics

Don Bosco Arts and Science College

Angadikadavu

DECLARATION

I **CYRIL SAJI** hereby declare that the project **'Power Series'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Athulya P, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

Name

CYRIL SAJI

Department of Mathematics

Don Bosco Arts and Science College,

Angadikkadavu

ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express my deepest gratitude to people along the way.

No words can adequately express the sense of gratitude; still, I try to express my heartfelt thanks through words. At the outset, I am deeply indebted to my project supervisor Ms. Athulya P, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu, for the invaluable guidance, loving encouragement and meticulous care towards me throughout my career. I express my deep sense of gratitude to all the faculty members of the Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu.

I can never forget the support and encouragement rendered by the Principal and the staff of Don Bosco Arts and Science College, Angadikkadavu.

I could not name many who sincerely supported and helped for the successful completion of this project. It is my pleasure and duty to thank each and every one of them who walked with me.

My greatest debt is always, to God Almighty.

CONTENTS

INTRODUCTION

A power series is a type of series with terms involving a variable. Power series are often used by calculators and computers to evaluate trigonometric, hyperbolic, exponential and logarithm functions. So any application of these kind of functions is a possible application of power series. Many interesting and important differential equations can be found in power series.

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PRELIMINERY

A. An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{1}$$

is called a *power series in x*. The series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

is a *power series in* $x - x_0$.

B. The series (1) is said to *converge* at a point *x* if the limit

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$$

exists, and in this case the *sum* of the series is the value of this limit.

Radius of convergence: Series in x has a radius of convergence R, where $0 \le R \le \infty$, with the property that the series converges if $|x| < R$ and diverges if $|x| > R$. It should be noted that if $R = 0$ then no x satisfies $|x| < R$, and if $R = \infty$ then no x satisfies $|x| > R$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \text{, if the limit exists.}$$

C. Suppose that (1) converges for $|x| < R$ with $R > 0$, and denote its sum by $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Then $f(x)$ is automatically continuous and has derivatives of all orders for $|x| < R$.

D. Let $f(x)$ be a continuous function that has derivatives of all orders for $|x| < R$ with $R > 0$. $f(x)$ be represented as power series using *Taylor's formula*:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x)$$

where the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(\bar{x})}{(n+1)!} x^{n+1}$$

for some point $\bar{x}$ between 0 and x.

E. A function $f(x)$ with the property that a power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is valid in some neighbourhood of the point x_0 is said to be *analytic* at x_0. In this case the a_n are necessarily given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

and is called the *Taylor series* of $f(x)$ at x_0.

Analytic functions: A function f defined on some open subset U of R or C is called analytic if it is locally given by a convergent power series. This means that every a ∈ U has an open neighbourhood V ⊆ U, such that there exists a power series with centre a that converges to f(x) for every x ∈ V.

CHAPTER 1

SERIES SOLUTION OF FIRST ORDER EQUATION

We have studied to solve linear equations with constants coefficient but with variable coefficient only specific cases are discussed. Now we turn to these latter cases and try to find a general method to solve this. The idea is to assume that the unknown function y can be explained into a power series. Our purpose in this section is to explain the procedures by showing how it works in the case of first order equation that are easy to solve by elementary methods.

Example 1: we consider the equation

$$y' = y$$

Consider the above equation as (1). Assume that y has a power series solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Which converges for $|x| < R,\ R > 0$

That is we assume that $y' = y$ has a solution that is analytic at origin. We have

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots \cdots$$

then

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots \ldots \ldots$$

$$\therefore (1) \Rightarrow a_1 + 2a_2 x + 3a_3 x^2 \cdots$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$\Rightarrow a_1 = a_0$

$2a_2 = a_1 \Rightarrow \qquad a_2 = \dfrac{a_1}{2} = \dfrac{a_0}{2}$

$3a_3 = a_2 \Rightarrow \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3} = \frac{a_0}{3!}$

$4a_4 = a_3 \Rightarrow \qquad a_4 = \frac{a_3}{4} = \frac{a_0}{2 \cdot 3 \cdot 4} = \frac{a_0}{4!}$

$\therefore$ we get

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{3!} x^3 + \frac{a_0}{4!} x^4 + \cdots$$

$$= a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right)$$

$$y = a_0 e^x$$

To find the actual function we have $y' = y$

$$\text{i.e., } \frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = dx$$

integrating

$$\log y = x + c$$

$$\text{i.e., } y = e^{x+c} = e^x \cdot e^c$$

$y = a_0 e^x$, where $a_0 = e^c$, a constant.

Example 2: solve $y' = 2xy$. Also find its actual solution.

<u>Solution:</u> $\qquad y' = 2xy \qquad (1)$

Assume that y has a power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Which converges for $|x| < R, R > 0$

We have

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= a_1 + 2a_2x + 3a_3x^2 + \cdots$$

Then (1) $\Rightarrow a_1 + 2a_2x + 3a_3x^2 + \cdots = \ 2x(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)$

$$= 2xa_0 + 2xa_1x + 2xa_2x^2 + 2xa_3x^3 + \cdots$$

$$= 2xa_0 + 2a_1x^2 + 2a_2x^3 + 2a_3x^4 + \cdots \ldots..$$

$\Rightarrow a_1 = 0 \qquad 2a_2 = 2a_0 \Rightarrow a_2 = \frac{2a_0}{z} = a_0$

$$3.a_3 = 2a_1 \Rightarrow a_3 = \frac{2a_1}{3} = 0$$

$$4a_4 = 2a_2 \Rightarrow a_4 = \frac{2a_2}{42} = \frac{a_0}{2}$$

$$5a_5 = 2a_3 = 0 \Rightarrow a_5 = 0$$

$$6a_6 = 2a_4 \Rightarrow a_6 = \frac{2a_4}{6} = \frac{a_4}{3} = \frac{a_0}{2\cdot 3} = \frac{a_0}{3!}$$

We get,

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ &= a_0 + 0 + a_0x^2 + 0x^3 + \frac{a_0}{2}x^4 + \cdots \\ &= a_0 + a_0x^2 + \frac{a_0}{2}x^4 + \cdots \\ &= a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) \end{aligned}$$

$$y = a_0e^{x^2}$$

To find an actual solution

$$y' = 2xy$$

$$\frac{dy}{dx} = 2xy$$

$$\Rightarrow \qquad \frac{dy}{y} = 2x \cdot dx$$

$$\log\, y = x^2 + c$$

$$y = e^{x^2} + c$$

$$\Rightarrow y = a_0e^{x^2}, \text{ where } a_0 = e^c$$

Example 3: Consider $y = (1 + x)^p$ where p is an arbitrary constant. Construct a differential equation from this and then find the solution using power series method.

Solution

First, we construct a differential equation

i.e. $y = (1 + x)^p$

$$y' = p(1 + x)^{p-1} = \frac{p(1+x)^p}{1+x} = \frac{py}{1+x}$$

$$\therefore (1 + x)y' = py, \quad y(0) = r$$

Assume that y has a power series solution of the form,

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + \dot{a}_2 x^2 + \cdots \ldots \ldots$$

Which converges for $|x| < \dot{R}, \quad R > 0$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots \ldots \ldots ..$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Then $(1 + x)y' = py$

$$\Rightarrow (1 + x)a_1 + 2a_2 x + 3a_3 x^2 + \cdots = p(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$\Rightarrow (a_1 + 2a_2 x + 3a_3 x^2 + \cdots) + (a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots)$$

$$= a_0 p + a_1 p x + a_2 p x^2 + \cdots$$

Equating the coefficients of $x, x^2, \ldots$

$$a_1 = a_0 p \text{ i.e. } a_1 = p, (\text{since } a_0 = 1)$$

$$\Rightarrow 2a_2 = a_1(\mathrm{p} - 1)$$

$$a_2 = \frac{a_1(\mathrm{p} - 1)}{2} = \frac{a_0 P(p - 1)}{2}$$

$$3a_3 + 2a_2 = a_2 p$$
$$sa_3 = a_2 p - 2a_2$$
$$= a_2(p-2)$$
$$a_3 = \frac{a_2(p-2)}{3} = \frac{a_0 p(p-1)(p-2)}{2 \cdot 3}$$

$$4a_4 + 3a_3 = a_3 p$$
$$4a_4 = a_3 p - 3a_3$$
$$= a_3(p-3)$$
$$a_4 = \frac{a_3(p-3)}{4} = \frac{a_0 p(p-1)(p-2)(p-3)}{2 \cdot 3 \cdot 4}$$

∴ we get,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= a_0 + a_0 p x + \frac{a_0 p(p-1)}{2} x^2 + \frac{a_0 p(p-1)(p-2)}{2 \cdot 3} x^3 + \cdots \ldots$$
$$= 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 +$$
$$\frac{p(p-1)(p-2)(p-3)}{4!} x^4 + \cdots + \frac{p(p-1)(p-2)(p-(n-1))}{n!} x^n$$

Since the initial problem y(0) = 1 has one solution the series converges for |x|<1

So this is a power solution,

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n$$

Which is binomial series.

Example 4: Solve the equation $y' = x - y,\ y\,(0) = 0$

Solution: Assume that y has a power series solution of the form

$$y = \sum_{n=0}^{\infty} \text{an } x^n$$

which converges for $|x| < R, R > 0$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Now $y' = x - y$

$$(a_1 + 2a_2x + 3a_3x^2 + \cdots) = x - (a_0 + a_1x + a_2x^2 + \cdots)$$

Equating the coefficients of x, x^2,

$$\begin{aligned} a_1 &= -a_0 = 0, \quad \text{Since} \quad y(0) = 0 \\ 2a_2 &= 1 - a_1 \\ &= 1 - 0 \end{aligned}$$

$$\Rightarrow a_2 = \frac{1}{2}$$

$$3a_3 = -a_2$$

$$a_3 = \frac{-a_2}{3} = -\frac{1}{2 \cdot 3}$$

$$4a_4 = -a_3$$

$$\Rightarrow a_4 = \frac{1}{2 \cdot 3 \cdot 4}$$

$$\therefore y = 0 + 0 + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \ldots \ldots$$

$$= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) + x - 1$$

$$= e^{-x} + x - 1$$

By direct method

$$y' = x - y$$

$$\frac{dy}{dx} = x - y \Rightarrow \frac{dy}{dx} + y = x \qquad \left(\frac{dy}{dx} + py = Q\ form\right)$$

here $P(x) = 1$, integrating factor

$$= e^{\int p(x) \cdot dx}$$

$$= e^x$$

$$\therefore ye^x = \int xe^x \cdot dx$$

$$ye^x = x \cdot e^x - \int e^x \cdot dx$$

$$= xe^x - e^x$$

$$ye^x = e^x(x - 1) + c$$

$$y = \frac{e^x(x-1) + c}{dx} = x - 1 + \frac{c}{e^x} = ce^{-x} + (x - 1)$$

$$\therefore y = (x - 1) + ce^{-x}$$

CHAPTER 2

SECOND ORDER LINEAR EQUATION, ORDINARY POINTS

Consider the general homogeneous second order linear equation,

$$y'' + P(x)y' + Q(x)y = 0 \qquad (1)$$

As we know, it is occasionally possible to solve such an equation in terms of familiar elementary functions. This is true, for instance, when $P(x)$ and $Q(x)$ are constants, and in a few other cases as well. For the most part, however, the equations of this type having the greatest significance in both pure and applied mathematics are beyond the reach of elementary methods, and can only be solved by means of power series.

P(x) and Q(x) are called coefficients of the equation. The behaviour of its solutions near a point x_0 depends on the behaviour of its coefficient functions $P(x)$ and $Q(x)$ near this point. we confine ourselves to the case in which $P(x)$ and $Q(x)$ are well behaved in the sense of being analytic at x0, which means that each has a power series expansion valid in some neighbourhood of this point. In this case x0 is called an *ordinary point* of equation (1). Any point that is not an ordinary point of (1) is called a *singular point.*

Consider the equation,

$$y'' + y = 0 \qquad (2)$$

the coefficient functions are $P(x) = 0$ and $Q(x) = 1$, These functions are analytic at all points, so we seek a solution of the form,

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \qquad (3)$$

Differentiating (3) we get,

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots \qquad (4)$$

And

$$y'' = 2a_2 + 2\cdot 3a_3x + 3\cdot 4a_4x^2 + \cdots + (n+1)(n+2)a_{n+2}x^n + \cdots \quad (5)$$

If we substitute (5) and (3) into (2) and add the two series term by term, we get

$$y'' + y = \begin{matrix}(2a_2 + a_0) + (2\cdot 3a_3 + a_1)x + (3\cdot 4a_4 + a_2)x^2 + \\ (4\cdot 5a_5 + a_3)x^3 \ + \cdots + [(n+1)(n+2)a_{n+2} + a_n]x^n + \cdots\end{matrix} = 0$$

and equating to zero the coefficients of successive powers of *x* gives

$$2a_2 + a_0 = 0, \qquad 2\cdot 3a_3 + a_1 = 0, \qquad 3\cdot 4a_4 + a_2 = 0$$

$$4\cdot 5a_5 + a_3 = 0, \dots\dots, \qquad (n+1)(n+2)a_{n+2} + a_n = 0, \dots$$

By means of these equations we can express a_n in terms of a_0 or a_0, according as *n* is even or odd:

$$a_2 = -\frac{a_0}{2}, \qquad a_3 = -\frac{a_1}{2\cdot 3}, \qquad a_4 = -\frac{a_2}{3\cdot 4} = \frac{a_0}{2\cdot 3\cdot 4}$$

$$a_5 = -\frac{a_3}{4\cdot 5} = \frac{a_1}{2\cdot 3\cdot 4\cdot 5}, \cdots$$

With these coefficients, (3) becomes

$$y = a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{2\cdot 3}x^3 + \frac{a_0}{2\cdot 3\cdot 4}x^4 + \frac{a_1}{2\cdot 3\cdot 4\cdot 5}x^5 - \cdots$$

$$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \qquad (6)$$

$i.e, \qquad y = a_0\cos\ x + a_1\sin\ x$

Since each of the series in the parenthesis converges for all *x*. This implies the series (2) for all *x*.

Solve the legenders equation,

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0$$

Solution

Consider $(1 - x^2)y'' - 2xy' + p(p+1)y = 0$ as equation (1)

Assume that y has a power series solution of the form

$$y = \sum a_n x^n$$

Which converges $|x| < R, R > 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

put $n = n + 2$ (Since y'' is not x^n form)

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+2-1) a_{n+2} x^{n+2-2}$$

$$\therefore y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

Now (1) $\Rightarrow$ $\quad y'' - x^2 y'' - 2xy' + p(p+1)y = 0$

$$\Rightarrow \sum (n+1)(n+2) a_{n+2} x^n - \sum n(n-1) a_n x^n - \sum 2n a_n x^n + \sum p(p+1) a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [((n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + p(p+1) a_n) x^n] = 0$$

for n = 0,1,2,3…….

$$\Rightarrow (n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + p(p+1) a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{[n(n-1) + 2n - p(p+1)]}{(n+1)(n+2)} a_n$$

$$= \frac{(n^2 - n + 2n - p^2 - p) a_n}{(n+1)(n+2)}$$

$$= \frac{(n^2 + n - p^2 - p) a_n}{(n+1)(n+2)}$$

$$\therefore a_{n+2} = \frac{-(p-n)(p+n+1)}{(n+1)(n+2)} a_n \,, \qquad n = 0,1,2 \ldots$$

This is an Recursion formula

put $n = 0,\qquad a_2 = \dfrac{-p(p+1)}{1 \cdot 2} a_0$

$n = 1,\qquad a_3 = \dfrac{-(p-1)(p+2)}{2 \cdot 3} \cdot a_1$

$n = 2,\qquad a_4 = \dfrac{-(p-2)(p+3)}{3i4} a_2$

$$= \frac{p(p-2)(p+1)(p+3)}{4!} a_0$$

$n = 3,\qquad a_5 = \dfrac{-(p-3)[p+4)}{4 \cdot 5} a_3$

$$= \frac{(p-1)(p-3)(p+2)(p+4)}{5!} a_1$$

$n = 4,\qquad a_6 = \dfrac{-(p-4)(p+5)}{5 \cdot 6} a_4$

$$= \frac{-p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} a_0$$

$n = 5,\qquad a_7 = -\dfrac{(p-5)(p+6)}{6 \cdot 7} a_5$

$$= -\frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} a_1$$

$$\begin{aligned} y = a_0 \Bigg[& 1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p-2)(p+1)(p+3)}{4!} x^4 \\ & - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} x^6 + \cdots \Bigg] \\ & + a_1 \Bigg[x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!} x^5 \\ & - \frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} x^7 + \cdots \Bigg] \end{aligned}$$

Find the general solution of $(1+x^2)y'' + 2xy' - 2y = 0$ in terms of power series in x. Can you express this solution by means of elementary functions?

Solution

Consider the equation $(1+x^2)y'' + 2xy' - 2y = 0$ as equation (1)

Assume that y has a power series solution of the form

$$y = \sum a_n x^n$$

Which converges $|x| < R, R > 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1+x^2)y'' = y'' + x^2 y''$$

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$\text{Now } y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

put $n = n + 2$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+2-1)a_n + 2x^{n+2=2}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n$$

$$(1) \Rightarrow \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x_n$$
$$+ \sum_{n=1}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\Rightarrow \sum[((n+1)(n+2)a_{n+2} + n(n-1)a_n + 2na_n - 2a_n)x^n] = 0$$

$$\Rightarrow (n+1)(n+2)a_{n+2} + n(n-1)a_n + 2na_n - 2a_n = 0$$

$$a_{n+2} = \frac{[-n(n-1) - 2n + 2]}{(n+1)(n+2)} a_n$$
$$= \frac{(-n^2 + n - 2n + 2)}{(n+1)(n+2)} a_n$$

put $n = 0$, $\quad a_2 = \frac{2}{1 \cdot 2} a_0 = \frac{2a_0}{2!} = a_0$

$n = 1$, $\quad a_3 = \frac{(1 - 1 - 2 + 2)}{2 \cdot 3} a_1 = 0$

$n = 2$, $\quad a_4 = \frac{2 - 4 - 4 + 2}{3 \cdot 4} a_2 \quad = \frac{-4}{3 \cdot 4} a_0 = \frac{-a_0}{3}$

$n = 3$, $\quad a_5 = \frac{3 - 9 - 16 + 2}{4.5} a_3 \quad = 0$

$n = 4$, $\quad a_6 = \frac{4 - 16 - 8 + 2}{5.6} a_4 \quad = \frac{-3}{5} a_4 \quad = \frac{3a_0}{3.5} = \frac{a_0}{5}$

$$\therefore y = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$= a_0 + a_1 x + a_0 x^2 - \frac{a_0}{3} x^4 + \frac{a_0}{5} x^6 \ldots.$$
$$= a_0 \left[1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \cdots \right] + a_1 x$$
$$= a_0 \left[1 + x\left(x - \frac{x^3}{3} + \frac{x^5}{5} \cdots \right)\right] + a_1 x$$
$$= a_0(1 + x\tan^{-1} x) + a_1 x$$

Consider the equation $y'' + xy' + y = 0$

(a) Find its general solution $y = \sum a_n x^n$ in the form $y = a_0 y_1(x) + a_1 y_2(x)$ where $y_1(x)$ and $y_2(x)$ are power series

(b) use the ratio test to verify that the two series $y_1(x)$ and $y_2(x)$ converges for all x.

Solution:

Given $y'' + xy' + y = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$

Assume that y has a power series solution the form $\sum \mathrm{a}_n x^n$ which converges for $|x|$ $R > 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$y' = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$
$$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n$$

$$(1) \Rightarrow \sum_{n=0}^{\infty} (n+1)(n+2)\mathrm{a}_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum \left[\left((n+1)(n+2) a_{n+2} + n a_n + a_n \right) x^n \right] = 0$$

$$\Rightarrow (n+1)(n+2) a_{n+2} + n a_n + a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{(-n-1) a_n}{(n+1)(n+2)} = \frac{-a_n}{n+2}$$

put $n = 0, a_2 = -\dfrac{a_0}{2}$

$n = 1, a_3 = \dfrac{-2a_1}{2 \cdot 3} = \dfrac{-a_1}{3}$

$$n = 2, \quad a_4 = \frac{-3a_2}{3 \cdot 4} = \frac{-a_2}{4} = \frac{a_0}{8}$$
$$n = 3, \quad a_5 = \frac{-4a_3}{4 \cdot 5} = \frac{a_1}{15}$$
$$n = 4, \quad a_6 = \frac{-5a_4}{5 \cdot 6} = \frac{-a_0}{48}$$

$\therefore$ we get $\quad y = a_0 + a_1 x + -\frac{a_0}{2}x^2 - \frac{a_1}{3}x^3 + \frac{a_0}{8}x^4 + \frac{a_1}{15}x^5 - \frac{a_0}{48}x^6 + \cdots$

$$= a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots\right] + a_1\left[x - \frac{x^3}{3} + \frac{x^5}{3.5} + \cdots\right]$$

where $\quad y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{\dot{x}^2}{2 \cdot 4 \cdot 6} +$

$$y_2(x) = x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \cdots$$

(b) $\quad y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$
$$= \lim_{n\to\infty} \left|\frac{(-1)^n}{2 \cdot 4 \cdot (2n)} \Big/ \frac{(-1)^{n+1}}{2 \cdot 4 \cdot\cdot (2n+2)}\right|$$
$$= \lim_{n\to\infty} \left|\frac{2(n+1)}{-1}\right|$$
$$= \lim_{n\to\infty} |-2n(1 + \frac{1}{n})| = \infty$$

$\therefore y_1(x)$ converges for all x

$$y_2(x) = x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} - \cdots$$

$$R = \lim_{n\to\infty} \left| \frac{(-1)^n}{3\cdot 5\cdots(2n+1)} \middle/ \frac{(-1)^{n+1}}{3\cdot 5\cdot\cdots(2n+3)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)\cdot 3\cdot 5\cdots(2n+1)(2n+3)}{3\cdot 5\cdot\cdots\cdot(2n+1)} \right|$$

$$= \lim_{n\to\infty} |(-1)n(2+3/n)| = \infty$$

$\therefore\ y_2(x)$ converges for all x

REGULAR SINGULAR POINTS

A singular point x_0 of equation

$$y'' + P(x)y' + Q(x)y = 0$$

is said to be regular if the functions $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic, and irregular otherwise. Roughly speaking, this means that the singularity in $P(x)$ cannot be worse than $1/(x-x_0)$, and that in $Q(x)$ cannot be worse than $1/(x-x_0)^2$.

If we consider Legendre's equation in the form

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$$

it is clear that $x = 1$ and $x = -1$ are singular points. The first is regular because

$$(x-1)P(x) = \frac{2x}{x+1} \text{ and } (x-1)^2Q(x) = -\frac{(x-1)p(p+1)}{x+1}$$

are analytic at $x = 1$, and the second is also regular for similar reasons.

Example: *Bessel's equation of order p*, where p is a nonnegative constant:

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

If this is written in the form

$$y'' + \frac{1}{x}y' + \frac{x^2-p^2}{x^2}y = 0,$$

it is apparent that the origin is a regular singular point because $xP(x) = 1$ and $x^2Q(x) = x^2 - p^2$ are analytic at $x = 0$.

CONCLUSION

The purpose of this project gives a simple account of series solution of first order equation, second order linear equation, ordinary points. The study of these topics given excellent introduction to the subject called ‘POWER SERIES’

we used application of power series extensively throughout this project. We take it for granted that most readers are reasonably well acquainted with these series from an earlier course in calculus. Nevertheless, for the benefit of those whose familiarity with this topic may have faded slightly, we presented a brief review of the main facts of power series.

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NORMED LINEAR SPACES

Project report submitted to
The Kannur University
for the award of the degree
of

Bachelor of Science

by

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DB18CMSR22

Under the guidance of

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CERTIFICATE

It is to certify that this project report **'Normed Linear Spaces'** is the bonafide project of **Delna Jomy** carried out the project work under my supervision.

Mrs. Riya Baby
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DECLARATION

I **Delna Jomy** hereby declare that the project **'Normed Linear Space'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Athulya P , Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

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CONTENTS

INTRODUCTION

This chapter gives an introduction to the theory of normed linear spaces. A skeptical reader may wonder why this topic in pure mathematics is useful in applied mathematics. The reason is quite simple: Many problems of applied mathematics can be formulated as a search for a certain function, such as the function that solves a given differential equation. Usually the function sought must belong to a definite family of acceptable functions that share some useful properties. For example, perhaps it must possess two continuous derivatives. The families that arise naturally in formulating problems are often linear spaces. This means that any linear combination of functions in the family will be another member of the family. It is common, in addition, that there is an appropriate means of measuring the “distance” between two functions in the family. This concept comes into play when the exact solution to a problem is inaccessible, while approximate solutions can be computed. We often measure how far apart the exact and approximate solutions are by using a norm. In this process we are led to a normed linear space, presumably one appropriate to the problem at hand. Some normed linear spaces occur over and over again in applied mathematics, and these, at least, should be familiar to the practitioner. Examples are the space of continuous functions on a given domain and the space of functions whose squares have a finite integral on a given domain.

PRELIMINARIES

1) LINEAR SPACES

We introduce an algebraic structure on a set X and study functions on X which are well behaved with respect to this structure. From now onwards , $\boldsymbol{K}$ will denote either $\boldsymbol{R}$, the set of all real numbers or $\boldsymbol{C}$, the set of all complex numbers. For $k \in \boldsymbol{C}$, *Re k* and *Im k* will denote the real and imaginary part of k.

A linear space(or a vector space) over $\boldsymbol{K}$ is a non-empty set X along with a function $+ : X \times X \to X$, called addition and a function $\cdot: \boldsymbol{K} \times X \to X$ called scalar multiplication, such that for all $x, y, z \in X$ and $k, l \in \boldsymbol{K}$, we have

$$x + y = y + x$$

$$x + (y + z) = (x + y) + z$$

$$\exists 0 \in X \text{ such that } x + 0 = x,$$

$$\exists - x \in X \text{ such that } x + (-x) = 0,$$

$$k \cdot (x + y) = k \cdot x + k \cdot y,$$

$$(k + l) \cdot x = k \cdot x + l \cdot x,$$

$$(kl) \cdot x = k \cdot (l \cdot x),$$

$$1 \cdot x = x.$$

We shall write kx in place of $k \cdot x$. We shall also adopt the following notations. For $x, y \in X, k \in \boldsymbol{K}$ and subsets E, F of X ,

$$x + F = \{x + y : y \in F\},$$

$$E + F = \{x + y : x \in E, y \in F\},`$$

$$kE = \{kx : x \in E\}.$$

2) BASIS

A nonempty subset E of X is said to be a subspace of X if $kx + ly \in E$ whenever $x, y \in E$ and $k, l \in \boldsymbol{K}$. If $\emptyset \neq E \subset X$, then the smallest subspace of X containing E is

$$spanE = \{k_1x_1 + \cdots + k_nx_n : x_1, ..., x_n \in E,\ k_1, ..., k_n \in K\}$$

It is called the span of E. If span $E = X$, we say that E spans X. A subset E of X is said to be linearly independent if for all $x_1, ..., x_n \in E$ and $k_1, ..., k_n \in \boldsymbol{K}$, the equation $k_1x_1 + \cdots + k_nx_n = 0$implies that $k_1 = \cdots = k_n = 0$. It is called linearly dependent if it is not linearly independent, that is, if there exist $x_1, ..., x_n \in E$ and $k_1, ..., k_n \in \boldsymbol{K}$ such that $k_1x_1 + \cdots + k_nx_n = 0$, where at least one $k_j$is nonzero.

A subset E of X is called a Hamel basis or simply basis for X if *span of* $E = X$ and E is linearly independent .

3) DIMENSION

If a linear space X has a basis consisting of a finite number of elements , then X is called finite dimensional and the number of elements in a basis for X is called the dimension of X, denoted as $dimX$. Every basis for a finite dimensional linear space has the same (finite) number of elements and hence the dimension is well-defined. The space $\{0\}$ is said to have zero dimension. Note that it has no basis !

If a linear space contains an infinite linearly independent subset, then it is said to be infinite dimensional.

4)METRIC SPACE

We introduce a distance structure on a set X and study functions on X which are well-behaved with respect to this structure.

A metric d on a nonempty set X is a function $d: X \times X \to \boldsymbol{R}$
such that for all $x, y, z \in X$

$$d(x, y) \geq 0 \text{ and } d(x , y) = 0 \text{ iff } x=y$$

$$d(y , x) = d(x , y)$$

$$d(x , y) \leq d(x , z) + d(z , y) .$$

The last condition is known as the triangle inequality. A metric space is a nonempty set X along with a metric on it.

5)CONTINUOUS FUNCTIONS

Roughly speaking, a function from a metric space to a metric space is continuous if it sends 'nearby' points to 'nearby' points. If X and Y are metric spaces with metrics d and e respectively, then a function $F: X \to Y$ is said to be continuous at $x_0 \in X$ if for every $\epsilon \rangle 0$, there is some $\delta > 0$ (possibly depending on ϵ and x_0) such that $e\left(F(x), F\left(x_0\right)\right) < \epsilon$ for all $x \in X$ satisfying $d\left(x, x_0\right) < \delta$. Further, F is said to be continuous on X if it is continuous at every point of X. It is easy to see that F is continuous on X if and only if the set $F^{-1}(E)$ is open in X whenever the set E is open in$Y$. Also , this happens iff $F(x_n) \to F(x)$ in Y whenever $x_n \to x$ in X.

6) UNIFORM CONTINUITY

We note that a continuous function $F: T \to S$ is, in fact, uniformly continuous, that is, for every $\epsilon > 0$, there exists some $\delta > 0$ such that $e(F(t), F(u)) < \epsilon$ whenever $d\left(t, u\ \right) < \delta$. This can be seen as follows. Let $t \in T$. By the continuity of F at $t \in T$, there is some δ_t, such that $e(F(t), F(u)) < \frac{\varepsilon}{2}$whenever $d(t, u) < \delta_t$.

7) FIELD

A ring is a set R together with two binary operations + and ·(which we call addition and multiplication) such that the following axioms are satisfied .

- R is an abelian group with respect to addition
- Multiplication is associative
- $\forall a , b, c \in R$the left distributive law $a(b + c) = (a \cdot b) + (a \cdot c)$ and the right distributive law $(a + b)c = (a \cdot c) + (b \cdot c)$, hold .

A field is a commutative division ring

CHAPTER 1

NORMED LINEAR SPACE

Let X be a linear space over $\boldsymbol{K}$. A norm on X is the function || || from X to $\boldsymbol{R}$ such that ∀ $x,y \in X$ *and* $k \in \boldsymbol{K}$,

$$||x|| \geq 0 \text{ and } ||x|| = 0 \text{ if and only if } x = 0 ,$$

$$||x + y|| \leq ||x|| + ||y|| ,$$

$$||kx|| = |k| \, ||x|| .$$

A norm is the formalization and generalization to real vector spaces of the intuitive notion of “ length” in the real world .

A normed space is a linear space with norm on it .

For x and y in X , let

$$d(x,y) = ||x - y||$$

Then d is a metric on X so that (X,d) is a metric space , thus every normed space is a metric space

- <u>*Every normed linear space is a metric space . But converse may not be true .*</u>

Example :

$$d(x,y) = \frac{|x-y|}{1+|x-y|} , \forall x , y \in X$$

$$\Rightarrow ||x - y|| = \frac{|x-y|}{1+|x-y|}$$

$$\Rightarrow ||z|| = \frac{|z|}{1+|z|} , z = x - y \in X$$

$$||\alpha z|| = \frac{|\alpha z|}{1 + |\alpha z|}$$

$$= \frac{|\alpha|\,|z|}{1 + |\alpha|\,|z|}$$

$$= |\alpha| \left(\frac{|z|}{1 + |\alpha|\,|z|} \right)$$

$$\neq |\alpha|\,||z|| .$$

➢ *Result*

Let X be a normed linear space . Then ,

$$\big|\, ||x|| - ||y|| \,\big| \leq ||x - y|| ,\ \forall x , y \in X$$

Proof :

$$||x|| = || (x - y) + y|| \leq ||x - y|| + ||y||$$

$$\Rightarrow ||x|| - ||y|| \leq ||x - y|| \rightarrow (1)$$

$x \leftrightarrow y$

$$||y|| - ||x|| \leq ||y - x||$$

$$\Rightarrow - (||x|| - ||y||) \leq ||x - y|| \rightarrow (2)$$

From (1) and (2)

$$\big|\,||x|| - ||y||\,\big| \leq ||x - y||$$

➢ *Norm is a continuous function*

Let $x_n \rightarrow x$, as $n \rightarrow \infty$

$\Rightarrow x_n - x \to 0$ *as* $n \to \infty$

$\Rightarrow ||x_n - x|| \to 0$ *as* $n \to \infty$

$|\ ||x_n|| - ||x||\ | \leq ||x_n - x|| \to 0$ *as* $n \to \infty$

$\Rightarrow ||x_n|| - ||x|| \to 0$, *as* $n \to \infty$

$\Rightarrow ||x||$ is continuous

- *Norm is a uniformly continuous function*

We have , $||\ || : X \to \boldsymbol{R}$. Let $x,y \in X$ and $\varepsilon > 0$

Then $\quad ||x|| = ||x - y + y||$

$\leq ||x - y|| + ||y||$

$\Rightarrow ||x|| - ||y|| \leq ||x - y|| \to (1)$

Interchanging x and y ,

$||y|| - ||x|| \leq ||y - x||$

$\Rightarrow -(\ ||x|| - ||y||\ ) \leq ||x - y||$

$\Rightarrow \quad ||x|| - ||y|| \geq -||x - y|| \to (2)$

Combining (1) and (2)

$-||x - y|| \leq ||x|| - ||y|| \leq ||x - y||$

That is ,

$|\ ||x|| - ||y||\ | \leq ||x - y||$

Take $\delta = \varepsilon$, then whenever $||x - y|| < \delta$, $|\ ||x|| - ||y||\ | < \varepsilon$

Therefore $|| \ ||$ is a uniformly continuous function .

➢ <u>*Continuity of addition and scalar multiplication*</u>

To show that $+ : X\times X \to X$ and $\cdot : \boldsymbol{K} \times X \to X$ are continuous functions.

Let $(x,y) \in X\times X$. To show that + is continuous at (x ,y) , that is , to show that for each $(x,y) \in X\times X$ *if* $x_n \to x$ and $y_n \to y$ in X , then

$$+(x_n , y_n) \to +(x , y) ;$$

That is ,

$$x_n + y_n \to x + y .$$

Consider

$$||(x_n + y_n) - (x + y)|| = ||x_n - x + y_n - y||$$

$$\leq ||x_n - x|| + ||y_n - y||$$

Given $x_n \to x$ *and* $y_n \to y$ *, for each* $\epsilon > 0$, $\exists N_1 \ni$

$$||x_n - x|| < \frac{\varepsilon}{2} \forall n \geq N_1 , \quad and \ \exists N_2 \ni$$

$$||y_n - y|| < \frac{\varepsilon}{2} \quad \forall n \geq N_2$$

\

Take $N = max \{ N_1, N_2\}$

Then $||x_n - x|| < \frac{\varepsilon}{2}$ and $||y_n - y|| < \frac{\varepsilon}{2} \forall n \geq N$

Therefore $||(x_n + y_n) - (x + y)|| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \forall n \geq N$

That is , $x_n + y_n \to x + y$

Now to show that $\cdot : \boldsymbol{K} \times X \to X$ is continuous

Let $(k , x) \in \boldsymbol{K}\times X$

To show that if $k_n \to k$ and $x_n \to x$, then $k_n x_n \to kx$

Since $k_n \to k$, $\forall\, \varepsilon > 0 \;\exists\; N_1 \ni |k_n - k| < \frac{\varepsilon}{2} \quad \forall\, n \geq N_1$

Since $x_n \to x$, $\forall\, \varepsilon > 0 \;\exists\; N_2 \ni ||x_n - x|| < \frac{\varepsilon}{2} \quad \forall\, n \geq N_2$

Consider $||k_n x_n - kx|| = ||k_n x_n - kx + x_n k - x_n k ||$

$$= ||x_n (k_n - k) + k(x_n - x)||$$

$$\leq ||x_n(k_n - k)|| + ||k(x_n - x)||$$

$$= ||x_n|| \, |k_n - k| + |k| \, ||x_n - x||$$

$$\leq ||x_n||\frac{\varepsilon}{2} + |k|\frac{\varepsilon}{2}$$

$\therefore \quad k_n x_n \to kx$

➢ <u>*Examples of normed space*</u>

1) Spaces $\boldsymbol{K}^n$ ($\boldsymbol{K} = \boldsymbol{R}$ *or* $\boldsymbol{C}$)

For $n = 1$, the absolute value of function | | is a norm on $\boldsymbol{K}$, since $\forall k \in \boldsymbol{K}$

We have ,

$||k|| = ||k \cdot 1|| = |k| \, ||1||$, by definition .

But $||1||$ is a positive scalar .

$\therefore$ $||k||$ is a positive scalar multiple of the absolute value function .

$\therefore$ any norm on $\boldsymbol{K}$ is a positive scalar multiple of the absolute value function

For $n > 1$, let $p \geq 1$ be a real number

$$K^n = \{ (x(1) , x(2) , \ldots , x(n)) : x(i) \in K , i = 1 , 2 , \ldots , n \}$$

For $x \in K^n$, that is , $x = (x(1) , x(2) , \ldots , x(n))$, define

$$||x||_p = (|x(1)|^p + \ldots + |x(n)|^p)^{1/p}$$

Then $|| \, ||_p$ is a norm on K^n

When $p = 1$,

Then , $||x||_1 = |x(1)| + |x(2)| + \ldots + |x(n)|$

Since $|x(i)| \geq 0 \;\; \forall \, i = 1 , 2 , \ldots , n$, $||x||_1 \geq 0$

And $||x||_1 = 0 \Leftrightarrow |x(1)| + \ldots + |x(n)| = 0$

$\Leftrightarrow |x(i)| = 0 \;\; \forall \, i$

$\Leftrightarrow x(i) = 0 \; \forall \, i$

$\Leftrightarrow x = (x(1) , \ldots , x(n)) = 0$

Now $||kx||_1 = |kx(1)| + |kx(2)| + \ldots + |kx(n)|$

$= |k| \, |x(1)| + \ldots + |k| \, |x(n)|$

$= |k| \, (|x(1)| + \ldots + |x(n)|)$

$= |k| \, ||x||_1$

$||x + y||_1 = |(x + y)(1)| + \ldots + |(x + y)(n)|$

$= |x(1) + y(1)| + \ldots + |x(n) + y(n)|$

$\leq |x(1)| + |y(1)| + \ldots + |x(n)| + |y(n)|$

$= |x(1)| + \ldots + |x(n)| + |y(1)| + \ldots + |y(n)|$

$= ||x||_1 + ||y||_1$

Consider $1<p<\infty$

Now , $||x||_p = (|x(1)|^p + . . . + |x(n)|^p)^{1/p}$

Since $|x(i)|^p \geq 0 \ \forall i$, we have $||x||_p \geq 0$

And $||x||_p = 0 \Leftrightarrow (|x(1)|^p + . . . + |x(n)|^p)^{1/p} = 0$

$\Leftrightarrow |x(i)|^p = 0 \ \forall i$

$\Leftrightarrow |x(i)| = 0 \ \forall i$

$\Leftrightarrow x(i) = 0 \ \forall i$

$\Leftrightarrow x = (x(1) , . . . , x(n)) = 0$.

Now

$$||kx||_p = (|kx(1)|^p + . . . + |kx(n)|^p)^{1/p}$$

$$= (|k|^p |x(1)|^p + . . . + |k|^p |x(n)|^p)^{1/p}$$

$$= |k| (|x(1)|^p + . . . + |x(n)|^p)^{1/p}$$

$$= |k| \, ||x||_p .$$

$$||x + y||_p = (|x(1) + y(1)|^p + . . . + |x(n) + y(n)|^p)^{1/p}$$

We have by Minkowski's inequality ,

$$\left(\sum_{i=1}^{n} |x(i) + y(i)|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y(i)|^p\right)^{1/p}$$

Then

$$||x + y||_p \leq \left(|x(1)|^p + \ldots + |x(n)|^p \right)^{1/p} + \left(|y(1)|^p + \ldots + |y(n)|^p \right)^{1/p}$$

$$= ||x||_p + ||y||_p$$

Then , for $1 \leq p < \infty$, $|| \, ||_p$ is a norm on $\boldsymbol{K^n}$

When $p = \infty$, define $||x||_\infty = max \{ |x(1)| , |x(2)| , \ldots , |x(n)| \}$

Then it is a norm on $\boldsymbol{K^n}$

$||x||_p \geq 0$ since each values $|x(i)| \geq 0$

So that

$$max \{|x(i)| , i=1, \ldots , n\} \geq 0$$

$$||x||_\infty = 0 \Leftrightarrow max \{ |x(i)| : i = 1, \ldots , n \} = 0$$

$$\Leftrightarrow |x(i)| = 0 \ \ \forall i$$

$$\Leftrightarrow x(i) = 0 , \forall i$$

$$\Leftrightarrow x = 0$$

$$||kx||_\infty = max \{ |kx(1)| , \ldots , |kx(n)| \}$$

$$= max \{ |k| \, |x(1)| , \ldots , |k| \, |x(n)| \}$$

$$= |k| \, max \{|x(1)| , \ldots , |x(n)|\}$$

$$= |k| \, ||x||_\infty$$

$$||x + y||_\infty = max \{ |x(1) + y(1)| , \ldots , |x(n) + y(n)| \}$$

$$\leq max \{ |x(1)| + |y(1)| , \ldots , |x(n)| + |y(n)| \}$$

$\leq max \{ |x(1)| , \ldots , |x(n)| \} + max \{ |y(1)| , \ldots , |y(n)| \}$

$= ||x||_\infty + ||y||_\infty$

2) Sequence space

Let $1 \leq p < \infty$, $l^p = \{ x = (x(1) , x(2) , \ldots) ; \ x(i) \in \boldsymbol{K} \ and \sum_{j=1}^{\infty} |x(j)|^p < \infty \}$, that is , l^p is the space of p-summable scalar sequences in $\boldsymbol{K}$. For $x = (x(1) , x(2) , \ldots) \in l^p$,

let $||x||_p = (|x(1)|^p + |x(2)|^p + \ldots)^{1/p}$. Then it is a norm on l^p.

That is , $|| \ ||_p$ is a function from l^p to $\boldsymbol{R}$.

If $p = 1$, then l^1 is a linear space and $||x||_1 = (|x(1)| + |x(2)| + \ldots)$ is a norm on l^1

Let $p = \infty$. Then l^∞ is the linear space of all bounded scalar sequences . And ,

$$||x||_\infty = sup \{ |x(j)| : j = 1, 2, 3, \ldots \}$$

Then $|| \ ||_\infty$ is a norm on l^∞

CHAPTER 2

THEOREMS ON NORMED SPACES

a) Let Y be a subspace of a normed space X, then Y and its closure $\overline{Y}$ are normed spaces with the induced norm.

b) Let Y be a closed subspace of a normed space X, for $x+Y$ in the quotient space X/Y, let $|||x+Y||| = inf\{\,||x+y|| : y \in Y\}$. Then $|||\ \ |||$ is a norm on X/Y, called the quotient norm.

A sequence $(x_n + Y)$ converges to $x + Y$ in X/Y iff there is a sequence (y_n) in Y, $(x_n + y_n)$ converges to x in X.

c) Let $||\ \ ||_p$ be a norm on the linear space X_p, $j = 1,2,\ldots$. Fix p such that $1 \leq p \leq \infty$

For $x = (x(1), x(2), \ldots, x(m))$ that is the product space $X = X_1 \times X_2 \times \ldots \times X_m$,

Let $||x||_p = \left(||x(1)||_1^p + ||x(2)||_2^p + \ldots + ||x(m)||_m^p\right)^{1/p}$, if $1 \leq p < \infty$

$||x||_p = max\{\,||x(1)||_1, \ldots, ||x(m)||_m\}$, if $p = \infty$.

Then $||\ \ ||_p$ is a norm on X.

A sequence (x_n) converges to x in X $\Leftrightarrow$ $(x_n(j))$ converges to $x(j)$ in X_j $\forall\, j=1,2,\ldots,m$.

Proof:

a) Since X is a normed space, there is a norm on X to Y. Since Y is a subspace of X,

$||\ \ ||_y : Y \to \boldsymbol{R}$ is a function. To show that $||\ \ ||_y$ is a norm on Y.

For $y \in Y$, $||y||_Y = ||y||$, then

$$||y||_Y \geq 0 \quad (\because ||\boldsymbol{y}|| \geq 0) \quad and \quad ||y||_Y = 0 \Leftrightarrow y = 0$$

$$||ky||_Y = ||ky|| = |k|\,||y|| = |k|\,||y||_y .$$

Let $y_1, y_2 \in Y$. Then ,

$$||y_1 + y_2||_y = ||y_1 + y_2|| \leq ||y_1|| + ||y_2|| = ||y_1||_y + ||y_2||_y$$

Now the continuity of addition and scalar multiplication shows that $\overline{Y}$ is a subspace of X, since if $x_n \to x$ and $y_n \to y$, $x_n, y_n \in \overline{Y}$, then

$x_n + y_n \to x + y$ (by continuity of addition) and

$kx_n \to kx$ (by continuity of scalar X^n) .

Since $\overline{Y}$ is closed , $x + y \in \overline{Y}$ and $kx \in \overline{Y}$. Therefore $\overline{Y} \leq X$.

$\therefore$ norm on X induces a norm on Y and $\overline{Y}$

b) X/Y , the quotient space equals $X/Y=\{ x + Y : x \in X \}$.

$$|||x + y||| = inf\{ ||x + y|| : y \in Y \}$$

Claim: $|||\ \ |||$ is a norm on X/Y , called quotient norm

- Let $x \in X$,

$$|||x + Y||| = inf\{ ||x + y|| : y \in Y \} \geq 0.$$

$\therefore$ $|||x + Y||| \geq 0$.

If $|||x + y||| = 0$ (0 in X/Y is Y) , then there is a sequence (y_n) in Y $\ni$

$$||x + y_n || \to 0$$

$$\Rightarrow \qquad x + y_n \to 0$$

$$\Rightarrow \qquad y_n \to -x$$

Since $y_n \in Y$ and Y is closed

$-x \in Y \Leftrightarrow x \in Y$ ($\because$ Y is a subspace)

$\Leftrightarrow x + Y = Y$, zero in X/Y.

- For $k \in \boldsymbol{K}$,

$$\begin{aligned} |||k(x + Y)||| &= |||kx + Y||| \\ &= inf\{ ||k(x + y)|| : y \in Y\} \\ &= inf\{ |k| \ ||x + y|| : y \in Y\} \\ &= |k| \ inf\{ ||x + y|| : y \in Y\} \\ &= |k| \ |||x + Y||| . \end{aligned}$$

- Let x_1 , $x_2 \in X$. Then

$|||x_1 + Y||| = inf\{ ||x_1 + y|| : y \in Y \}$. Then $\exists$ $y_1 \in Y \ni$

$$|||x_1 + Y||| + \frac{\varepsilon}{2} > ||x_1 + y_1|| , \ and$$

$|||x_2 + Y||| = inf\{ ||x_2 + y|| : y \in Y\}$, *Then* $\exists y_2 \in Y \ni$

$$|||x_2 + Y||| + \frac{\varepsilon}{2} > ||x_2 + y_2|| .$$

$$||x_1 + y_1 + x_2 + y_2|| \leq ||x_1 + y_1|| + ||x_2 + y_2||$$

$$\leq |||x_1 + Y||| + \frac{\varepsilon}{2} + |||x_2 + Y||| + \frac{\varepsilon}{2}$$

Let $y = y_1 + y_2 \in Y$. Then ,

$$||(x_1+x_2) + y|| \leq |||x_1 + Y||| + |||x_2 + Y||| + \varepsilon \quad \text{—(1)}$$

Now , $|||(x_1 + Y) + (x_2 + Y)||| = |||x_1 + x_2 + Y|||$

$$=inf\{ ||x_1 + x_2 + y|| : y \in Y \}$$

$$< ||x_1 + x_2 + y||$$

$$\leq |||x_1 + Y||| + |||x_2 + Y||| + \varepsilon \qquad \text{(by (1))}$$

since ε is arbitrary , we have

$$|||(x_1 + Y) + (x_2 + Y)||| \leq |||x_1 + Y||| + |||x_2 + Y|||$$

$\therefore$ ||| ||| is a norm on X/Y.

Let $(x_n + Y)$ be a sequence in X/Y . Assume that (y_n) is a sequence in $Y \ni (x_n + y_n)$ converges to x in X.

That is , $(x_n - x + y_n)$ converges to 0 . —(1)

Claim: $(x_n + Y)$ converges to $x + Y$.

Consider

$$|||x_n + Y - (x+Y)||| = |||(x_n - x) + Y|||$$

$$= inf\{ ||x_n - x + y_n|| : y \in Y \}$$

$$\leq ||x_n - x + y_n|| \quad \forall y_n \in Y .$$

Then by (1) , $x_n + Y$ converges to $x + Y$ in X/Y.

Conversely assume that the sequence $(x_n + Y) \rightarrow x + Y$ in X/Y.

Consider $|||x_n + Y - (x + Y)||| = |||x_n - x + Y|||$

$$= inf\{ ||x_n -x + y|| : y \in Y \}$$

Then we can choose $y_n \in Y \ni$

$$||x_n - x + y_n|| < |||(x_n - x) + Y||| + \frac{1}{n} \ , \ n=1,2,3,\ldots.$$

Since $x_n+Y \to x+Y$, we get

$(x_n - x + y_n)$ converges to zero as $n \to\infty$

That is , $(x_n + y_n)$ converges to x in X as $n \to\infty$

c) Consider $1\leq p < \infty$

Given that

$$||x||_p = (||x(1)||_1^p + ||x(2)||_2^p +. . . + ||x(m)||_m^p)^{1/p}$$

Clearly , $||x||_p \geq 0$.

Since each $||x(i)||_i^p \geq 0$.

$$||x||_p = 0 \Leftrightarrow |x(j)|_j^p = 0 \quad \forall\ j = 1, . . . ,m$$

$$\Leftrightarrow x(j) = 0 \qquad \forall\ j.$$

$$\Leftrightarrow x = (x(1), . . . ,x(m)) = 0$$

$$||kx||_p = \left(||kx(1)||_1^p +. . .+ ||kx(m)||_m^p \right)^{1/p}$$

$$= \left(|k|^p||x(1)||_1^p +. . .+ |k|^p||x(m)||_m^p \right)^{1/p}$$

$$= |k| \left(||x(1)||_1^p +. . .+ ||x(m)||_m^p \right)^{1/p}$$

$$= |k|\ ||x||_p \quad , k\in \boldsymbol{K} \text{ and } x\in X$$

Now, $||x + y||_p = \left(||x(1) + y(1)||_1^p +. . .+ ||x(m) + y(m)||_m^p \right)^{1/p}$

(by Minkowski's inequality)

$$\leq$$

$$\left(\left(||x(1)||_1 + ||y(1)||_1\right)^p +. . . + \left(||x(m)||_m + ||y(m)||_m\right)^p \right)^{1/p}$$

$$\leq \left(\sum_{j=1}^{m} ||x(j)||_j^p\right)^{1/p} + \left(\sum_{j=1}^{m} ||y(j)||_j^p\right)^{1/p} \qquad \text{(Minkowski's inequality)}$$

$$= \left(||x(1)||_1^p + . . . + ||x(m)||_m^p \right)^{1/p}$$

$$= ||x||_p + ||y||_p$$

Now suppose $p = \infty$

$$||x||_\infty = max \{ ||x(1)||_1 , . . . , ||x(m)||_m \}$$

$||x||_\infty \geq 0$ Since $||x(j)|| \geq 0$, $\forall\ j$

$$||x||_\infty = 0 \quad \Leftrightarrow \ ||x(m)|| = 0 \quad \forall\ m$$

$$\Leftrightarrow \ x(m) = 0 \quad \forall\ m$$

$$\Leftrightarrow x = 0$$

$$||kx||_\infty = max \{ ||kx(1)||_1 , . . . , ||kx(m)||_m \}$$

$$= |k|\ max \{ ||x(1)||_1 , . . . , ||x(m)||_m \}$$

$$= |k|\ ||x||_\infty$$

$$||x + y||_\infty = max \{ ||x(1) + y(1)||_1, . . . , ||x(m) + y(m)||_m \}$$

$$\leq max \{ ||x(1)||_1 + ||y(1)||_1 , . . . , ||x(m)||_m + ||y(m)||_m \}$$

$$= max \{ ||x(1)||_1 , . . . , ||x(m)||_m \} + max \{ ||y(1)||_1 , . . . , ||y(m)||_m \}$$

$$= ||x||_\infty + ||y||_\infty$$

We now consider ,

$$||x_n - x(1)||_p = \left(||x_n(1) - x(1)\right)||_1^p + . . . + ||x_n(m) - x(m)||_m^p)^{1/p}$$

Then

$$x_n \to x \ in \ X \quad \Leftrightarrow \ ||x_n - x ||_p \to 0$$

$$\Leftrightarrow \ ||x_n(j) - x(j)||_j^p \to 0$$

$$\Leftrightarrow \ x_n(j) - x(j) \to 0$$

$$\Leftrightarrow \ x_n(j) \to x(j) \text{ in } X\ \forall j\ .$$

RIESZ LEMMA

Let X be a normed space . Y be a closed subspace of X and $X \neq Y$. Let r be a real number such that $0 < r < 1$. Then there exist some $x_r \in X$ such that $||x_r|| = 1$ and

$r < dist\,(x_r\,,\,Y) \leq 1$

Proof :

We have ,

$$dist\,(x\,,\,Y) = inf\,\{\,d(x\,,\,y) : y \in Y\}$$

$$= inf\,\{\,||x - y|| : y \in Y\}$$

Since $Y \neq X$, consider $x \in X \ni x \notin Y$.

If $dist(x\,,\,Y) = 0$, then $||x - y|| = 0 \Rightarrow x \in \overline{Y} = Y$ ($\because$ Y is closed)

Therefore ,

$$dist\,(x\,,\,Y) \neq 0$$

That is ,

$$dist\,(x\,,\,Y) > 0$$

Since $0 < r < 1$, $\frac{1}{r} > 1$

$$\Rightarrow \frac{dist\,(x\,,Y)}{r} > dist\,(x\,,\,Y)$$

That is , $\frac{dist\,(x\,,Y)}{r}$ is not a lower bound of $\{\,||x - y|| : y \in Y\,\}$

Then $\exists$ $y_0 \in Y \ni ||x - y_0|| < \frac{dist(x\,,Y)}{r}$ $\rightarrow$(1)

Let $x_r = \frac{x - y_0}{||x - y_0||}$. Then $x_r \in X$

($\because y_0 \in Y$, $x \notin Y \Rightarrow x - y_0 \in X$ and $||x - y_0|| \neq 0$)

Then $||x_r|| = \left|\left|\dfrac{x - y_0}{||x - y_0||}\right|\right| = \dfrac{||x - y_0||}{||x - y_0||} = 1$

Now to prove $r < dist(x_r, Y) \leq 1$

We have $dist(x_r , Y) = inf\{ ||x_r - y|| : y \in Y \}$

$\leq ||x_r - y|| \quad \forall y \in Y$

In particular, $0 \in Y$, so that $dist(x_r , Y) \leq ||x_r - 0|| = 1$

That is ,

$$dist\ (x_r , Y) \leq 1$$

Now ,

$$dist\ (x_r , Y) = dist\left(\frac{x - y_0}{||x - y_0||}, Y\right)$$

$$= \frac{1}{||x - y_0||} dist\ (x - y_0 , Y)$$

$$= \frac{1}{||x - y_0||} inf\{ ||x - y_0 - y|| : y \in Y\}$$

$$= \frac{1}{||x - y_0||} inf\{||x - (y_0 + y)|| : y_0 + y \in Y\}$$

$$= \frac{1}{||x - y_0||} dist\ (x , Y)$$

$$> \frac{r}{dist\ (x , Y)} dist\ (x , Y) \quad by\ (1)$$

$\Rightarrow dist\ (x_r , Y) > r$

That is ,

$$r < dist\ (x_r , Y) \leq 1$$

CONCLUSION

This project discusses the concept of normed linear space that is fundamental to functional analysis . A normed linear space is a vector space over a real or complex numbers ,on which the norm is defined . A norm is a formalization and generalization to real vector spaces of the intuitive notion of “length” in real world

In this project , the concept of a norm on a linear space is introduced and thus illustrated . It mostly includes the properties of normed linear spaces and different proofs related to the topic.

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NUMBER THEORETIC FUNCTION

Project report submitted to
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by

DILNA TERECE JOSE

DB18CMSR03

Under the guidance of

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CERTIFICATE

Certified that this project **'Number Theoretic Function'** is a bona fide project of **DILNA TERECE JOSE** carried out the project work under my supervision.

Mrs. Riya Baby
Head of Department

Ms. Ajeena joseph
Supervisor

Department of Mathematics

Don Bosco Arts and Science College

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DECLARATION

I **DILNA TERECE JOSE** hereby declare that the project **'Number Theoretic Function'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Ajeena joseph, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

Name

DILNA TERECE JOSE

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Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express my deepest gratitude to people along the way.

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My greatest debt is always, to God Almighty.

CONTENTS

INTRODUCTION

A Number Theoretic Function is a complex valued function defined for all positive integers. In Number Theory, there exist many number theoretic functions. This includes Divisor Function, Sigma Function, Euler's-Phi Function and Mobius Function. All these functions play a very important role in the field of Number Theory.

In the first chapter we will discuss about Arithmetic Function. In the second chapter we will introduce Euler's-Phi Function and Mobius Function.

PRELIMINARY

Let n be a fixed positive integer. Two integers a and b are said to be *congruent modulo n,* symbolized by

$$a \equiv b \pmod{n}$$

if n divides the difference $a - b$; that is, provided that $a - b = kn$ for some integer k.

Example:

To fix the idea, consider $n = 7$. It is routine to check that

$3 \equiv 24 \pmod{7}$ $\quad - 31 \equiv 11 \pmod{7}$ $\quad - 15 \equiv -64 \pmod{7}$

Because $3 - 24 = (-3)7$, $-31 - 11 = (-6)7$ and $-15 - (-64) = 77$. When n does not divide $(a - b)$, we say that a is *incongruent to b modulo n*, and in this case we write

$a \not\equiv b \pmod{n}$. For a simple example: $25 \not\equiv 12 \pmod{7}$, because 7 fails to divide

$25 - 12 = 13$.

It is to be noted that any two integers are congruent modulo 1, whereas two integers are congruent modulo 2 when they are both even or both odd. In as much as congruence modulo 1 is not particularly interesting, the usual practice is to assume that $n > 1$.

Remark:

Given an integer a, let q and r be its quotient and remainder upon division by n, so that

$a = qn + r \quad 0 \leq r < n$

Then, by definition of congruence, $a \equiv r \pmod{n}$. Because there are n choices for r , we see that every integer is congruent modulo n to exactly one of the values $0, 1, 2, \ldots , n - 1$; in particular, $a \equiv 0 \pmod{n}$ if and only if $n \mid a$.

Fundamental Theorem of Arithmetic

is Every integer $n > 1$ can be represented as Product of prime factor in only one way, apart from the order of the factors.

Residue

If a is an integer and m is a positive integer then the residue class of a modulo m is denoted by $\hat{a}$ and is given by

$$\begin{aligned} \hat{a} \quad &= \{x : x \equiv a (mod m)\} \\ &= \{x : x = a + mk,\ k = 0, \pm 1, \pm 2, \cdots\} \end{aligned}$$

CHAPTER 1

ARITHMETIC FUNCTION

An arithmetic Function is a function defined on the positive integers which take values in the real or complex numbers. i.e., A function f: N→ C is called an arithmetic function.

An arithmetic function is called multiplicative if f(mn) = f(m)f(n) for all coprime natural numbers m and n.

Examples

a) Sum of divisors σ(n)
b) Number of divisors τ(n)
c) Euler's function ϕ(n)
d) Mobius function μ(n)

Definition 1.1

Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of n and $\sigma(n)$ denote the sum of positive divisors of n.

Example

Consider $n = 12$. Since 12 has the positive divisors 1, 2, 3, 4, 6, 12, we find that

$\tau(12) = 6$ and $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$

For the first few integers,

$\tau(1) = 1$ $\tau(2) = 2$ $\tau(3) = 2$ $\tau(4) = 3$ $\tau(5) = 2$ $\tau(6) = 4, \ldots$

$\sigma(1) = 1,$ $\sigma(2) = 3,$ $\sigma(3) = 4,$ $\sigma(4) = 7,$ $\sigma(5) = 6,$ $\sigma(6) = 12, \ldots$

It is not difficult to see that $\tau(n) = 2$ if and only if n is a prime number; also, $\sigma(n) = n + 1$ if and only if n is a prime.

Theorem 1.1

If n = $p_1^{k_1} \ldots \ldots \ldots \ldots . p_r^{k_r}$ is the prime factorization of $n > 1$, then

(a) $\tau(n) = (k_1 + 1)(k_2 + 1) \cdot \cdot \cdot (k_r + 1)$, and

(b) $\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1} \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \frac{p_r^{k_r+1}-1}{p_r-1}$

Proof

The positive divisors of n are precisely those integers

$$d = p_1^{a_1}\, p_2^{a_2} \ldots \ldots . . p_r^{a_r}$$

where $0 \le a_i \le k_i$. There are $k_1 + 1$ choices for the exponent a_1; $k_2 + 1$ choices for a_2, . . . ; and $k_r + 1$ choices for a_r . Hence, there are

$(k_1 + 1)(k_2 + 1) \cdot \cdot \cdot (k_r + 1)$

possible divisors of n.

To evaluate σ(n), consider the product

$$\left(1 + p_1 + P_1^2 + \cdots \ldots \ldots \ldots P_1^{K_1}\right) \left(1 + p_2 + P_2^2 + \cdots \ldots \ldots \ldots P_2^{K_2}\right) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \left(1 + p_r + P_r^2 + \cdots \ldots \ldots \ldots P_r^{K_r}\right)$$

Each positive divisor of n appears once and only once as a term in the expansion of this product, so that

$$\sigma(n) = \left(1 + p_1 + P_1^2 + \cdots \ldots \ldots \ldots P_1^{K_1}\right) \left(1 + p_2 + P_2^2 + \cdots \ldots \ldots \ldots P_2^{K_2}\right) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \left(1 + p_r + P_r^2 + \cdots \ldots \ldots \ldots P_r^{K_r}\right)$$

Applying the formula for the sum of a finite geometric series to the ith factor on the right-hand side, we get

$$\left(1 + p_i + P_i^2 + \cdots \ldots \ldots \ldots P_i^{K_i}\right) = \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

It follows that

$$\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \frac{p_r^{k_r+1}-1}{p_r-1} .$$

Corresponding to the $\sum$ notation for sums, the notation for products may be defined using $\prod$, the Greek capital letter pi. The restriction delimiting the numbers over which the product is to be made is usually put under the $\prod$ sign.

Examples

$$\prod_{1\le d\le 5} f(d) = f(1)f(2)f(3)f(4)f(5)$$

$$\prod_{d\,|\,9} f(d) = f(1)f(3)f(9)$$

$$\prod_{\substack{p\,|\,30 \\ p \text{ prime}}} f(p) = f(2)f(3)f(5)$$

With this convention, the conclusion to Theorem 1.1 takes the compact form: if $n = p_1^{k_1}\, p_2^{k_2} \ldots \ldots p_r^{k_r}$ is the prime factorization of n > 1, then

$$\tau(n) = \prod_{1\le i\le r} (k_i + 1)$$

and

$$\sigma(n) = \prod_{1\le i\le r} \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

Theorem 1.2

The functions τ and σ are both multiplicative functions

Proof

Let m and n be relatively prime integers. Because the result is trivially true if either m or n is equal to 1, we may assume that m > 1 and n > 1. If

$$m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \qquad \text{and} \qquad n = q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$$

are the prime factorizations of m and n . It follows that the prime factorization of the product mn is given by

$$mn = p_1^{k_1} \cdots p_r^{k_r} q_1^{j_1} \cdots q_s^{j_s}$$

Applying to theorem 1.1, we obtain

$$\begin{aligned} \tau(mn) &= [(k_1 + 1) \cdots (k_r + 1)][(j_1 + 1) \cdots (j_s + 1)] \\ &= \tau(m)\tau(n) \end{aligned}$$

In a similar fashion, theorem 1.1 gives

$$\begin{aligned} \sigma(mn) &= \left[\frac{p_1^{k_1+1} - 1}{p_1 - 1} \cdots \frac{p_r^{k_r+1} - 1}{p_r - 1} \right] \left[\frac{q_1^{j_1+1} - 1}{q_1 - 1} \cdots \frac{q_s^{j_s+1} - 1}{q_s - 1} \right] \\ &= \sigma(m)\sigma(n) \end{aligned}$$

Thus, τ and σ are multiplicative functions.

Theorem 1.3

If f is a multiplicative function and F is defined by

$$F(n) = \sum_{d \mid n} f(d)$$

then F is also multiplicative.

Proof

Let m and n be relatively prime positive integers. Then

$$F(mn) = \sum_{d \mid mn} f(d)$$

$$= \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1 d_2)$$

because every divisor d of mn can be uniquely written as a product of a divisor d_1 of m and a divisor d_2 of n, where $\gcd(d_1, d_2) = 1$. By the definition of a multiplicative function,

$$f(d_1 d_2) = f(d_1)\, f(d_2)$$

It follows that

$$F(mn) = \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1) f(d_2)$$

$$= \left(\sum_{d_1 \mid m} f(d_1)\right)\left(\sum_{d_2 \mid n} f(d_2)\right)$$

$$= F(m)F(n)$$

It might be helpful to take time out and run through the proof of Theorem 1.3 in a concrete case. Letting m = 8 and n = 3, we have

$$F(8 \cdot 3) = \sum_{d \mid 24} f(d)$$

$= f(1) + f(2) + f(3) + f(4) + f(6) + f(8) + f(12) + f(24)$

$= f(1 \cdot 1) + f(2 \cdot 1) + f(1 \cdot 3) + f(4 \cdot 1) + f(2 \cdot 3) + f(8 \cdot 1) + f(4 \cdot 3) + f(8 \cdot 3)$

$= f(1)\, f(1) + f(2)\, f(1) + f(1)\, f(3) + f(4)\, f(1) + f(2)\, f(3) + f(8)\, f(1)$

$+ f(4) f(3) + f(8)\, f(3)$

$= [f(1) + f(2) + f(4) + f(8)][f(1) + f(3)]$

$$= \sum_{d \mid 8} f(d) \cdot \sum_{d \mid 3} f(d)$$

$= F(8)F(3)$

Theorem 1.3 provides a deceptively short way of drawing the conclusion that τ and σ are multiplicative

The Mangoldt function $\Lambda(n)$

Definition 1.2

For every integer $n \geq 1$ we define

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here is a short table of values of $\Lambda(n)$:

n:	1	2	3	4	5	6	7	8	9	10
$\Lambda(n)$:	0	log 2	log 3	log 2	log 5	0	log 7	log 2	log 3	0

The proof of the next theorem shows how this function arises naturally from the fundamental theorem of arithmetic.

Theorem 1.4

If $n \geq 1$ we have

$$\log n = \sum_{d \mid n} \Lambda(d) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

Proof

The theorem is true if $n = 1$ since both members are 0 . Therefore, assume that $n > 1$ and write

$$n = \prod_{k=1}^{r} p_k{}^{a_k}$$

Taking logarithms we have

$$\log\ n = \sum_{k=1}^{r} a_k \log\ p_k$$

Now consider the sum on the right of (1). The only nonzero terms in the sum come from those divisors d of the form $p_k{}^m$ for $m = 1,2,\dots,a_k$ and $k = 1,2,\dots,r$. Hence

$$\sum_{d|n} \Lambda(d) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \Lambda(p_k^m) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \log\ p_k = \sum_{k=1}^{r} a_k \log\ p_k = \log\ n$$

which proves (1).

CHAPTER 2

EULER'S ϕ FUNCTION

Let n be positive integer. Let U_n denote the set of all positive integers less than n and coprime to it

For example,

$$U_6 = \{1,5\}$$

$$U_{10} = \{1,3,7,9\}$$

$$U_{18} = \{1,5,7,11,13,17\}$$

Definition 2.1

Euler's ϕ function is a function ϕ: N→N such that for any n ∈ N, ϕ (n) is the number of integers less than n and coprime to it

In other words

'Euler's ϕ function counts the number of elements in U_n'

For example,

$$\phi(1) = 1, \phi(2) = 1, \phi(3) = 2, \phi(4) = 2, \phi(5) = 4$$
$$\phi(6) = 2 \dots .$$

Theorem 2.1

Let p be a prime. Then ϕ (p) = p-1

Proof:

By definition, any natural number strictly less than p is coprime to p, hence

$$\phi \text{ (p)} = \text{p-1}$$

Theorem 2.2

If p is a prime and $k > 0$, then

$$\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1)$$

Proof:

Consider the successive p^k natural numbers not greater than p^k arranged in the following rectangular array of p columns and p^{k-1} rows

1	2	.	.	p
p+1	p+2	.	.	2p
.	.	.	.	.
.	.	.	.	.
p^k-p+1	p^k-p+2	.	.	p^k

among these numbers only the ones at the rightmost sides are not coprime to p^k and there are p^{k-1} members in that column. So

$$\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1).$$

For example, $\phi(8) = 2^3 - 2^2 = 4$ which counts the number of elements in the set $U_8 = \{1,3,5,7\}$

By the fundamental theorem of arithmetic, we can write any natural number n as

$$n = p_1^{k_1} \dots \dots \dots . . p_r^{k_r}$$

where P_i 's are distinct prime and $k_i \geq 1$ are integers. We already know how to find $\phi\left(p_i^{k_i}\right)$ we would lie to see how $\phi(n)$ is related to $\phi\left(p_i^{k_i}\right)$. This follows from a very important property of Euler's ϕ Function

Multiplicativity of Euler's ϕ Function

Theorem 2.3

$\phi(mn) = \phi(m)\phi(n)$ if m and n are coprime natural numbers.

Proof:

Consider the array of natural numbers not greater than mn arranged in m columns and n rows in the following manner

$$\begin{array}{cccccc} 1 & 2 & \cdots & r & \cdots & m \\ m+1 & m+2 & & m+r & & 2m \\ 2m+1 & 2m+2 & & 2m+r & & 3m \\ \vdots & \vdots & & \vdots & & \vdots \\ (n-1)m+1 & (n-1)m+2 & & (n-1)m+r & & nm \end{array}$$

Clearly each row of the above array has m distinct residues modulo m. Each column has n distinct residues modulo n: for $1 \leq i, i \leq n-1$

im +j $\equiv$ $im + j$ (mod n)

$\Rightarrow im \equiv im$ (mod n)

$\Rightarrow i \equiv i$ (mod n)   (as gcd(m,n) = 1)

$\Rightarrow i \equiv i$

Each row has ϕ(m) residues coprime to m, and each column has ϕ(n) residues coprime to n. Hence in total ϕ(m)ϕ(n) elements in the above array which are coprime to both m and n, it follows that

$$\phi(\text{mn}) = \phi(\text{m})\phi(\text{n})$$

Theorem 2.4

Let n be any natural numbers, then

$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$

Proof:

By fundamental theorem of arithmetic, we can write

$$n = P_1^{k_1}P_2^{k_2} \ldots \ldots . \ldots P_r^{k_r}$$

Where p_i are the distinct prime factor of n, and k_i are the non negative integers. By previous theorem and proposition,

$$\phi(n) = \phi\left(p_1^{k_1}\right) \cdot \ldots, \phi\left(p_r^{k_r}\right)$$

$$= P_1^{k_1-1}(P_1-1)\cdots P_r^{k_{r-1}}(P_r-1)$$

$$= p_1^{k_1}\left(1-\frac{1}{P_1}\right)\cdots P_r^{k_r}\left(1-\frac{1}{P_r}\right)$$

$$= n\left(1-\frac{1}{p_1}\right)\cdots\cdot\left(1-\frac{1}{p_r}\right)$$

Theorem 2.5

For $n > 2$, $\phi(n)$ is an even integer.

Proof:

First, assume that n is a power of 2, let us say that $n = 2^k$, with $k \geq 2$. By

theorem 2.2,

$$\phi(n) = \phi(2^k) = 2^k\left(1-\frac{1}{2}\right) = 2^{k-1}$$

an even integer. If n does not happen to be a power of 2, then it is divisible by an odd prime p; we therefore may write n as $n = p^k m$, where $k \geq 1$ and gcd $(p^k, m) = 1$. Exploiting the multiplicative nature of the phi-function, we obtain

$$\phi(n) = \phi(p^k)\phi(m) = p^{k-1}(p-1)\phi(m)$$

which again is even because $2 \mid p - 1$.

Theorem 2.6

For each positive integer n,

$$n = \sum_{d\mid n} \phi(d)$$

Proof:

Let us partition the set $\{1,2,\ldots\ldots,n\}$ into mutually disjoint subsets S_d for each d/n, where

$$S_d = \{1 \leq m \leq n \mid \text{gcd}\,(m,n) = d\}$$

$$= \{1 \leq \frac{m}{d} \leq \frac{n}{d} \mid \text{gcd}\,(\frac{m}{d},\frac{n}{d}) = 1\}$$

Then

$$\{1,2,\ldots\ldots,n\} = \sum_{d|n} S_d$$

$$\Rightarrow \qquad n = \sum_{d|n} \phi\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \phi(d)$$

As for each divisor of n, n/d is also a divisor of n

MOBIUS FUNCTION

Definition 2.2

The Mobius function $\mu: N \longrightarrow \{0, \pm 1\}$ is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2/n \text{ for some prime } p \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r, \text{ where } p_i \text{ are distinct primes} \end{cases}$$

For example,

$\mu(1) = 1$ $\quad$ $\mu(2) = -1$ $\quad$ $\mu(3) = -1$

$\mu(4) = 0$ $\quad$ $\mu(5) = -1$ $\quad$ $\mu(6) = 1$

If p is a prime number, it is clear that $\mu(p) = -1$; in addition, $\mu(p^e) = 0$ for $e \geq 2$.

Theorem 2.7

The Mobius function is a multiplicative function i.e.

$\mu(mn) = \mu(m)\mu(n)$, if m and n are relatively prime

Proof:

Let m and n be coprime integers, we can consider the following to cases

Case 1: let $\mu(mn) = 0$ then there is a prime p such that $p^2/_{mn}$. As m and n are coprime p cannot divide both m and n hence either $p^2/_m$ or $p^2/_n$. Therefore either $\mu(m) = 0$ or $\mu(n) = 0$ and we have $\mu(mn) = \mu(m)\mu(n)$

Case 2: suppose that $\mu(mn) \neq 0$ then mn is square free, hence so are m and n. let

$m = p_1 \dots \dots \dots p_r$ and $n = q_1 \dots \dots \dots q_s$ where p_i *and* q_j are all distinct primes then mn = $p_1 \dots \dots \dots p_r q_1 \dots \dots \dots q_s$ where all the primes occurring in the factorization of mn are distinct. Hence

$$\mu(mn) = (-1)^{r+s}$$

$$= (-1)^r(-1)^s$$

$$= \mu(m)\mu(n)$$

Theorem 2.8

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Where d runs through all the positive divisors of n.

Proof:

$$Let \quad F(n) = \sum_{d|n} \mu(d)$$

As μ is multiplicative, so is F(n) by the theorem (F be a multiplicative arithmetic function $F(n) = \sum_{d|n} f(d)$ then F is also a multiplicative arthmetic function)

Clearly

$$F(1) = \sum_{d|n} \mu(d)$$

$$= \mu(1)$$

$$= 1$$

For integers which are prime power, i.e. of the form p^k for some $k \geq 1$

$$F(p^2) = \mu(1) + \mu(p) + \mu(p^2) + \cdots \dots \dots \dots .. + \mu(p^k)$$

$$= 1 + (-1) + 0 \dots\dots\dots\dots + 0$$

$$= 0$$

Now consider any integer n, and consider its prime factorization. Then

$$n = p_1^{k_1} \dots \dots \dots \dots \dots . p_r^{k_r}, \qquad k_i \geq 1$$

$$\Rightarrow \quad F(n) = \prod F(p_i^{k_i})$$

$$= 0$$

Mobius inversion formula

The following theorem is known as Mobius inversion formula

Theorem 2.9

Let F and f be two function from the set N of natural number to the field complex number C such that

$$F(n) = \sum_{d|n} f(d)$$

Then we can express f(n) as

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d)$$

Proof:

First observe that if d is divisor of n so is n/d. Hence both the summation in the last line of the theorem are same. Now

$$\sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \left(\mu(d) \sum_{c|(n/d)} f(c) \right)$$

The crucial step in the proof is to observe that the set of S of pairs of integers (c,d) with d|n and c|n/d is the same as the set T of pairs (c,d) with c/n and d|n/c.

$$\sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \left(\mu(d) \sum_{c|(n/d)} f(c) \right)$$

$$= \sum_{d|n} \left(\sum_{c|(n/d)} \mu(d) f(c) \right)$$

$$= \sum_{(c,d)\in S} f(c)\mu(d)$$

$$= \sum_{(c,d)\in T} f(c)\mu(d)$$

$$= \sum_{c|n} \left(f(c) \sum_{d|(n/c)} \mu(d) \right)$$

= F(n)

$As \sum_{d|n} \mu(d) = 0 \quad unless\ {}^{n}/_{c} = 1,$ which happens when c = n

Let us demonstrate this with n = 15

$$\sum_{d|15} \mu(d)F\left(\frac{15}{d}\right) = \mu(1)[\,f\,(1) +\ f\,(3) +\ f\,(5) +\ f\,(15)] + \mu(3)[\,f\,(1) +\ f\,(5)]$$

$$+\mu(5)[\,f\,(1) +\ f\,(3)] + \mu(15)[\,f\,(1)]$$

= f (1)[μ(1) + μ(3) + μ(5) + μ(15)] + f (3)[μ(1) + μ(5)] + f (5)[μ(1) + μ (5)] + f(15) μ(1)

= f(1).0 + f(3).0 + f(5).0 + f(15)

= f(15)

The above theorem leads to the following interesting identities

1. we know that for any positive integer n,

$$\sum_{d|n} \phi(d) = n$$

Where $\phi(n)$ is Euler’s ϕ function. Hence

$$\phi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d$$

For example,

$$\phi(10) = \mu(1)10 + \mu(2)5 + \mu(5)2 + \mu(10)1$$

$= 10 - 5 – 2 + 1$

$= 4$

2. similarly

$$\sigma(n) = \sum_{d|n} d$$
$$n \ = \sum_{d|n} \mu\left(\frac{n}{d}\right)\sigma(d)$$

For example,

With n = 10

$$\mu(10).1 + \mu(2)(1+5) + \mu(5)(1+3) + \mu(1)(1+3+5+10)$$

$= 1 – 1 – 5 – 1 – 3 + 1 + 3 +5 + 10$

$= 10$

We have seen before that if multiplicative so is $F(n) = \sum_{d|n} f(d)$. But we can now

Prove that converse applying the Mobius inversion formula

Theorem 2.10

If F is a multiplicative function and

$$F(n) = \sum_{d|n} f(d)$$

then f is also multiplicative.

Proof:

By the Mobius inversion formula we know that

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right)F(d)$$

Let m and n be relatively prime positive integers. We recall that any divisor

d of mn can be uniquely written as $d = d_1, d_2$, where $d_1 \mid m$, $d_2 \mid n$, and

$\gcd(d_1, d_2) = 1 = \gcd(\frac{m}{d_1}, \frac{n}{d_2})$.

Conversely if d_1/m and d_2/n then d_1d_2/mn thus,

$$\begin{aligned}
f(mn) &= \sum_{d|mn} \mu(d)F\left(\frac{mn}{d}\right) \\
&= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1d_2)F\left(\frac{mn}{d_1d_2}\right) \\
&= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1)\mu(d_2)F\left(\frac{m}{d_1}\right)F\left(\frac{n}{d_2}\right) \\
&= \sum_{d_1|m} \mu(d_1)F\left(\frac{m}{d_1}\right)\sum_{d_2|n} \mu(d_2)F\left(\frac{n}{d_2}\right) \\
&= f(m)f(n)
\end{aligned}$$

In view of the above theorem we can say that as N(n) = n is a multiplicative function so is $\phi(n)$ because

$$\sum_{d|n} \phi(d) = n = \mathrm{N(n)}$$

CONCLUSION

The purpose of this project gives a simple account of Arithmetic function, Euler's phi function and Mobius Function. The study of these topics given excellent introduction to the subject called 'NUMBER THEORETIC FUNCTION'

Number Theoretic Function demands a high standard of rigor. Thus, our presentation necessarily has its formal aspect with care taken to present clear and detailed argument. An understanding of the statement of the theorem, number theory proof is the important issue. In the first chapter we discuss about function τ and σ are both multiplicative function. If f is a multiplicative function and F is defined by

$F(n) = \sum_{d|n} f(d)$, then F is also multiplicative. In the second chapter 2 we discuss about that if p is prime the $\phi(p) = p - 1$, $\phi(mn) = \phi(m)\phi(n)$. The Mobius function is multiplicative function if f is multiplicative function and $F(n)$=$\sum_{d|n} f(d)$,

then F is also multiplicative.

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GRAPH COLORING

Project report submitted to

The Kannur University

for the award of the degree

of

Bachelor of Science

by

DONA ROSE STEPHEN

DB18CMSR21

Under the guidance of

MRS. Riya Baby

Department of Mathematics
Don Bosco Arts and Science College
Angadikadavu
March 2021

Examiners 1: Examiner 2:

CERTIFICATE

It is to certify that this project report '**GRAPH COLORING**' is the bona fide project of DONA ROSE STEPHEN who carried out the project under my supervision.

Mrs. Riya Baby

Supervisor, HOD

Department of Mathematics

Don Bosco Arts and Science College

Angadikadavu

DECLARATION

I, DONA ROSE STEPHEN, hereby declare that this project report entitled "**GRAPH COLORING**" is an original record of studies and bona fide project carried out by me during the period from November 2019 to March 2020, under the guidance of **Mrs. Riya Baby**, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not been submitted by me elsewhere for the award of any degree, diploma, title or recognition, before.

DONA ROSE STEPHEN

Department of Mathematics

Don Bosco Arts and Science College

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DONA ROSE STEPHEN

CONTENTS

INTRODUCTION

A proper coloring of a graph is an assignment of colors to the vertices of the graph so that no two adjacent vertices have the same color.

Usually we drop the word “proper” unless other types of coloring are also under discussion. Of course, the “colors” don’t have to be actual colors ; may can be any distinct labels - integers ,for examples , if a graph is not connected , each connected component can be colored independently; except where otherwise noted , we assume graphs are connected. We also assume graphs are simple in this section. Graph coloring has many applications in addition to its intrinsic interest.

In the same way the most important concept of graph coloring is utilized in resource allocation, scheduling. Also, paths, walks and circuits in graph theory are used in tremendous applications say travelling salesman problem, database design concepts, resource networking.

This project deals with coloring which is one of the most important topics in graph theory. In this project there are three chapters. First chapter is coloring . The second chapter is chromatic number. The last chapter deals with application of graph coloring.

BASIC CONCEPTS

1. GRAPH

A graph is an ordered triplet. G=(V(G), E(G), I(G)); V(G) is a non empty set, E(G) is a set disjoint from V(G) and I(G) is an incidence map that associates each element of E(G) and unrecorded pair of element of V(G). The elements of V(G) are called vertices (or nodes or points) of G and the elements of E(G) are Called edges or lines of G.

2. MULTIPLE EDGE / PARALLEL EDGE

A set of 2 or more edges of a graph G is called a multiple edge or parallel edge if they have the same end vertices.

3. LOOP

An edge for which the 2 end vertices are same is called a loop.

4. SIMPLE GRAPH

A graph is simple if it has no loop and no multiple edges.

5. DEGREE

Let G be a graph and v € V the number of edge incident at V in G is called the degree or vacancy of the vertex v in G.

CHAPTER - 1

COLORING

Graph coloring is nothing but a simple way of labeling graph components such as vertices , edges and regions under some constraints. In a graph, no two adjacent vertices, adjacent edges , or adjacent regions are colored with minimum number of colors .This number is called the chromatic number and the graph is called properly colored graph.

In graph theory coloring is a special case of graph labeling; it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. In it simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color, it is called vertex coloring. Similarly, edge coloring assigns a color to each edge so that no two adjacent edges share the common color.

While graph coloring , the constraints that are set on the graph are colors , order of coloring , the way of assigning color , etc. A coloring is given to a vertex or a particular region . Thus, the vertices or regions having same colors form independent sets.

VERTEX COLORING

Vertex coloring is an assignment of colors to the vertices of a graph ‘G’ such that no two adjacent vertices have the same color .Simply put , no two vertices of an edge should be of the same color.

The most common type of vertex coloring seeks to minimize the number of colors for a given graph . Such a coloring is known as a minimum vertex coloring , and the minimum number of colors which with the vertices of a graph may be colored is called the chromatic number .

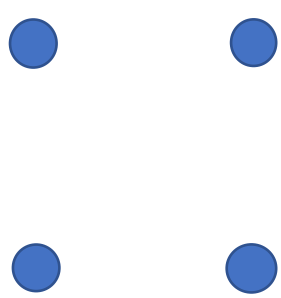
CHROMATIC NUMBER:

The minimum number of colors required for vertex coloring of graph ‘ G ’ is called as the chromatic number of G , denoted by X (G) .

X(G) = 1 iff ‘ G ’ is a null graph. If ‘G ’ is not a null graph , then X(G) ≥ 2.

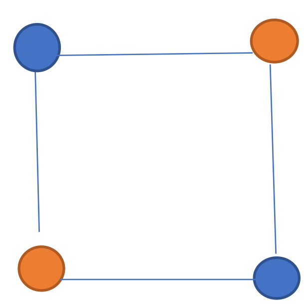
EXAMPLES;

1.



Null Graph (X (G) = 1)

2.

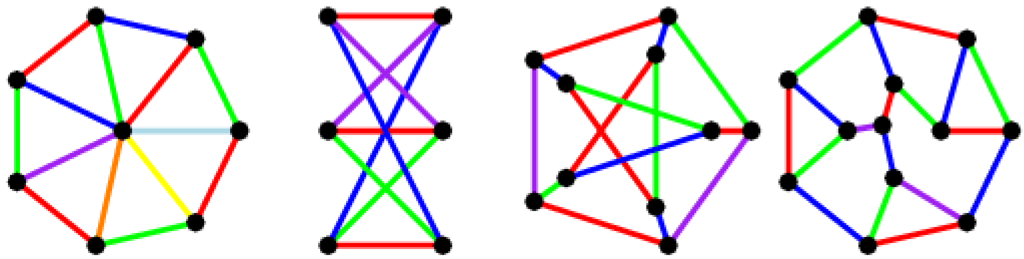


Not Null Graph (X (G) = 2)

EDGE COLORING

An edge coloring of a graph G is a coloring of the edges of G such that adjacent edges (or the edges bounding different regions) receive different colors. An edge coloring containing the smallest possible number of colors for a given graph is known as a minimum edge coloring.

The edge chromatic number gives the minimum number of colours with which graph’s edges can be colored.



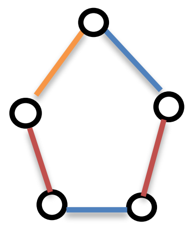
CHROMATIC INDEX

The minimum number of colors required for proper edge coloring of graph is called chromatic index.

A complete graph is the one in which each vertex is directly connected with all other vertices with an edge. If the number of vertices of a complete graph is n, then the chromatic index for an odd number of vertices will be n and the chromatic index for even number of vertices will be n-1.

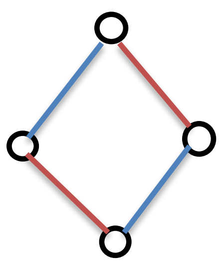
EXAMPLES;

1.



The given graph will require 3 unique colors so that no two incident edges have the Same color. So its chromatic index will be 3.

2.



The given graph will require 2 unique colors so that no two incident edges have the same color. So its chromatic index will be 2.

CHAPTER 2

Chromatic Number

The chromatic number of a graph is the smallest number of colors needed to color the vertices of so that no two adjacent vertices share the same color. That is the smallest value of possible to obtain a k-coloring.

- Graph Coloring is a process of assigning colors to the vertices of a graph.
- It ensures that no two adjacent vertices of the graph are colored with the same color.
- Chromatic Number is the minimum number of colors required to properly color any graph.

Graph Coloring Algorithm

- There exists no efficient algorithm for coloring a graph with minimum number of colors.

However, a following greedy algorithm is known for finding the chromatic number of any given graph.

Greedy Algorithm

Step-01:

Color first vertex with the first color.

Step-02:

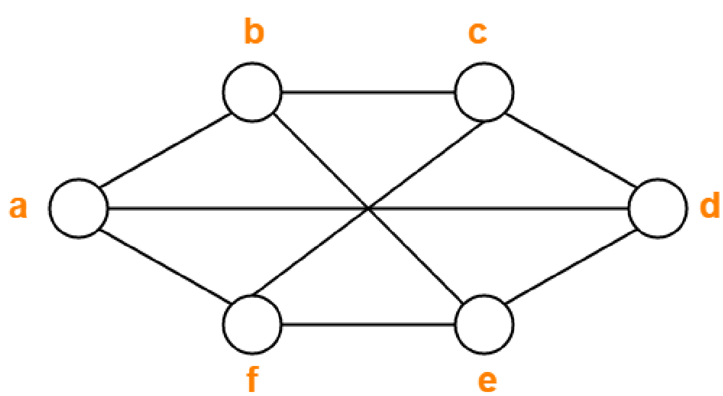
Now, consider the remaining (V-1) vertices one by one and do the following-

- Color the currently picked vertex with the lowest numbered color if it has not been used to color any of its adjacent vertices.
- If it has been used, then choose the next least numbered color.
- If all the previously used colors have been used, then assign a new color to the currently picked vertex.

Problems Based On Finding Chromatic Number of a Graph

Problem-01:

Find chromatic number of the following graph-



Solution-

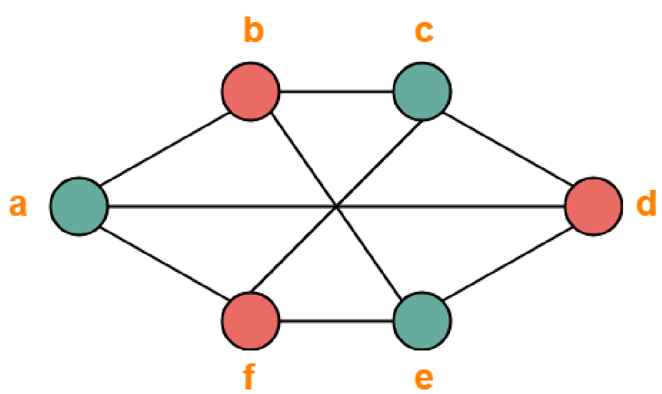
Applying Greedy Algorithm, we have

Vertex	a	b	C	d	e	f
Color	C1	C2	C1	C2	C1	C2

From here,

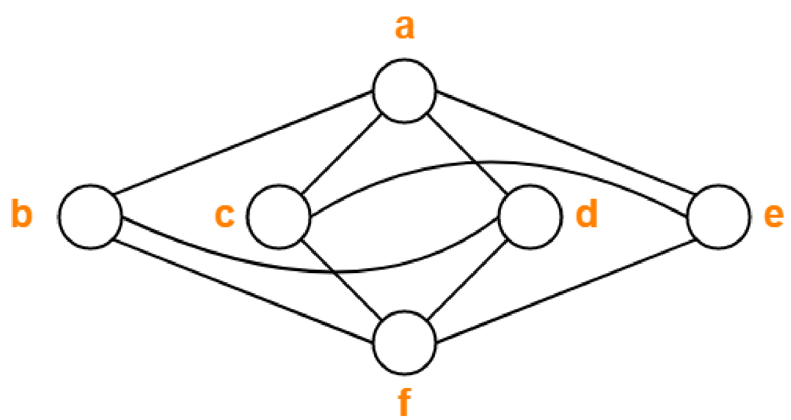
- Minimum numbers of colors used to color the given graph are 2.
- Therefore, Chromatic Number of the given graph = 2.

The given graph may be properly colored using 2 colors as shown below-



Problem-02:

Find chromatic number of the following graph-



Solution-

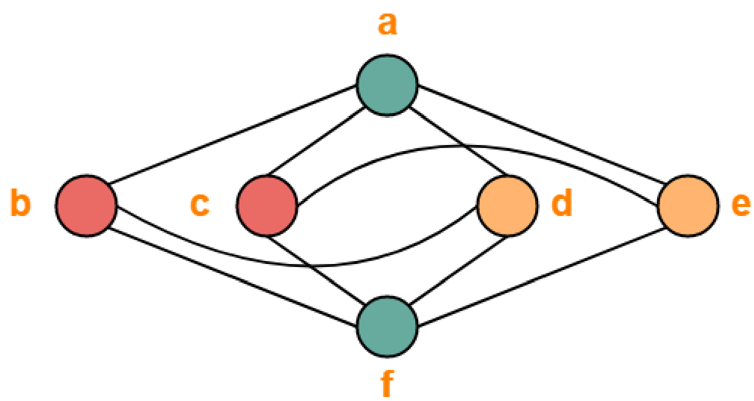
Applying Greedy Algorithm, we have-

Vertex	a	b	C	d	e	f
Color	C1	C2	C2	C3	C3	C1

From here,

- Minimum numbers of colors used to color the given graph are 3.
- Therefore, Chromatic Number of the given graph = 3.

The given graph may be properly colored using 3 colors as shown below-



Chromatic Number of Graphs

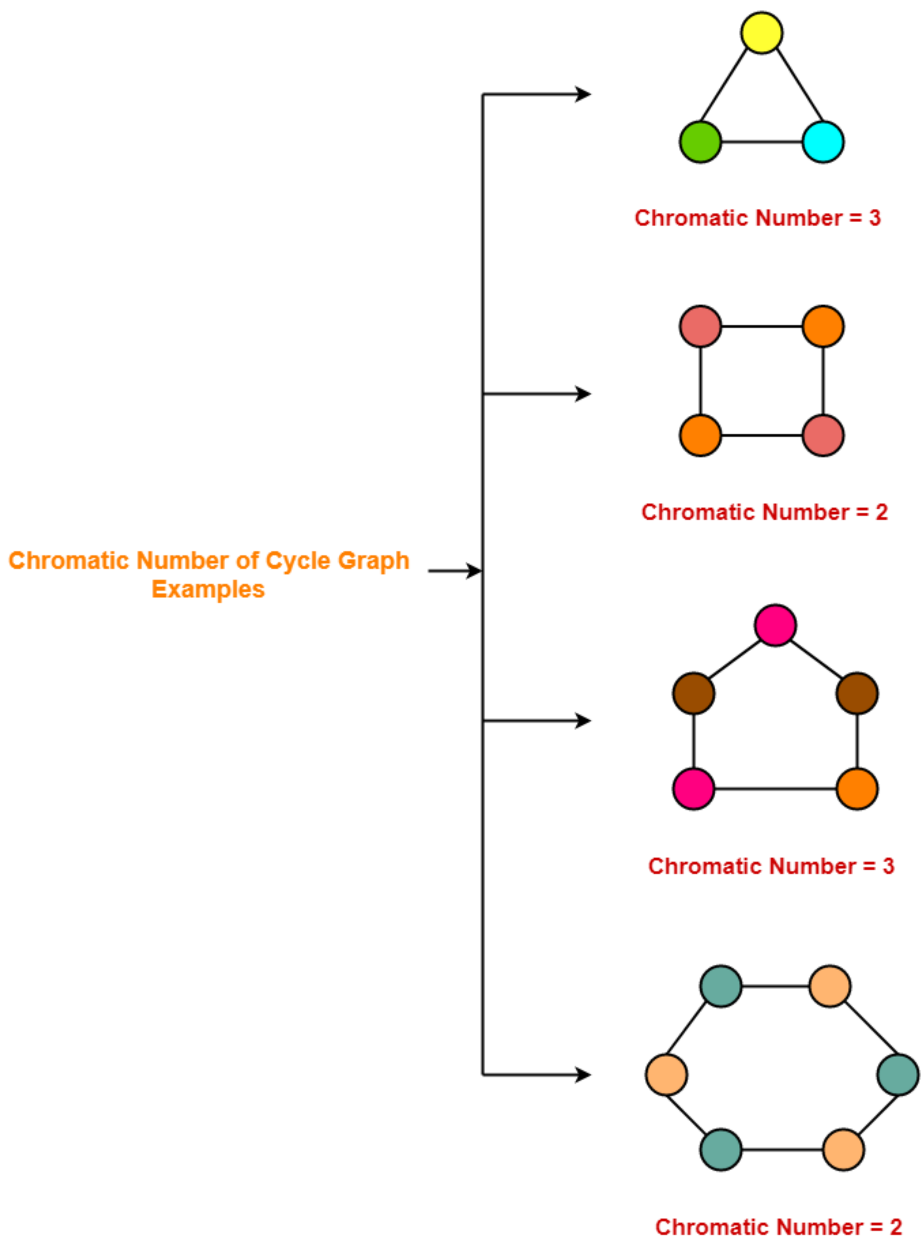
Chromatic Number of some common types of graphs are as follows-

1. Cycle Graph-

- A simple graph of ‘n’ vertices (n>=3) and ‘n’ edges forming a cycle of length ‘n’ is called as a cycle graph.
- In a cycle graph, all the vertices are of degree 2.

Chromatic Number
• If number of vertices in cycle graph is even, then its chromatic number = 2. • If number of vertices in cycle graph is odd, then its chromatic number = 3.

Examples-



2. Planar Graphs-

A planar graph is a graph that can be embedded in the plane, that is it can be drawn on the plane in such a way that its edges intersect only at their endpoint. In other words, it can be drawn in such a way that no edges cross each other.

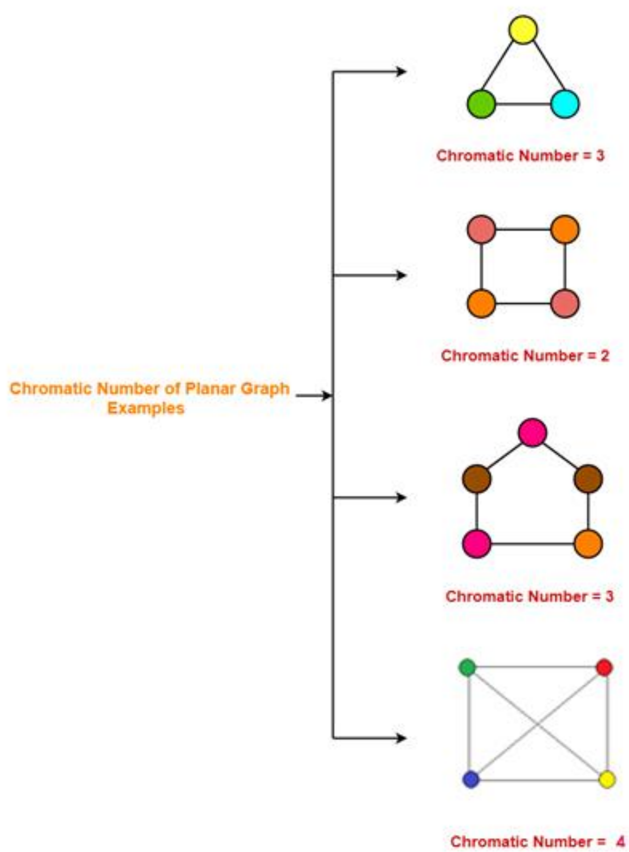
A **Planar Graph** is a graph that can be drawn in a plane such that none of its edges cross each other.

Chromatic Number Chromatic Number of any Planar Graph is less than or equal to 4

Examples-

+

- All the above cycle graphs are also planar graphs.
- Chromatic number of each graph is less than or equal to 4.

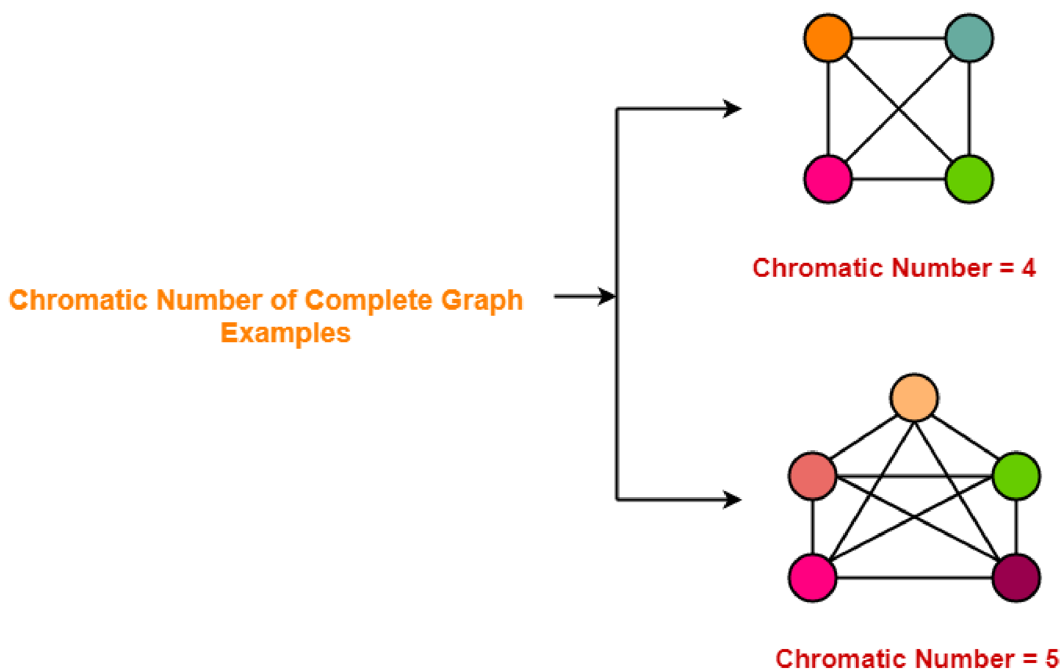


3. Complete Graphs-

- A complete graph is a graph in which every two distinct vertices are joined by exactly one edge.
- In a complete graph, each vertex is connected with every other vertex.
- So to properly it, as many different colors are needed as there are number of vertices in the given graph.

Chromatic Number Chromatic Number of any Complete Graph = Number of vertices in that Complete Graph

Examples-



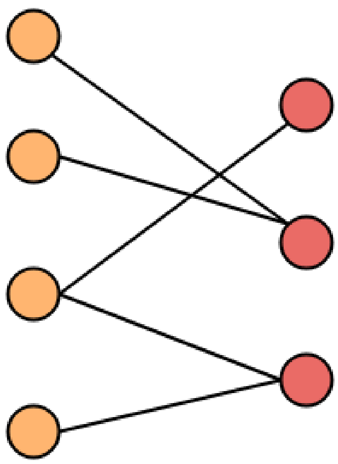
4. Bipartite Graphs-

A bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to one in V. Vertex sets U and V are usually called the parts of the graph.

- A **Bipartite Graph** consists of two sets of vertices X and Y.
- The edges only join vertices in X to vertices in Y, not vertices within a set.

Chromatic Number Chromatic Number of any Bipartite Graph = 2

Example-



Chromatic Number = 2

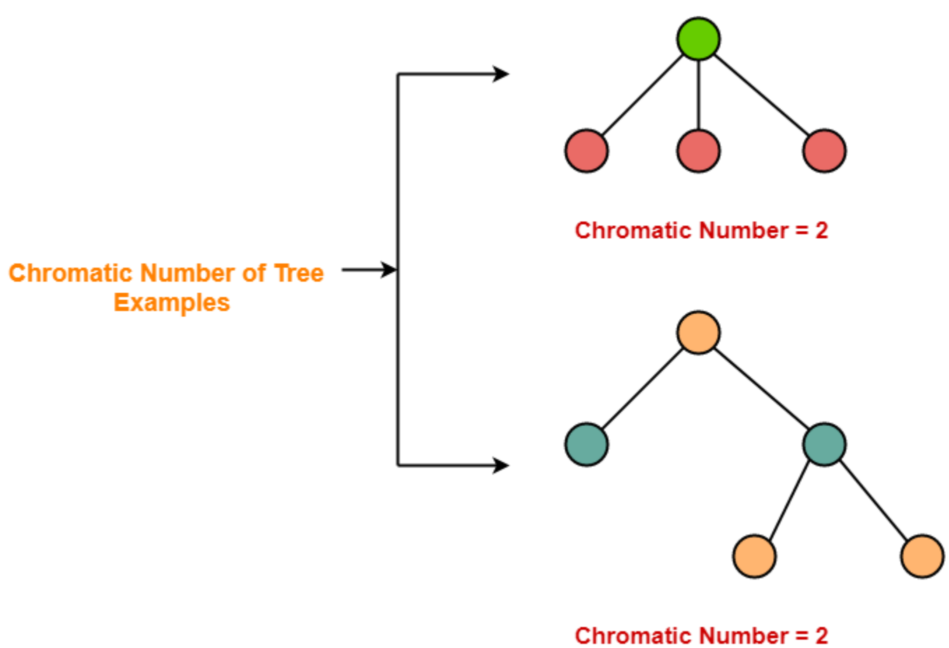
5. Trees-

A tree is an undirected graph in which any two vertices are connected by exactly one path, or equivalently a connected acyclic undirected graph.

- A **Tree** is a special type of connected graph in which there are no circuits.
- Every tree is a bipartite graph.
- So, chromatic number of a tree with any number of vertices = 2.

Chromatic Number Chromatic Number of any tree = 2

Examples-



CHAPTER-3

APPLICATIONS OF GRAPH COLORING

1) Making Schedule or Time Table:

Suppose we want to make an exam schedule for a university. We have list different subjects and students enrolled in every subject. Many subjects would have common students (of same batch, some backlog students, etc). How do we schedule the exam so that no two exams with a common student are scheduled at same time? How many minimum time slots are needed to schedule all exams? This problem can be represented as a graph where every vertex is a subject and an edge between two vertices mean there is a common student. So this is a graph coloring problem where minimum number of time slots is equal to the chromatic number of the graph.

2) Mobile Radio Frequency Assignment:

When frequencies are assigned to towers, frequencies assigned to all towers at the same location must be different. How to assign frequencies with this constraint? What is the minimum number of frequencies needed? This problem is also an instance of graph coloring problem where every tower represents a vertex and an edge between two towers represents that they are in range of each other.

3) Register Allocation:

In compiler optimization, register allocation is the process of assigning a large number of target program variables onto a small number of CPU registers. This problem is also a graph coloring problem.

4) Sudoku:

Sudoku is also a variation of Graph coloring problem where every cell represents a vertex. There is an edge between two vertices if they are in same row or same column or same block.

5) Map Coloring:

Geographical maps of countries or states where no two adjacent cities cannot be assigned same color. Four colors are sufficient to color any map.

6) Bipartite Graphs:

We can check if a graph is bipartite or not by coloring the graph using two colors. If a given graph is 2-colorable, then it is Bipartite, otherwise not. See this for more details.

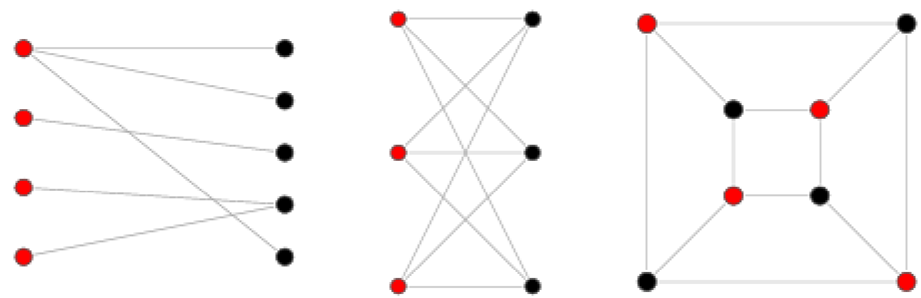
Explanation;

Algorithm:

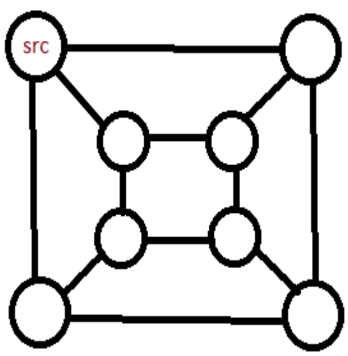
A bipartite graph is possible if it is possible to assign a color to each vertex such that no two neighbour vertices are assigned the same color. Only two colors can be used in this process.

Steps:

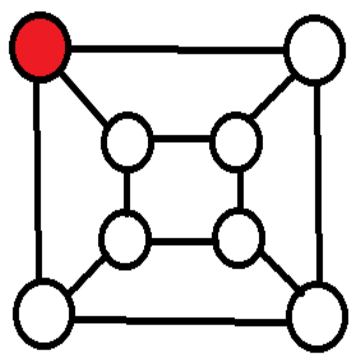
1. Assign a color (say red) to the source vertex.
2. Assign all the neighbours of the above vertex another color (say blue).
3. Taking one neighbour at a time, assign all the neighbour's neighbours the color red.
4. Continue in this manner till all the vertices have been assigned a color.
5. If at any stage, we find a neighbour which has been assigned the same color as that of the current vertex, stop the process. The graph cannot be colored using two colors. Thus the graph is not bipartite.



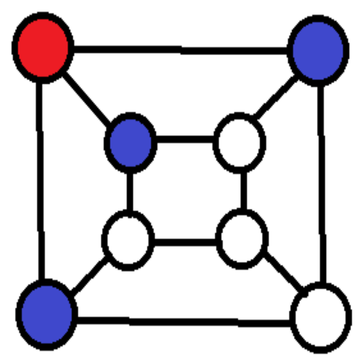
Example:



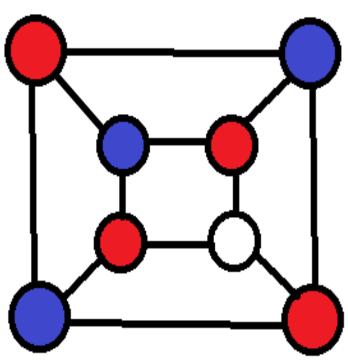
given a graph with source vertex



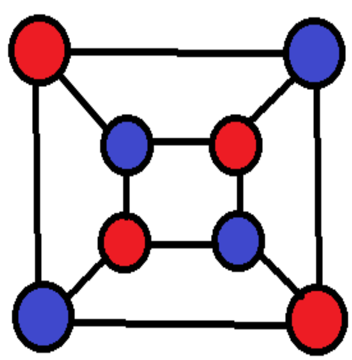
colour src vertex, say red



assign another colour to the neighbours, say blue



assign the neighbours of the vertices of the previous step the colour red



repeat till all vertices are coloured, or a conflicting colour assignment occurs.

set U: red colour
set V: blue colour

CONCLUSION

This project aims to provide a solid background in the basic topics of graph coloring. Graph coloring problem is to assign colors to certain elements of a graph subject to certain constraints. The nature of coloring problem depends on the number of colors but not on what they are.

The study of this topic gives excellent introduction to the subject called “Graph Coloring”.

This project includes two important topics such as vertex coloring and edge coloring and came to know about different ways and importance of coloring.

Graph coloring enjoys many practical applications as well as theoretical challenges. Besides the applications, different limitations can also be set on the graph or on the away a color is assigned or even on the color itself. It has been reached popularity with the general public in the form of the popular number puzzle Sudoku and it is also use in the making of time management which is an important application of coloring. So graph coloring is still a very active field of research.

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NORMED LINEAR SPACES

Project report submitted to
The Kannur University
for the award of the degree
of

Bachelor of Science

by

FREDIN JOSHY

DB18CMSR13

Under the guidance of

Ms. Athulya P

Department of Mathematics
Don Bosco Arts and Science College
Angadikadavu

CERTIFICATE

It is to certify that this project report **'Normed Linear Spaces'** is the bonafide project of **Fredin Joshy** carried out the project work under my supervision.

Mrs. Riya Baby
Head Of Department

Ms. Athulya P
Supervisor

Department Of Mathematics
Don Bosco Arts And Science College
Angadikadavu

DECLARATION

I **Fredin Joshy** hereby declare that the project **'Normed Linear Space'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Athulya P , Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

Fredin Joshy

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Department of Mathematics

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I express my gratitude to all who directly or indirectly helped me to complete this project successfully. Their guidance and support was very helpful in bringing this work to conclusion.

CONTENTS

INTRODUCTION

This chapter gives an introduction to the theory of normed linear spaces. A skeptical reader may wonder why this topic in pure mathematics is useful in applied mathematics. The reason is quite simple: Many problems of applied mathematics can be formulated as a search for a certain function, such as the function that solves a given differential equation. Usually the function sought must belong to a definite family of acceptable functions that share some useful properties. For example, perhaps it must possess two continuous derivatives. The families that arise naturally in formulating problems are often linear spaces. This means that any linear combination of functions in the family will be another member of the family. It is common, in addition, that there is an appropriate means of measuring the "distance" between two functions in the family. This concept comes into play when the exact solution to a problem is inaccessible, while approximate solutions can be computed. We often measure how far apart the exact and approximate solutions are by using a norm. In this process we are led to a normed linear space, presumably one appropriate to the problem at hand. Some normed linear spaces occur over and over again in applied mathematics, and these, at least, should be familiar to the practitioner. Examples are the space of continuous functions on a given domain and the space of functions whose squares have a finite integral on a given domain.

PRELIMINARIES

1) LINEAR SPACES

We introduce an algebraic structure on a set $X$and study functions on X which are well behaved with respect to this structure. From now onwards , $\boldsymbol{K}$ will denote either $\boldsymbol{R}$, the set of all real numbers or $\boldsymbol{C}$**,** the set of all complex numbers. For $k \in \boldsymbol{C}$, *Re k* and *Im k* will denote the real and imaginary part of k.

A linear space(or a vector space) over $\boldsymbol{K}$ is a non-empty set X along with a function $+ \ : X \times X \to X$, called addition and a function $\cdot$: $\boldsymbol{K}\times X \to X$ called scalar multiplication, such that for all $x\,,\ y\,,\ z\ \in\ X$and $k\,,\ l \in \boldsymbol{K}$, we have

$$x + y = y + x$$

$$x + (y + z) = (x + y) + z$$

$$\exists 0 \in X \text{ such that } x + 0 = x,$$

$$\exists - x \in X \text{ such that } x + (- x) = 0\,,$$

$$k \cdot (x + y) = k \cdot x + k \cdot y,$$

$$(k + l)\cdot x = k \cdot x + l \cdot x,$$

$$(kl)\cdot x = k \cdot (l \cdot x),$$

$$1\cdot x = x.$$

We shall write kx in place of $k \cdot x$. We shall also adopt the following notations. For $x, y \in X, k \in \boldsymbol{K}$ and subsets $E, F$of X ,

$$x + F = \{x + y : y \in F\},$$

$$E + F = \{x + y : x \in E, y \in F\},\grave{}$$

$$kE = \{kx : x \in E\}.$$

2) BASIS

A nonempty subset E of X is said to be a subspace of X if $kx + ly \in E$ whenever $x, y \in E$ and $k, l \in \boldsymbol{K}$. If $\emptyset \neq E \subset X$, then the smallest subspace of X containing E is

$$spanE = \left\{k_1 x_1 + \cdots + k_n x_n : x_1, \ldots, x_n \in E\ ,\ k_1, \ldots, k_n \in K\right\}$$

It is called the span of E. If span $E = X$, we say that E spans X. A subset E of X is said to be linearly independent if for all $x_1, \ldots, x_n \in E$ and $k_1, \ldots, k_n \in \boldsymbol{K}$, the equation $k_1 x_1 + \cdots + k_n x_n = 0$implies that $k_1 = \cdots = k_n = 0$. It is called linearly dependent if it is not linearly independent, that is, if there exist $x_1, \ldots, x_n \in E$ and $k_1, \ldots, k_n \in \boldsymbol{K}$ such that $k_1 x_1 + \cdots + k_n x_n = 0$, where at least one $k_j$is nonzero.

A subset E of X is called a Hamel basis or simply basis for X if *span of* $E = X$ and E is linearly independent .

3) DIMENSION

If a linear space X has a basis consisting of a finite number of elements , then X is called finite dimensional and the number of elements in a basis for X is called the dimension of X, denoted as $dimX$. Every basis for a finite dimensional linear space has the same (finite) number of elements and hence the dimension is well-defined. The space $\{0\}$ is said to have zero dimension. Note that it has no basis !

If a linear space contains an infinite linearly independent subset, then it is said to be infinite dimensional.

4)METRIC SPACE

We introduce a distance structure on a set X and study functions on X which are well-behaved with respect to this structure.

A metric d on a nonempty set X is a function $d: X \times X \rightarrow \boldsymbol{R}$
such that for all $x, y, z \in X$

$$d(x, y) \geq 0 \text{ and } d(x, y) = 0 \text{ iff } x=y$$

$$d(y, x) = d(x, y)$$

$$d(x, y) \leq d(x, z) + d(z, y) .$$

The last condition is known as the triangle inequality. A metric space is a nonempty set X along with a metric on it.

5)CONTINUOUS FUNCTIONS

Roughly speaking, a function from a metric space to a metric space is continuous if it sends 'nearby' points to 'nearby' points. If X and Y are metric spaces with metrics d and e respectively, then a function $F: X \rightarrow Y$ is said to be continuous at $x_0 \in X$ if for every $\epsilon \rangle\, 0$, there is some $\delta > 0$ (possibly depending on ϵ and x_0) such that $e\left(F(x), F\left(x_0\right)\right) < \epsilon$ for all $x \in X$ satisfying $d\left(x, x_0\right) < \delta$. Further, F is said to be continuous on X if it is continuous at every point of X. It is easy to see that F is continuous on X if and only if the set $F^{-1}(E)$ is open in X whenever the set E is open in Y. Also , this happens iff $F(x_n) \rightarrow F(x)$ in Y whenever $x_n \rightarrow x$ in X.

6) UNIFORM CONTINUITY

We note that a continuous function $F: T \rightarrow S$ is, in fact, uniformly continuous, that is, for every $\epsilon > 0$, there exists some $\delta > 0$ such that $e(F(t), F(u)) < \epsilon$ whenever $d\left(t, u\ \right) < \delta$. This can be seen as follows. Let $t \in T$. By the continuity of F at $t \in T$, there is some δ_t, such that $e(F(t), F(u)) < \frac{\varepsilon}{2}$ whenever $d(t, u) < \delta_t$.

7) FIELD

A ring is a set R together with two binary operations + and ·(which we call addition and multiplication) such that the following axioms are satisfied .

- R is an abelian group with respect to addition
- Multiplication is associative
- $\forall a , b, c \in R$the left distributive law $a(b + c) = (a\cdot b) + (a\cdot c)$ and the right distributive law $(a + b)c = (a\cdot c) + (b\cdot c)$, hold .

A field is a commutative division ring

CHAPTER 1

NORMED LINEAR SPACE

Let X be a linear space over $\boldsymbol{K}$. A norm on X is the function $\|\ \|$ from X to $\boldsymbol{R}$ such that $\forall$ $x,y \in X$ and $k \in \boldsymbol{K}$,

$$||x|| \geq 0 \text{ and } ||x|| = 0 \text{ if and only if } x = 0 ,$$

$$||x + y|| \leq ||x|| + ||y|| ,$$

$$||kx|| = |k|\ ||x|| .$$

A norm is the formalization and generalization to real vector spaces of the intuitive notion of " length" in the real world .

A normed space is a linear space with norm on it .

For x and y in X, let

$$d(x,y) = ||x - y||$$

Then d is a metric on X so that (X,d) is a metric space , thus every normed space is a metric space

- *Every normed linear space is a metric space . But converse may not be true .*

Example :

$$d(x,y) = \frac{|x-y|}{1+|x-y|} , \forall x , y \in X$$

$$\Rightarrow ||x - y|| = \frac{|x-y|}{1+|x-y|}$$

$$\Rightarrow ||z|| = \frac{|z|}{1+|z|} , z = x - y \in X$$

$$||\alpha z|| = \frac{|\alpha z|}{1+|\alpha z|}$$

$$= \frac{|\alpha|\,|z|}{1+|\alpha|\,|z|}$$

$$= |\alpha| \left(\frac{|z|}{1+|\alpha|\,|z|}\right)$$

$$\neq |\alpha|\,||z||\,.$$

➢ ***Result***

Let X be a normed linear space . Then ,

$$|\,||x|| - ||y||\,| \leq ||x-y||\,,\ \forall x, y \in X$$

Proof :

$$||x|| = ||\,(x-y)+y|| \leq ||x-y|| + ||y||$$

$$\Rightarrow ||x|| - ||y|| \leq ||x-y|| \rightarrow (1)$$

$x \leftrightarrow y$

$$||y|| - ||x|| \leq ||y-x||$$

$$\Rightarrow -(\,||x|| - ||y||\,) \leq ||x-y|| \rightarrow (2)$$

From (1) and (2)

$$|\,||x|| - ||y||\,| \leq ||x-y||$$

➢ ***Norm is a continuous function***

Let $x_n \rightarrow x$, as $n \rightarrow \infty$

$\Rightarrow x_n - x \to 0 \text{ as } n \to \infty$

$\Rightarrow ||x_n - x|| \to 0 \text{ as } n \to \infty$

$\big|\, ||x_n|| - ||x|| \,\big| \leq ||x_n - x|| \to 0 \text{ as } n \to \infty$

$\Rightarrow ||x_n|| - ||x|| \to 0 \text{ , as } n \to \infty$

$\Rightarrow ||x||$ is continuous

➢ *Norm is a uniformly continuous function*

We have , $||\ || : X \to \boldsymbol{R}$. Let $x, y \in X$ and $\varepsilon > 0$

Then $\quad ||x|| = ||x - y + y||$

$\leq ||x - y|| + ||y||$

$\Rightarrow ||x|| - ||y|| \leq ||x - y|| \to (1)$

Interchanging x and y ,

$||y|| - ||x|| \leq ||y - x||$

$\Rightarrow -(\, ||x|| - ||y|| \,) \leq ||x - y||$

$\Rightarrow \quad ||x|| - ||y|| \geq -\, ||x - y|| \to (2)$

Combining (1) and (2)

$- ||x - y|| \leq ||x|| - ||y|| \leq ||x - y||$

That is ,

$\big|\, ||x|| - ||y|| \,\big| \leq ||x - y||$

Take $\delta = \varepsilon$, then whenever $||x - y|| < \delta$, $\big|\, ||x|| - ||y|| \,\big| < \varepsilon$

Therefore $|| \; ||$ is a uniformly continuous function .

➢ <u>*Continuity of addition and scalar multiplication*</u>

To show that $+ : X\times X \to X$ and $\cdot : \boldsymbol{K} \times X \to X$ are continuous functions.

Let $(x,y) \in X\times X$. To show that + is continuous at (x ,y) , that is , to show that for each $(x,y) \in X\times X$ *if* $x_n \to x$ and $y_n \to y$ in X , then

$$+(x_n , y_n) \to +(x , y) ;$$

That is ,

$$x_n + y_n \to x + y .$$

Consider

$$||(x_n + y_n) - (x + y)|| = ||x_n - x + y_n - y||$$

$$\leq ||x_n - x|| + ||y_n - y||$$

Given $x_n \to x$ *and* $y_n \to y$ *, for each* $\epsilon > 0$, $\exists N_1 \ni$

$$||x_n - x|| < \frac{\varepsilon}{2} \forall n \geq N_1 , \quad \text{and } \exists N_2 \ni$$

$$||y_n - y|| < \frac{\varepsilon}{2} \quad \forall n \geq N_2$$

\

Take $N = max \{ N_1, N_2\}$

Then $||x_n - x|| < \frac{\varepsilon}{2}$ and $||y_n - y|| < \frac{\varepsilon}{2} \; \forall n \geq N$

Therefore $||(x_n + y_n) - (x + y)|| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \; \forall n \geq N$

That is , $x_n + y_n \to x + y$

Now to show that $\cdot : \boldsymbol{K} \times X \to X$ is continuous

Let $(k , x) \in \boldsymbol{K}\times X$

To show that if $k_n \to k$ and $x_n \to x$, then $k_n x_n \to kx$

Since $k_n \to k$, $\forall\, \varepsilon > 0\ \exists\ N_1 \ni |k_n - k| < \frac{\varepsilon}{2}\ \ \forall\, n \geq N_1$

Since $x_n \to x$, $\forall\, \varepsilon > 0\ \exists\ N_2 \ni ||x_n - x|| < \frac{\varepsilon}{2}\ \ \forall\, n \geq N_2$

Consider $||k_n x_n - kx|| = ||k_n x_n - kx + x_n k - x_n k||$

$$= ||x_n (k_n - k) + k(x_n - x)||$$

$$\leq ||x_n(k_n - k)|| + ||k(x_n - x)||$$

$$= ||x_n||\, |k_n - k| + |k|\, ||x_n - x||$$

$$\leq ||x_n|| \frac{\varepsilon}{2} + |k| \frac{\varepsilon}{2}$$

$\therefore\ \ k_n x_n \to kx$

➢ <u>*Examples of normed space*</u>

1) Spaces $\boldsymbol{K}^n$ ($\boldsymbol{K} = \boldsymbol{R}$ *or* $\boldsymbol{C}$)

For $n = 1$, the absolute value of function | | is a norm on $\boldsymbol{K}$, since $\forall k \in \boldsymbol{K}$

We have ,

$||k|| = ||k \cdot 1|| = |k|\ ||1||$, by definition .

But $||1||$ is a positive scalar .

$\therefore\ ||k||$ is a positive scalar multiple of the absolute value function .

$\therefore$ any norm on $\boldsymbol{K}$ is a positive scalar multiple of the absolute value function

For $n > 1$, let $p \geq 1$ be a real number

$$K^n = \{ (x(1) , x(2) , . . . , x(n)) : x(i) \in K , i = 1 , 2 , . . . , n \}$$

For $x \in K^n$, that is , $x = (x(1) , x(2) , . . . , x(n))$, define

$$||x||_p = (|x(1)|^p + . . . + |x(n)|^p)^{1/p}$$

Then $|| \, ||_p$ is a norm on K^n

When $p = 1$,

Then , $||x||_1 = |x(1)| + |x(2)| + . . . + |x(n)|$

Since $|x(i)| \geq 0 \;\; \forall \, i = 1 , 2 , . . . , n$, $||x||_1 \geq 0$

And $||x||_1 = 0 \Leftrightarrow |x(1)| + . . . +|x(n)| = 0$

$\Leftrightarrow |x(i)| = 0 \;\; \forall \, i$

$\Leftrightarrow x(i) = 0 \; \forall \, i$

$\Leftrightarrow x = (x(1) , . . . , x(n)) = 0$

Now $||kx||_1 = |kx(1)| + |kx(2)| + . . . + |kx(n)|$

$= |k| \, |x(1)| + . . . + |k| \, |x(n)|$

$= |k| \, (|x(1)| + . . . + |x(n)|)$

$= |k| \, ||x||_1$

$||x + y||_1 = |(x + y)(1)| + . . . + |(x + y)(n)|$

$= |x(1) + y(1)| + . . . + |x(n) + y(n)|$

$\leq |x(1)| + |y(1)| + . . . + |x(n)| + |y(n)|$

$= |x(1)| + . . . + |x(n)| + |y(1)| + . . . + |y(n)|$

$= ||x||_1 + ||y||_1$

Consider $1<p<\infty$

Now , $||x||_p = (|x(1)|^p +. . . + |x(n)|^p)^{1/p}$

Since $|x(i)|^p \geq 0 \ \forall i$, we have $||x||_p \geq 0$

And $||x||_p = 0 \Leftrightarrow (|x(1)|^p +. . . + |x(n)|^p)^{1/p} = 0$

$$\Leftrightarrow |x(i)|^p = 0 \ \forall i$$

$$\Leftrightarrow |x(i)| = 0 \ \forall i$$

$$\Leftrightarrow x(i) = 0 \ \forall i$$

$$\Leftrightarrow x = (x(1) , . . . , x(n)) = 0 .$$

Now

$$||kx||_p = (|kx(1)|^p +. . . + |kx(n)|^p)^{1/p}$$

$$= (|k|^p |x(1)|^p +. . . + |k|^p |x(n)|^p)^{1/p}$$

$$= |k| (|x(1)|^p +. . . + |x(n)|^p)^{1/p}$$

$$= |k| \, ||x||_p .$$

$$||x + y||_p = (|x(1) + y(1)|^p +. . . + |x(n) + y(n)|^p)^{1/p}$$

We have by Minkowski's inequality ,

$$\left(\sum_{i=1}^{n} |x(i) + y(i)|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y(i)|^p\right)^{1/p}$$

Then

$$||x + y||_p \leq \left(|x(1)|^p + \ldots + |x(n)|^p \right)^{1/p} + \left(|y(1)|^p + \ldots + |y(n)|^p \right)^{1/p}$$

$$= ||x||_p + ||y||_p$$

Then , for $1 \leq p < \infty$,$|| \; ||_p$ is a norm on $\boldsymbol{K^n}$

When $p = \infty$, define $||x||_\infty = max \{ |x(1)| , |x(2)| ,. \ldots , |x(n)| \}$

Then it is a norm on $\boldsymbol{K^n}$

$||x||_p \geq 0$ since each values $|x(i)| \geq 0$

So that

$$max \{|x(i)| , i=1, \ldots , n\} \geq 0$$

$$||x||_\infty = 0 \Leftrightarrow max \{ |x(i)| : i = 1,. \ldots , n \} = 0$$

$$\Leftrightarrow |x(i)| = 0 \;\; \forall i$$

$$\Leftrightarrow x(i) = 0 , \forall i$$

$$\Leftrightarrow x = 0$$

$$||kx||_\infty = max \{ |kx(1)| ,. \ldots , |kx(n)| \}$$

$$= max \{ |k| \; |x(1)| , \ldots , |k| \; |x(n)| \}$$

$$= |k| \; max \{|x(1)| , \ldots , |x(n)|\}$$

$$= |k| \; ||x||_\infty$$

$$||x + y||_\infty = max \{ |x(1) + y(1)| , \ldots , |x(n) + y(n)| \}$$

$$\leq max \{ |x(1)| + |y(1)| , \ldots , |x(n)| + |y(n)| \}$$

$\leq max \{ |x(1)| , \ldots , |x(n)| \} + max \{ |y(1)| , \ldots , |y(n)| \}$

$= ||x||_\infty + ||y||_\infty$

2) Sequence space

Let $1 \leq p < \infty$, $l^p = \{ x = (x(1) , x(2) , \ldots) ;\ x(i) \in \boldsymbol{K}$ and $\sum_{j=1}^{\infty} |x(j)|^p < \infty \}$, that is , l^p is the space of p-summable scalar sequences in $\boldsymbol{K}$. For $x = (x(1) , x(2) , \ldots) \in l^p$,

let $||x||_p = (|x(1)|^p + |x(2)|^p + \ldots)^{1/p}$. Then it is a norm on l^p.

That is , $||\ ||_p$ is a function from l^p to $\boldsymbol{R}$.

If $p = 1$, then l^1 is a linear space and $||x||_1 = (|x(1)| + |x(2)| + \ldots)$ is a norm on l^1

Let $p = \infty$. Then l^∞ is the linear space of all bounded scalar sequences . And ,

$$||x||_\infty = sup \{ |x(j)| : j = 1, 2, 3, \ldots \}$$

Then $||\ ||_\infty$ is a norm on l^∞

CHAPTER 2

THEOREMS ON NORMED SPACES

a) Let Y be a subspace of a normed space X, then Y and its closure $\overline{Y}$ are normed spaces with the induced norm.

b) Let Y be a closed subspace of a normed space X, for $x+Y$ in the quotient space X/Y, let $|||x+Y||| = inf\{ ||x+y|| : y \in Y\}$. Then $|||\ \ |||$ is a norm on X/Y, called the quotient norm.

A sequence $(x_n + Y)$ converges to $x + Y$ in X/Y iff there is a sequence (y_n) in Y, $(x_n + y_n)$ converges to x in X.

c) Let $||\ \ ||_p$ be a norm on the linear space X_p, $j = 1,2,\ldots$. Fix p such that $1 \leq p \leq \infty$

For $x = (x(1), x(2), \ldots, x(m))$ that is the product space $X = X_1 \times X_2 \times \ldots \times X_m$,

Let $||x||_p = \left(||x(1)||_1^p + ||x(2)||_2^p + \ldots + ||x(m)||_m^p \right)^{1/p}$, if $1 \leq p < \infty$

$||x||_p = max\{ ||x(1)||_1, \ldots, ||x(m)||_m\}$, if $p = \infty$.

Then $||\ \ ||_p$ is a norm on X.

A sequence (x_n) converges to x in X $\Leftrightarrow$ $(x_n(j))$ converges to $x(j)$ in X_j $\forall j=1,2,\ldots,m$.

Proof:

a) Since X is a normed space, there is a norm on X to Y. Since Y is a subspace of X,

$||\ \ ||_y : Y \to \boldsymbol{R}$ is a function. To show that $||\ \ ||_y$ is a norm on Y.

For $y \in Y$, $||y||_Y = ||y||$, then

$$||y||_Y \geq 0 \quad (\because ||y|| \geq 0) \quad and \quad ||y||_Y = 0 \Leftrightarrow y = 0$$

$$||ky||_Y = ||ky|| = |k|\ ||y|| = |k|\ ||y||_y .$$

Let $y_1, y_2 \in Y$. Then ,

$$||y_1 + y_2||_y = ||y_1 + y_2|| \leq ||y_1|| + ||y_2|| = ||y_1||_y + ||y_2||_y$$

Now the continuity of addition and scalar multiplication shows that $\overline{Y}$ is a subspace of X, since if $x_n \to x$ and $y_n \to y$, $x_n, y_n \in \overline{Y}$, then

$x_n + y_n \to x + y$ (by continuity of addition) and

$kx_n \to kx$ (by continuity of scalar X^n) .

Since $\overline{Y}$ is closed , x + y ∈ $\overline{Y}$ and kx ∈ $\overline{Y}$. Therefore $\overline{Y} \leq$ X.

∴ norm on X induces a norm on Y and $\overline{Y}$

b) X/Y , the quotient space equals $X/Y=\{ x + Y : x \in X \}$.

$$|||x + y||| = inf \{ ||x + y|| : y \in Y \}$$

Claim: ||| ||| is a norm on X/Y , called quotient norm

- Let $x \in X$,

$$|||x + Y||| = inf \{ ||x + y|| : y \in Y \} \geq 0.$$

∴ $|||x + Y||| \geq 0$.

If $|||x + y||| = 0$ (0 in X/Y is Y) , then there is a sequence (y_n) in Y ∋

$$||x + y_n || \to 0$$

$$\Rightarrow \quad x + y_n \to 0$$

$$\Rightarrow \quad y_n \to -x$$

Since $y_n \in Y$ and Y is closed

$-x \in Y \Leftrightarrow x \in Y$ (∵ Y is a subspace)

$\Leftrightarrow x + Y = Y$, zero in X/Y.

- For $k \in \boldsymbol{K}$,

$$|||k(x + Y)||| = |||kx + Y|||$$

$$= inf \{ ||k(x + y)|| : y \in Y\}$$

$$= inf \{ |k| \ ||x + y|| : y \in Y\}$$

$$= |k| \ inf \{ ||x + y|| : y \in Y\}$$

$$= |k| \ |||x + Y||| .$$

- Let $x_1 , x_2 \in X$. Then

$|||x_1 + Y||| = inf \{ ||x_1 + y|| : y \in Y \}$. Then ∃ $y_1 \in Y$ ∋

$$|||x_1 + Y||| + \frac{\varepsilon}{2} > ||x_1 + y_1|| , \ and$$

$$|||x_2 + Y||| = inf\{ ||x_2 + y|| : y \in Y\} \text{ , Then } \exists y_2 \in Y \ni$$

$$|||x_2 + Y||| + \frac{\varepsilon}{2} > ||x_2 + y_2|| .$$

$$||x_1 + y_1 + x_2 + y_2|| \leq ||x_1 + y_1|| + ||x_2 + y_2||$$

$$\leq |||x_1 + Y||| + \frac{\varepsilon}{2} + |||x_2 + Y||| + \frac{\varepsilon}{2}$$

Let $y = y_1 + y_2 \in Y$. Then ,

$$||(x_1+x_2) + y|| \leq |||x_1 + Y||| + |||x_2 + Y||| + \varepsilon \quad \text{—(1)}$$

Now , $|||(x_1 + Y) + (x_2 + Y)||| = |||x_1 + x_2 + Y|||$

$$= inf\{ ||x_1 + x_2 + y|| : y \in Y \}$$

$$< ||x_1 + x_2 + y||$$

$$\leq |||x_1 + Y||| + |||x_2 + Y||| + \varepsilon \qquad \text{(by (1))}$$

since ε is arbitrary , we have

$$|||(x_1 + Y) + (x_2 + Y)||| \leq |||x_1 + Y||| + |||x_2 + Y|||$$

$\therefore$ ||| ||| is a norm on X/Y.

Let $(x_n + Y)$ be a sequence in X/Y . Assume that (y_n) is a sequence in $Y \ni (x_n + y_n)$ converges to x in X.

That is , $(x_n - x + y_n)$ converges to 0 . —(1)

Claim: $(x_n + Y)$ converges to $x + Y$.

Consider

$$|||x_n + Y - (x+Y)||| = |||(x_n - x) + Y|||$$

$$= inf\{ ||x_n - x + y_n|| : y \in Y \}$$

$$\leq ||x_n - x + y_n|| \quad \forall y_n \in Y .$$

Then by (1) , $x_n + Y$ converges to $x + Y$ in X/Y.

Conversely assume that the sequence $(x_n + Y) \rightarrow x + Y$ in X/Y.

Consider $|||x_n + Y - (x + Y)||| = |||x_n - x + Y|||$

$$= inf\{ ||x_n -x + y|| : y \in Y \}$$

Then we can choose $y_n \in Y \ni$

$$||x_n - x + y_n|| < |||(x_n - x) + Y||| + \frac{1}{n}, \quad n=1,2,3,....$$

Since $x_n+Y \to x+Y$, we get

$(x_n - x + y_n)$ converges to zero as $n \to\infty$

That is, $(x_n + y_n)$ converges to x in X as $n \to\infty$

c) Consider $1\leq p < \infty$

Given that

$$||x||_p = (||x(1)||_1^p + ||x(2)||_2^p + . . . + ||x(m)||_m^p)^{1/p}$$

Clearly, $||x||_p \geq 0$.

Since each $||x(i)||_i^p \geq 0$.

$$||x||_p = 0 \Leftrightarrow |x(j)|_j^p = 0 \quad \forall\, j = 1, . . . ,m$$

$$\Leftrightarrow x(j) = 0 \qquad \forall\, j.$$

$$\Leftrightarrow x = (x(1), . . . ,x(m)) = 0$$

$$||kx||_p = \left(||kx(1)||_1^p + . . . + ||kx(m)||_m^p \right)^{1/p}$$

$$= \left(|k|^p||x(1)||_1^p + . . . + |k|^p||x(m)||_m^p \right)^{1/p}$$

$$= |k| \left(||x(1)||_1^p + . . . + ||x(m)||_m^p \right)^{1/p}$$

$$= |k|\ ||x||_p \quad , k\in\boldsymbol{K} \text{ and } x\in X$$

Now, $||x + y||_p = \left(||x(1) + y(1)||_1^p + . . . + ||x(m) + y(m)||_m^p \right)^{1/p}$

(by Minkowski's inequality)

$$\leq$$

$$\left(\left(||x(1)||_1 + ||y(1)||_1\right)^p + . . . + \left(||x(m)||_m + ||y(m)||_m\right)^p\right)^{1/p}$$

$$\leq \left(\sum_{j=1}^{m} ||x(j)||_j^p\right)^{1/p} + \left(\sum_{j=1}^{m} ||y(j)||_j^p\right)^{1/p} \qquad \text{(Minkowski's inequality)}$$

$$= \left(||x(1)||_1^p + . . . + ||x(m)||_m^p \right)^{1/p}$$

$$= ||x||_p + ||y||_p$$

Now suppose $p = \infty$

$$||x||_\infty = max \{ ||x(1)||_1 , . . . , ||x(m)||_m \}$$

$||x||_\infty \geq 0$ Since $||x(j)|| \geq 0$, $\forall j$

$$||x||_\infty = 0 \Leftrightarrow ||x(m)|| = 0 \quad \forall m$$

$$\Leftrightarrow x(m) = 0 \quad \forall m$$

$$\Leftrightarrow x = 0$$

$$||kx||_\infty = max \{ ||kx(1)||_1 , . . . , ||kx(m)||_m \}$$

$$= |k| \; max \{ ||x(1)||_1 , . . . , ||x(m)||_m \}$$

$$= |k| \; ||x||_\infty$$

$$||x + y||_\infty = max \{ ||x(1) + y(1)||_1, . . . , ||x(m) + y(m)||_m \}$$

$$\leq max \{ ||x(1)||_1 + ||y(1)||_1 , . . . , ||x(m)||_m + ||y(m)||_m \}$$

$$= max \{ ||x(1)||_1 , . . . , ||x(m)||_m \} + max \{ ||y(1)||_1 , . . . , ||y(m)||_m \}$$

$$= ||x||_\infty + ||y||_\infty$$

We now consider ,

$$||x_n - x(1)||_p = \left(||x_n(1) - x(1)||_1^p + . . . + ||x_n(m) - x(m)||_m^p \right)^{1/p}$$

Then

$$x_n \rightarrow x \text{ in } X \Leftrightarrow ||x_n - x||_p \rightarrow 0$$

$$\Leftrightarrow ||x_n(j) - x(j)||_j^p \rightarrow 0$$

$$\Leftrightarrow x_n(j) - x(j) \rightarrow 0$$

$$\Leftrightarrow x_n(j) \rightarrow x(j) \text{ in } X \; \forall j .$$

RIESZ LEMMA

Let X be a normed space . Y be a closed subspace of X and $X \neq Y$. Let r be a real number such that $0 < r < 1$. Then there exist some $x_r \in X$ such that $||x_r|| = 1$ and

$r < dist\,(x_r, Y) \leq 1$

Proof :

We have ,

$$dist\,(x, Y) = inf\{ d(x, y) : y \in Y\}$$

$$= inf\{ ||x - y|| : y \in Y\}$$

Since $Y \neq X$, consider $x \in X \ni x \notin Y$.

If $dist(x, Y) = 0$, then $||x - y|| = 0 \Rightarrow x \in \overline{Y} = Y$ ($\because$ Y is closed)

Therefore ,

$$dist\,(x, Y) \neq 0$$

That is ,

$$dist\,(x, Y) > 0$$

Since $0 < r < 1$, $\frac{1}{r} > 1$

$$\Rightarrow \frac{dist\,(x, Y)}{r} > dist\,(x, Y)$$

That is , $\frac{dist\,(x, Y)}{r}$ is not a lower bound of $\{ ||x - y|| : y \in Y \}$

Then $\exists$ $y_0 \in Y \ni ||x - y_0|| < \frac{dist(x, Y)}{r}$ $\rightarrow$(1)

Let $x_r = \frac{x - y_0}{||x - y_0||}$. Then $x_r \in X$

($\because y_0 \in Y$, $x \notin Y \Rightarrow x - y_0 \in X$ and $||x - y_0|| \neq 0$)

Then $||x_r|| = \left|\left|\dfrac{x - y_0}{||x - y_0||}\right|\right| = \dfrac{||x - y_0||}{||x - y_0||} = 1$

Now to prove $r < dist(x_r, Y) \leq 1$

We have $dist(x_r , Y) = inf\{ ||x_r - y|| : y \in Y \}$

$\leq ||x_r - y|| \quad \forall y \in Y$

In particular, $0 \in Y$, so that $dist(x_r , Y) \leq ||x_r - 0|| = 1$

That is ,

$$dist\ (x_r , Y) \leq 1$$

Now ,

$$dist\ (x_r , Y) = dist\left(\frac{x - y_0}{||x - y_0||}, Y\right)$$

$$= \frac{1}{||x - y_0||} dist\ (x - y_0 , Y)$$

$$= \frac{1}{||x - y_0||} inf\{ ||x - y_0 - y|| : y \in Y\}$$

$$= \frac{1}{||x - y_0||} inf\{||x - (y_0 + y)|| : y_0 + y \in Y\}$$

$$= \frac{1}{||x - y_0||} dist\ (x , Y)$$

$$> \frac{r}{dist\ (x , Y)}\ dist\ (x , Y) \quad by\ (1)$$

$$\Rightarrow dist\ (x_r , Y) > r$$

That is ,

$$r < dist\ (x_r , Y) \leq 1$$

CONCLUSION

This project discusses the concept of normed linear space that is fundamental to functional analysis . A normed linear space is a vector space over a real or complex numbers ,on which the norm is defined . A norm is a formalization and generalization to real vector spaces of the intuitive notion of "length" in real world

In this project , the concept of a norm on a linear space is introduced and thus illustrated . It mostly includes the properties of normed linear spaces and different proofs related to the topic.

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DON BOSCO ARTS & SCIENCE COLLEGE

ANGADIKADAVU

DEPARTMENT OF MATHEMATICS

2018-2021

Project Report on

INNER PRODUCT SPACES

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DON BOSCO ARTS & SCIENCE COLLEGE
ANGADIKADAVU

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Project Report on

INNER PRODUCT SPACES

Dissertation submitted in the partial fulfilment
of the requirement for the award of

Bachelor of Science in Mathematics of

Kannur University

Name : JEROM DOMINIC
Roll No. : DB18CMSR14

Examiners: 1.

2.

KANNUR UNIVERSITY

BONAFIDE CERTIFICATE

Certified that this project report on " INNER PRODUCT SPACES” is the bonafide work of JEROM DOMINIC who carried out the project work under my supervision.

Mrs. Riya Baby

Head of Department

Mr. Anil M V

Supervisor

DECLARATION

I, JEROM DOMINIC hereby declare that the project work entitled ‘INNER PRODUCT SPACES′ has been prepared by me and submitted to Kannur University in partial fulfilment of requirement for the award of Bachelor of Science is a record of original work done by me under the supervision of Mr. ANIL M V, Assistant Professor, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu.

I, also declare that this Project work has been submitted by me fully or partially for the award of any Degree, Diploma, Title or recognition before any authority.

Place : Angadikadavu

JEROM DOMINIC

Date :

DB18CMSR14

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Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express mydeepest gratitude to people along the way.

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JEROM DOMINIC

Contents

INTODUCTION

In linear algebra, an inner product space is a vector space with an additional structure called an inner product. This additional structure associates each pair of vectors in the space with a scalar quantity known as the inner product of the vectors. Inner products allow the rigorous introduction of intuitive geometrical notions such as the length of a vector or the angle between two vectors. They also provide the means of defining orthogonality between vectors (zero inner product). Inner product spaces generalize Euclidean spaces (in which the inner product is the dot product, also known as the scalar product) to vector spaces of any (possibly infinite) dimension and are studied in functional analysis. The first usage of the concept of a vector space with an inner product is due to Peano, in 1898.

An inner product naturally induces an associated norm, thus an inner product space is also a normed vector space. A complete space with an inner product is called a Hilbert space. An (incomplete) space with an inner product is called a pre-Hilbert space.

PRELIMINARIES

LINEAR SPACES

Definition 1: A linear (vector) space X over a field **F** is a set of elements together with a function, called addition, from $X \times X$ into X and a function called scalar multiplication, from $\mathbf{F} \times X$ into X which satisfy the following conditions for all $x, y, z \in X$ and $\alpha, \beta \in \mathbf{F}$;

i. $(x + y) + z = x + (y + z)$

ii. $x + y = y + x$

iii. There is an element 0 in X such that $x + 0 = x$ for all $x \in X$.

iv. For each $x \in X$ there is an element $-x \in X$ such that $x + (-x) = 0$.

v. $(x + y) = \alpha x + \alpha y$

vi. $(\alpha + \beta)x = \alpha x + \beta x$

vii. $\alpha(\beta x) = (\alpha\beta)x$

viii. $1 \cdot x = x$.

Properties i to iv imply that X is an abelian group under addition and v to vi relate the operation of scalar multiplication to addition X and to addition and multiplication in **F**.

Examples:

(a) $V_n(\mathbf{R})$. The vectors are n-tuples of real numbers and the scalars are real

numbers with addition and scalar multiplication defined by

$$(\alpha_1,\cdots,\alpha_n)+(\beta_1,\cdots,\beta_n)=(\alpha_1+\beta_1,\cdots,\alpha_n+\beta_n) \quad (1)$$

$$\beta(\alpha_1,\cdots,\alpha_n)=(\beta\alpha_1,\cdots,\beta\alpha_n) \quad (2)$$

$V_n(\mathbf{R})$ is a linear space over **R**. Similarly, the set of all n-tuples of complex numbers with the above definition of addition and multiplication is a linear space over **C** and is denoted as $V_n(\mathbf{C})$.

(b) The set of all functions from a nonempty set X into a field **F** with addition and scalar multiplication defined by

$$[f+g](t)=f(t)+g(t) \quad \text{and} \quad [\alpha f](t)=\alpha f(t);\ f,g\in X,\ t\in T \quad (3)$$

is a linear space.

Let T = **N** the set of all positive integers and X is the set of all sequences of elements **F** with addition and scalar multiplication defined by

$$(\alpha_n+\beta_n)=\ (\alpha_n+\beta_n) \quad (4)$$

$$\beta\left(\alpha_n\right)=\left(\beta\alpha_n\right) \quad (5)$$

denoted as $V_\infty(\mathbf{F})$, form a linear space.

METRIC SPACES

Remember the distance function in the Euclidean space $\mathbf{R}^n$.

Let $x,y,z \in \mathbf{R}^n$, then

(1) $|x - y| \geq 0$; $|x - y| = 0$ if and only if $x = y$;

(2) $|x - y| = |y - x|$;

(3) $|x - y| \leq |x - z| + z - y|$.

Definition 2: A **metric** or **distance function** on a set X is a real valued function d defined on $X \times X$ which has the following properties: for all $x, y, z \in X$.

(1) $d(x,y) \geq 0$; $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq d(x, z) + d(z, y)$

A **metric space** (X, d) is a nonempty set X and a metric d defined on X.

Examples: In addition to the Euclidean spaces let us have the following examples. *Here all functions are assumed to be continuous*. Let L^p denotes a set of complex valued functions in $\mathbf{R}^n$ such that $|f|^p$ is integrable. Let us recall some results concerning such functions.

Höder's Inequality: If $p > 1$, $1/q = 1 - 1/p$

$$\int | fg | \leq [\int | f |^p]^{1/p} [\int | g |^q]^{1/q} .$$

Minkowski's Inequality: If $p \geq 1$,

$$[\int | f + g |^p]^{1/p} \leq [\int | f |^p]^{1/p} + [\int | g |^p]^{1/p}$$

If x_k and y_k for $k = 1, \ldots, m$ are complex numbers, let $f(t) = |x_k|$ and $g(t) = |y_k|$ for $t \in [k, k+1]$ and $f(t) = 0 = g(t)$ for $t \in [1, m+1]$. Then we obtain the summation form of the above inequalities from the integral form

Hölder's Inequality

$$\sum_{k=1}^{m} |x_k y_k| \le \left[\sum_{k=1}^{m} |x_k|^p \right]^{1/p} \left[\sum_{k=1}^{m} |y_k|^q \right]^{1/q}$$

Minkowski's Inequality:

$$\left[\sum_{k=1}^{m} |x_k + y_k|^p \right]^{1/p} \le \left[\sum_{k=1}^{m} |x_k|^p \right]^{1/p} + \left[\sum_{k=1}^{m} |y_k|^p \right]^{1/p}$$

NORMED LINEAR SPACES

Definition 3. A norm on X is a real valued function, whose value at x is denoted by $||x||$, satisfying the following conditions for all $x, y \in X$ and $\alpha \in \mathbf{F}$;

(1) $||x|| > 0$ if $x \neq 0$

(2) $||\alpha x|| = |\alpha| ||x||$

(3) $||x + y|| \le ||x|| + ||y||$.

A linear space X with a norm defined on it is called a **normed linear space**.

Example: l^p **space**. On the linear space $V_n(\mathbf{F})$, define

$$|| x || = [\sum_{k=1}^{n} | \alpha_i |^p]^{1/p}$$

where $p \ge 1$ is any real number and $x = (\alpha_1, \cdots, \alpha_n)$. This defines a norm (called p-norm) on $V_n(\mathbf{F})$. This space is called l^p space.

CHAPTER 1

INNER PRODUCT SPACES

INNER PRODUCTS

Let F be the field of real numbers or the field of complex numbers, and V a vector space over F an inner product on V is a function which assigns to each ordered' pair of vectors α, β in V a scalar $(\alpha|\beta)$ in F in such a way that for all α, β, γ in V and all scalars c.

(a) $(\alpha + \beta|\gamma) = (\alpha|\gamma) + (\beta|\gamma)$;

(b) $(\mathrm{c}\alpha|\beta) = c(\alpha|\beta)$;

(c) $(\beta|\alpha) = (\overline{\alpha|\beta})$, the bar denoting complex conjugation

(d) $(\alpha|\alpha) > 0$ if $\alpha \neq 0$

It should be observed that conditions (a), (b) and (c) implies that

$(e) = (\alpha \mid c\beta + \gamma) = (\bar{c}(\alpha|\beta) + (\alpha|\gamma)$

One other point should be made. When F is the field R of real nunbers. The complex conjugates appearing in (c) and (e) are superflom. However, in the complex case they are necessary for the consistency of the conditions. Without these complex conjugates we would have the contradiction

$$(\alpha|\alpha) > 0 \text{ and } (i\alpha \mid i\alpha) = -1(\alpha|\alpha)$$

Example 1:

On F^n there is an inner product which we call the standard inner product. It is defined on $\alpha = (x_1, \cdots x_n)$ and $\beta = (y_1, \dots, y_n)$, by

$$(\alpha|\beta) = \sum_i x_i \overline{y_i}$$

When F is R this may be also written as

$$(\alpha|\beta) = \sum_i x_i y_i$$

In the real case, the standard inner product is often called the dot or scalar product and denoted by $\alpha \cdot \beta$.

INNER PRODUCTS SPACES

An inner product space is a real or complex vector space together with a specified inner product on that space.

- A finite-dimensional real inner product space is often called a Euclidean spare. A complex inner product spare often referred to as a unitary spare.
- Every inner product space is a normed linear space and every normed space is a metric space. Hence , every inner product space is a metric space.

Theorem

If V is an inner product space, then for any vector's α, β in V and any scalar c

(1) $||c\alpha|| = |c|||\alpha||$;

(ii) $\left||\alpha|\right| > 0$ for $\alpha \neq 0$

(iii) $|(\alpha \mid \beta)| \leqslant ||\alpha|| \, ||\beta||$

(iv) $\| \alpha + \beta|| \leqslant \| \alpha \| + \| \beta||$

Proof:

Statements (i) and (ii) follow almost immediately form the various definitions involved. The inequality in (iii) is clearly valid when $\alpha = 0.$ if $\alpha \neq 0$, put

$$\gamma = \beta - \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha$$

Then, $(\gamma \mid \alpha) = 0$ and

$$0 \leqslant \| \gamma \|^2 = \left(\beta - \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha \middle/ \beta - \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha\right)$$

$$= (\beta|\beta) - \frac{(\beta|\alpha)(\alpha|\beta)}{\| \alpha \|^2}$$

$$= \| \beta \|^2 - \frac{|(\alpha|\beta)|^2}{\| \alpha \|^2}$$

Hence,

$$|(\alpha \mid \beta)|^2 \leqslant \| \alpha \|^2 \| \beta \|^2$$

Now using (c) we find that

$$\| \alpha + \beta \|^2 = \| \alpha \|^2 + (\alpha \mid \beta) + (\beta \mid \alpha) + \| \beta \|^2$$

$$= \| \alpha \|^2 + 2\mathrm{Re}\,(\alpha \mid \beta) + \| \beta \|^2$$
$$\leqslant \| \alpha \|^2 + 2 \| \alpha \| \| \beta \| + \| \beta \|^2$$
$$= (\| \alpha \| + \| \beta \|)^2$$

Thus,

$$\| \alpha + \beta \| \leqslant \| \alpha \| + \| \beta \|$$

the inequality (iii) is called the Cauchy -Schwarz inequality. It has a wide variety of application

the proof shows that if $\alpha$is non-zero then

$$\big((\alpha \mid \beta)\big) < \| \alpha \| \| \beta \|, \text{unless}$$

$$\beta = \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha$$

Then equality occurs in (iii) if and only if α and β are linearly independent.

CHAPTER 2

ORTHOGONAL SETS

Definition

Let α and β be the vectors in an inner product space V. Then α is orthogonal to β if (α | β) = 0. We simply say that and are orthogonal.

Definition

If S is a set of vectors in V, S is called an orthogonal set provided all set pairs of distinct vectors in S are orthogonal.

Definition

An orthogonal set is an orthogonal set S with the additional property that $\parallel \alpha \parallel = 1$ for every α in S.

- The zero vectors are orthogonal to every vector in V and is the only vector with this property.
- It is an appropriate to think of an orthonormal set as a set of mutually perpendicular vectors each having length l.

Example: the vector (x , y) is R^2 is orthogonal to (−y , x) with respect to the standard inner product, for,

$$((x , y)|(-y , x)) = -xy + yx = 0$$

- The standard basis of either R^n or C^n is an orthonormal set with respect to the standard inner product.

Theorem : An orthogonal set of nonzero vectors is linearly independent.

Proof:

Let S be a finite or infinite orthogonal set of nonzero vectors in a given inner product space suppose $\alpha_{1,}\alpha_2, \ldots \alpha_n$ are distinct vectors in S and that β=$c_1\alpha_{1+} + \cdots c_n\alpha_n$

Then (β|α_k)=($c_1\alpha_{1+} + \cdots c_n\alpha_n|\alpha_k$)

$= c_1(\alpha_1|\alpha_k) + c_2(\alpha_2|\alpha_k) + \ldots + c_n(\alpha_n|\alpha_k)$

$= c_k(\alpha_n|\alpha_k)$, since $(\alpha_i\,|\alpha_j) = 0$,if i ≠ j and $(\alpha_i\,|\alpha_j) = 1$,if i=j

Hence, c_k= (β $|\alpha_k)\,/(\alpha_k\,,\alpha_k)$)

c_k=(β $|\alpha_k)/||\alpha_k||^2$,1≤ k ≤ m

Thus, when β=0 each c_k=0; so S is a linearly independent set.

Corollary:

If $\{\alpha_{1,}\alpha_2, \ldots \alpha_m\}$ is an orthogonal set of nonzero vectors in a finite dimensional inner product space V, then m ≤ dimV.

That is number of mutually orthogonal vectors in V cannot exceed the dimensional V.

Corollary:

If a vector β is linear combination of an orthogonal of nonzero vectors $\alpha_{1,}\alpha_2, \ldots \alpha_n$, then β is the particular linear combination

$$\beta = \sum_{k=1}^{m} \frac{(\beta\,|\alpha_k)}{||\alpha_k||^2}\alpha_k$$

Proof :

Since β is the linear combination of an orthogonal sequence of nonzero vectors $\alpha_{1,}\alpha_2, \ldots \alpha_n$, we can write β =$c_1\alpha_1 + \cdots c_n\alpha_n$.

Where $c_k = \frac{(\beta|\alpha_k)}{||\alpha_k||^2}$, 1 ≤ k ≤ m (ref. by previous theorem)

Hence, β= $\sum_{k=1}^{m} \frac{(\beta|\alpha_k)}{||\alpha_k||^2}$

Theorem (Gram Schmidt Orthogonalization Process)

Let V be an inner product space and $\{\beta_{1,} \dots, \beta_n\}$ be any linearly independent vectors in V. Then one may construct orthogonal vectors $\{\alpha_{1,}\alpha_2, \dots \alpha_n\}$ in V, such that for each k = 1, 2, …n, the set $\{\alpha_{1,}\alpha_2, \dots \alpha_k\}$is an orthogonal basis for the subspace of V spanned by $\beta_{1,} \dots, \beta_n$.

Proof :

The vectors are obtained by means of a construction known as the Gram Schmidt orthogonalization process.

First let $\alpha_1 = \beta_1$ The other vectors are then given inductively as follows:

Suppose $\alpha_{1,}\alpha_2, \dots \alpha_m$ (1 ≤ m ≤ n) have been chosen so that for every k

$$\{\alpha_{1,}\alpha_2, \dots \alpha_k\} \ (1 \leq k \leq m)$$

is an orthogonal basis for the space of v that is spanned by $\beta_{1,} \dots, \beta_n$

To construct the next vector $\alpha_{m+1,}$ let

$$\alpha_{m+1,} = \beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2} \alpha_k$$

Then $\alpha_{m+1} \neq 0$. For otherwise, $\beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2} \alpha_k = 0$, implies,

$\beta_{m+1} = \beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2} \alpha_k$,implies, β_{m+1} is a linear combination of $\alpha_{1,}\alpha_2, \dots \alpha_m$ and hence a linear combination of $\beta_1, \beta_2, \dots, \beta_m$, a contradiction.

Furthermore, if 1≤j≤m, then,

$$(\alpha_{m+1}|\ \alpha_j) = (\ \beta_{m+1}|\ \alpha_j) - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2} (\alpha_k\ |\alpha_j)$$

$$= (\ \beta_{m+1}|\ \alpha_m) - (\ \beta_{m+1}|\ \alpha_j)$$, using the orthonormality of $\{\alpha_{1,}\alpha_2, \dots \alpha_m\}$.

Therefore $\{\alpha_{1,}\alpha_2, \dots,\alpha_{m+1}\}$ is an orthogonal set consisting of m+1 nonzero vectors in the subspace spanned by $\beta_{1,} \dots, \beta_{m+1}$. Hence by an earlier Theorem , it is a basis for this subspace .Thus the vectors , $\alpha_{1,}\alpha_2, \dots \alpha_n$ may be constructed using the formula

$$\alpha_{m+1} = \beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}\alpha_k$$

In particular, when n=3 ,we have

$$\alpha_1 = \beta_1$$

$$\alpha_2 = \beta_2 - \frac{(\alpha_2|\beta_2)}{||\alpha_k||^2}\alpha 1$$

$$\alpha_3 = \beta_3 - \frac{(\beta_3|\alpha_1)}{||\alpha_1||^2}\alpha 1 - \frac{(\alpha_2|\beta_3)}{||\alpha_k||^2}\alpha_2$$

Corollary :

Every finite dimensional inner product space has an orthonormal basis.

Proof :

Let V be a finite dimensional inner product space and $\{\beta_{1,} \dots, \beta_n\}$ a basis for V. Apply the gram Schmidt orthogonalization process to construct an orthogonal basis , simply replace each vector α_n by $\frac{\alpha_k}{||\alpha_k||}$.

Gram-Schmidt process can be used to test for linear dependence . For suppose $\beta_{1,} \dots, \beta_n$ are linearly independent vectors in an inner product space; to exclude a trivial case , assume that β≠0. Let m be largest integers for which $\beta_{1,} \dots, \beta_m$are independent. Then 1≤m<n.

Let $\alpha_1, \alpha_2, \dots \alpha_m$ be the vectors obtained by applying the orthogonalization process to $\beta_{1,} \dots, \beta_m$. Then the vector α_{m+1} given by $\alpha_{m+1} = \beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}\alpha_k$ is necessarily 0.

For α_{m+1} is in the subspace spanned by $\alpha_1, \alpha_2, \dots \alpha_m$ and orthogonal to each of the vectors , hence it is 0 as $\beta = \sum_{k=1}^{m} \frac{(\beta|\alpha_k)}{||\alpha_k||^2}\alpha_k$. Conversely, if $\alpha_1, \alpha_2, \dots \alpha_m$ are different from 0 and α_{m+1} =0, then $\beta_{1,} \dots, \beta_{m+1}$ are linearly independent .

Definition:

A best approximation to β ∈ V by vectors in a subspace W of V is a vector $\alpha \in W$ such that

$$\|\beta - \alpha\| \leq \|\beta - \gamma\| \text{ for every vector } \gamma \in W .$$

Theorem

Let W be a subspace of an inner product space V and let $\beta \in V$.

1. The vector $\alpha \in W$ is a best approximation to $\beta \in V$ by vectors in W if and only if $\beta - \alpha$ is orthogonal to every vector in W .

2. If a best approximation to $\beta \in V$ by vectors in W exists, it is unique.

3. If W is finite-dimensional and $\{\alpha_{1,}\alpha_2, \dots \alpha_n\}$ is any orthonormal basis for W ,

   then the vector

$$\alpha = \sum_{k=1}^{n} \frac{(\beta \mid \alpha_k)}{\|\alpha_k\|^2}\alpha_k$$

   is the (unique) best approximation to β by vectors in W.

Definition:

Let V be an inner product space and S be any set of vectors in V. The orthogonal complement of S is the set $S^{\perp}$ of all vectors in V which are orthogonal to every vector in S.

That is, $S^{\perp} = \{\alpha \in V : (\alpha \mid \beta) = 0, \forall \beta \in S\}$

Definition:

Whenever the vector α in the above theorem exists it is called the orthogonal projection of β on W. If every vector in V has an orthogonal projection on W, the mapping that assigns to each vector in V its orthogonal projection on W is called the orthogonal projection of V on W.

Corollary :

Let V be an inner product space and W a finite dimensional subspace and E be the orthogonal projection of V on W. Then the mapping

$$\beta \to \beta - E\beta$$

is the orthogonal projection of V on $W^{\perp}$.

Proof :

Let $\beta \in V$. Then $\beta - E\beta \in W^{\perp}$, and for any $\gamma \in W^{\perp}$, $\beta - \gamma = E\beta + (\beta - E\beta - \gamma)$

Since $E\beta \in W$ and $\beta - E\beta - \gamma \in W^{\perp}$,

It follows that

$$||\beta - \gamma||^2 = (E\beta+(\beta - E\beta - \gamma), E\beta+(\beta - E\beta - \gamma))$$

$$= ||E\beta||^2 + ||\beta - E\beta - \gamma||^2$$

$$\geq ||\beta - (\beta - E\beta)||^2$$

with strict inequality when $\gamma \neq \beta - E\beta$. Therefore, $\beta - E\beta$ is the best approximation to β by vectors in $W^{\perp}$.

Theorem

Let W be a finite dimensional subspace of an inner product space V and let E be the orthogonal projection of V on W. Then E is an idempotent linear transformation of V onto W, $W^{\perp}$ is the null space of E, and $V = W \oplus W^{\perp}$.

Proof

Let β be an arbitrary vector in V. Then $E\beta$ is the best approximation to β that lies in W. In particular, $E\beta = \beta$ when β is in W. Therefore, $E(E\beta) = E\beta$ for every β in V; that is, E is idempotent : $E^2 = E$. To prove that E is linear transformation, let α and β be any vectors in V and c an arbitrary scalar ,Then by theorem,

$\alpha - E\alpha$ and $\beta - E\beta$ are each orthogonal to every vector in W. Hence the vector

$c(\alpha - E\alpha) + (\beta - E\beta) = (c\alpha + \beta) - (cE\alpha + E\beta)$

Also belongs to $W^{\perp}$. Since $cE\alpha + E\beta$ is a vector in W, it follows from theorem that

$E(c\alpha + \beta) = cE\alpha + E\beta$.

Again let β be any vector in V. Then $E\beta$ is the unique vector in W such that $\beta - E\beta$ is in $W^{\perp}$. Thus $E\beta = 0$ when β is in $W^{\perp}$.

Conversely, β is in $W^{\perp}$ when $E\beta = 0$. Thus $W^{\perp}$ is the null space of E.

The equation ,

$$\beta = E\beta + \beta - E\beta$$

shows that V = $W + W^{\perp}$; moreover $W \cap W^{\perp} = \{0\}$; for if α is a vector in $W \cap W^{\perp}$,then

$(\alpha | \alpha)$=0. Therefore, α=0 and V is the direct sum of W and $W^{\perp}$.

Corollary :

Under the conditions of theorem, $I - E$ is the orthogonal projection of V on $W^{\perp}$.

It is an independent linear transformation of V onto $W^{\perp}$ with null space W.

Proof :

We have seen that the mapping $\beta \to \beta - E\beta$ is the orthogonal projection of V on $W^{\perp}$.

Since E is a linear transformation , this projection $W^{\perp}$ is the linear transformation $I - E$ from its geometric properties one sees that $I - E$ is an idempotent .Transformation of V onto W.

This also follows from the computation $(I - E)(I - E) = I - E - E + E^2$

$$= I - E$$

Moreover , $(I - E)\beta = 0$ If and only if $\beta = E\beta$, and this is the case if and only if β is in W .

Therefore W is the null space of $I - E$.

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PLANARITY IN GRAPH THEORY

Project report submitted to
The Kannur University
for the award of the degree
of

Bachelor of Science

by

JOYAL SIBY

DB18CMSR15

Under the guidance of

Mrs. PRIJA V

Department of Mathematics
Don Bosco Arts and Science College
Angadikkadavu
June 2021

CERTIFICATE

Certified that this project **'Planar Graph'** is a bona fide project of **Joyal Siby** carried out the project work under my supervision.

Mrs. Riya Baby
Head of Department

Mrs. Prija V
Supervisor

Department of Mathematics
Don Bosco Arts and Science College
Angadikkadavu

DECLARATION

I **Joyal Siby** hereby declare that the project **'Planar Graph'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Mrs. Prija V, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu and has not submitted by me elsewhere for the award of my degree, diploma, title, or recognition, before.

Joyal Siby

DBI8CMSR15

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Angadikkadavu

ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express my deepest gratitude to people along the way.

No words can adequately express the sense of gratitude; still, I try to express my heartfelt thanks through words. At the outset, I am deeply indebted to my project supervisor Mrs. Prija V, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu, for the invaluable guidance, loving encouragement, and meticulous care towards me throughout my career. I express my deep sense of gratitude to all the faculty members of the Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu.

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My greatest debt is always, to God Almighty.

CONTENTS

INTRODUCTION

In recent years, Graph Theory has established itself as an important mathematical tool in a wide variety of subjects, ranging from Operational Research and Chemistry to Genetics and Linguistics, and from Electrical Engineering and Geography to Sociology and Architecture. At the same time, it has also emerged as a worthwhile mathematical discipline in its own right.

A great mathematician, Euler become the Father of Graph Theory, when in 1736, he solved a famous unsolved problem of his days, called Konigsberg Bridge Problem. This is today, called as the First Problem of the Graph theory. This problem leads to the concept of the planar graph as well as Eulerian Graphs, while planar graphs were introduced for practical reasons, they pose many remarkable mathematical properties. In 1936, the psychologist Lewin used planar graphs to represent the life space of an individual.

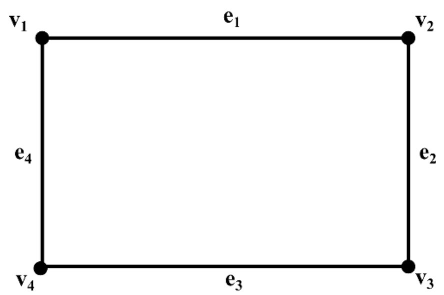
Chapter 1

BASIC CONCEPTS

Graph

A **graph** is an ordered triple $G = \{V(G), E(G), I_G\}$ where $V(G)$ is a non-empty set, $E(G)$ is a set disjoint from $V(G)$ and $I(G)$ is an incidence map that associates each element of $E(G)$ and unordered pair of elements of $V(G)$. The elements of $V(G)$ are called vertices (or nodes or points) of G and the element of $E(G)$ are called edges or lines of G.

Example:



Here $V(G) = \{v_1, v_2, v_3, v_4\}$

$E(G) = \{e_1, e_2, e_3, e_4\}$

$I_G(e_1) = \{v_1, v_2\}$ or $\{v_2, v_1\}$

$\mathrm{I}_G(e_2) = \{v_2, v_3\}$ or $\{v_3, v_2\}$

$I_G(e_3) = \{v_3, v_4\}$ or $\{v_4, v_3\}$

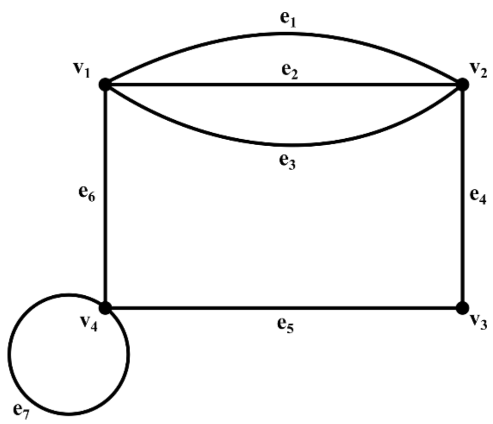
$I_G(e_4) = \{v_4, v_1\}$ or $\{v_1, v_4\}$

Multiple edges

A set of two or more edges of a graph G is called multiple edges or parallel edges if they have the same end vertices.

Loop

An edge for which the two end vertices are same is called a loop.



Here $\{e_1, e_2, e_3, e_4\}$ form the parallel edges.

e_7 is the Loop.

Simple Graph

A graph is simple if it has no loops and no multiple edges.

Finite & Infinite Graphs

A graph is called finite if both $V(G)$ & $E(G)$ are finite. A graph that is not finite is called infinite graph.

Adjacent Vertices

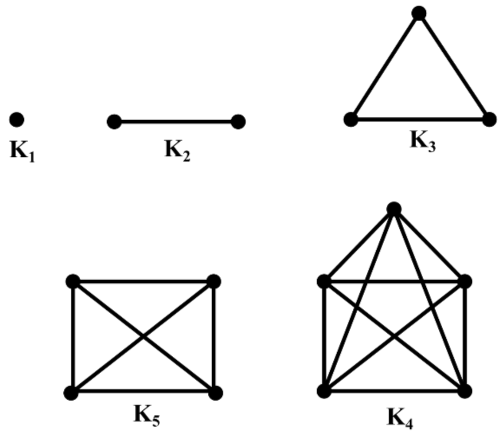
Two vertices u and v are said to be adjacent vertices if and only if there is an edge with u and v as its end vertices.

Adjacent Edges

Two distinct edges are said to be adjacent edges if and only if they have a continuous end vertex.

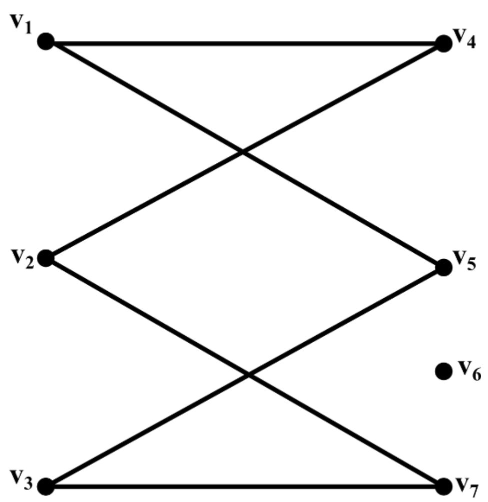
Complete Graph

A simple graph G is said to be complete if every pair of distinct vertices of G are adjacent in G. A complete graph with n vertices is denoted by K_n.



Bipartite Graph

A graph is bipartite if its vertex set can be partitioned into two non-empty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a bipartition of the bipartite graph G. The bipartite graph G with bipartition (X, Y) denoted by $G\ (X, Y)$.



Here $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$

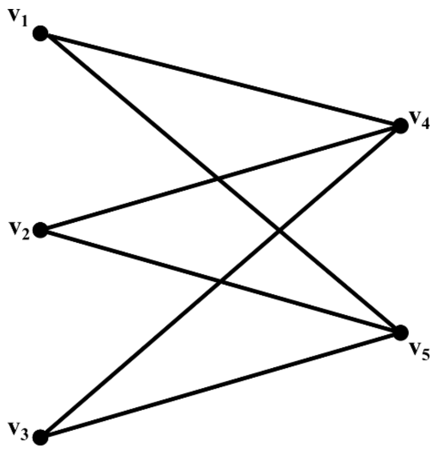
The Bipartition is

$X = \{v_1, v_2, v_3\}$

$Y = \{v_4, v_5, v_6, v_7\}$

Complete Bipartite Graph

A simple bipartite graph $G\ (X, Y)$ is complete if each vertex X is adjacent to all the vertices of Y.

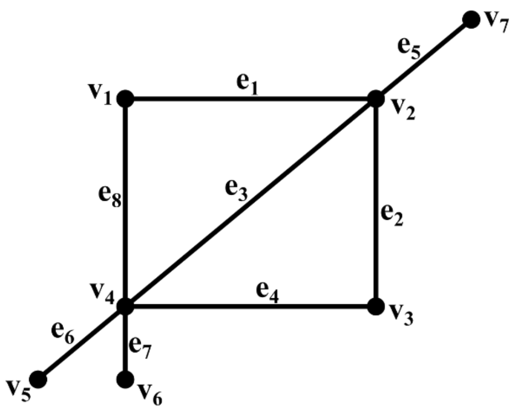


Here $X = \{v_1, v_2, v_3\}$

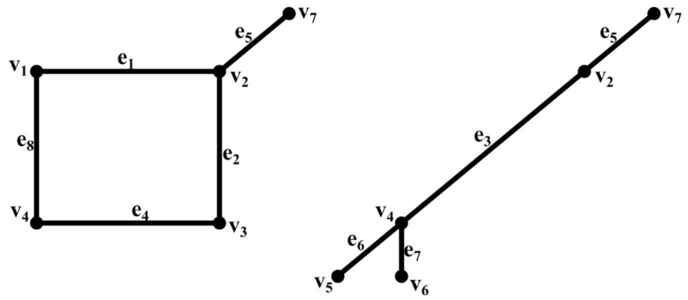
$Y = \{v_4, v_5\}$

Subgraph

A graph H is called subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and I_H is the restriction of I_G to $E(H)$ [ie, $I_H(e) = I_G(e)$ whenever $e \in E(H)$.



G



Subgraphs

Degrees of Vertices

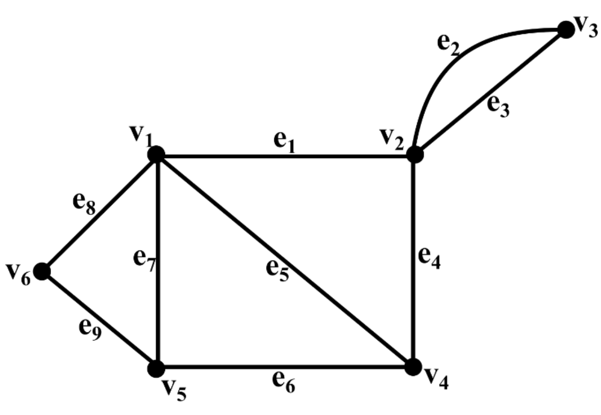
The number of edges incident with vertex V is called degree of a vertex or valency of a vertex and it is denoted by $d(v)$.

Isomorphism of Graph

A graph isomorphism from a graph G to a graph H is a pair (ϕ, θ), where $\phi : V(G) \to V(H)$ and $\theta : E(G) \to E(H)$ are bijection with a property that $I_G(e) = \{u, v\}$ and $I_H(\theta(e)) = \{\phi(u), \phi(v)\}$.

Walk

A walk in a graph G is an alternative sequence $W = v_0 v_1 e_1 v_2 e_2 \ldots v_n e_n$ vertices and edges, beginning and ending with vertices where v_0 is the origin and v_n is the terminus of W.



$W = v_6 e_8 v_1 e_1 v_2 e_2 v_3 e_3 v_2 e_1 v_1$

Closed Walk

A walk to begin and ends at the same vertex is called a closed walk. That is, the walk W is closed if $v_0 = v_n$.

Open Walk

If the origin of the walk and terminus of the walk are different vertices, then it is called an open walk.

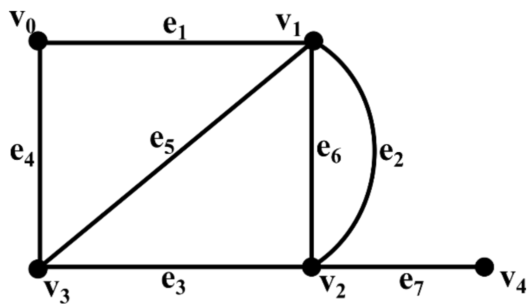
Trail

A walk is called a trail if all the edges in the walk are distinct.

Path

A walk is called a path if all the vertices are distinct.

Example:



$v_0 e_1 v_1 e_2 v_2 e_6 v_1 \rightarrow$ A trail

$v_0 e_1 v_1 e_2 v_2 e_3 v_3 \rightarrow$ A path

$v_0 e_1 v_1 e_2 v_2 e_3 v_3 e_5 v_1 \rightarrow$ A trail, but not a path

Euler's Theorem

The sum of the degrees of the vertices of a graph is equal to the twice the number of edges.

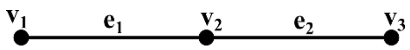
ie: $\sum_{i=1}^{n} d(v_i) = 2m$

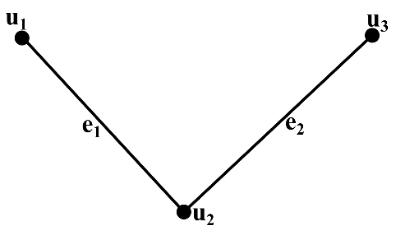
Isomorphic Graph

$$I_H(\theta(e)) = \{\phi(u), \phi(v)\}$$

A graph $G_1 = (V_1, E_1)$ is said to be isomorphic to graph $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and v_1 in G_1 then the corresponding edge e_2 in G_2 has its end vertices u_2 and v_2 in G_2. This correspondence is called a graph isomorphism.

Example:

$G =$ 

$H =$ 

ie: G and H are isomorphic.

Components

A connected component of a graph is a maximal connected subgraph. The term is also used for maximal subgraph or subset of a graph 's vertices that have some higher order of connectivity, including bi-connected components, tri-connected components and strongly connected components.

Tree

A connected graph without cycles is called a tree.

Vertex Cut

Let G be a connected graph. The set V' subset of $V(G)$ is called a Vertex cut of G, if $G - V'$ is a disconnected graph.

Cut Vertex

If $V' = \{v\}$ is a Vertex cut of the connected Graph G, then the vertex $\boldsymbol{v}$ is called a Cut vertex.

Edge Cut

Let G be a non-trivial connected graph with vertex set V and let S be a non-empty subset of V and $\bar{S} = V - S$. Let $E' = [S, \bar{S}]$ denote the set of all edges of G that have one end vertex is S and the other is $\bar{S}$. Then $G - E'$ is a disconnected graph and $E' = [S, \bar{S}]$ is called an edge cut of G.

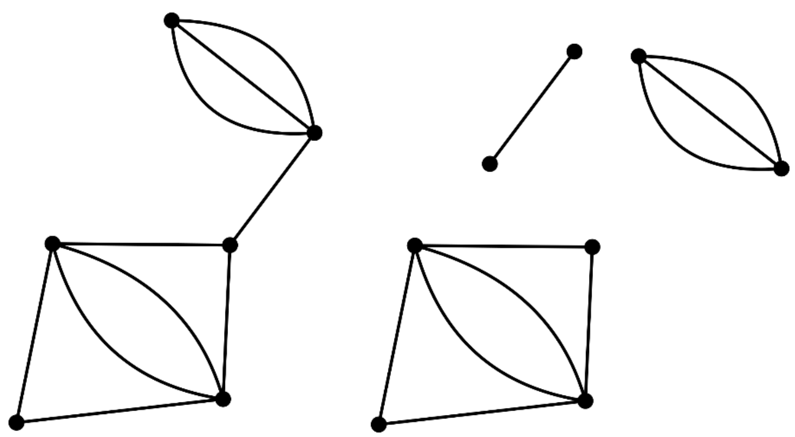
Cut Edge

If $E' = \{e\}$ is an edge cut of G then e is called a cut edge of G.

Block

A block is a Connected graph without any cut vertices.

Eg:



Graph G Blocks of G

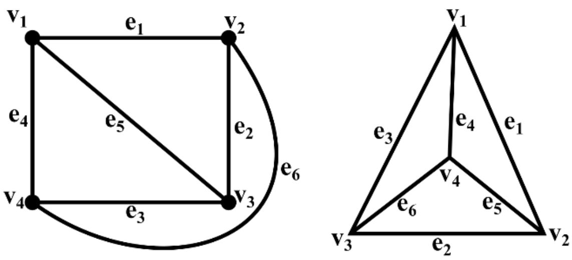
Chapter 2

PLANAR GRAPHS

Plane Graph

A plane graph is a graph drawn in the plane, such a way that any pair of edges meet only at their end vertices.

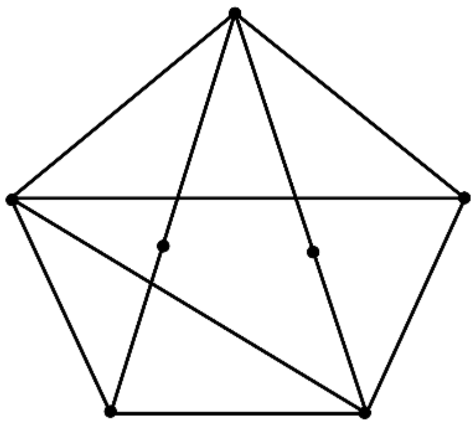
Example:



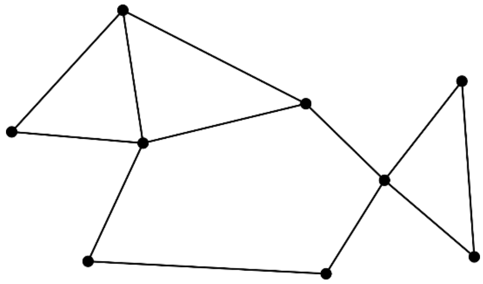
Planar Graph

A planar graph is a graph which is isomorphic to a plane graph, ie: it can be drawn as a plane graph.

A plane graph is a graph that can be drawn in the plane without any edge crossing.



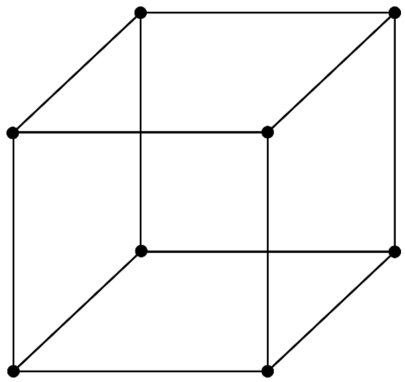
Example of Planar graph:



Planar Representation

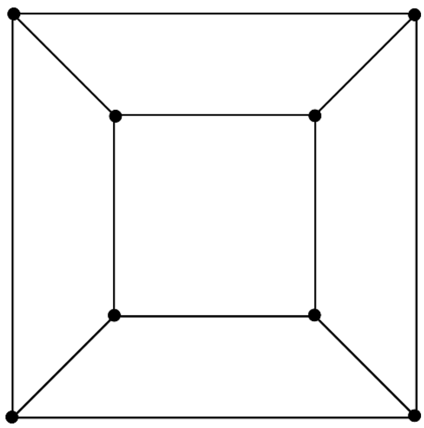
The pictorial representation of a planar graph as a plane graph is called a planar representation.

Eg: Is Q_3 shown below, planar?



The graph Q_3

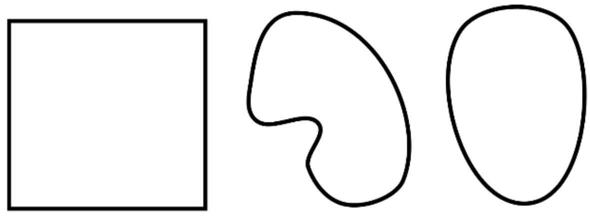
Planar representation of Q_3 is:



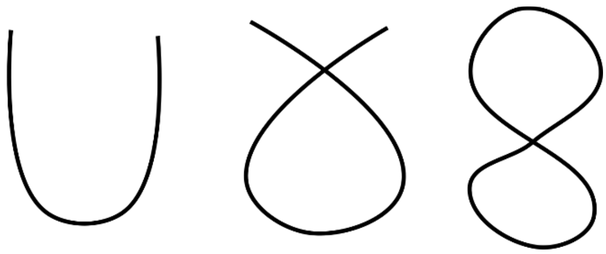
Jordan Curve

A Jordan Curve in the plane is a continuous non-self-intersecting curve where Origin and Terminals coincide.

Example:



Jordan Curves

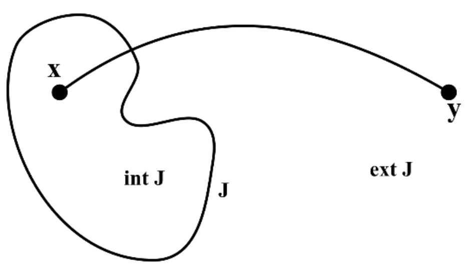


Non-Jordan Curves

Remark

If J is a Jordan Curve in the plane, then the part of the plane enclosed by J is called interior of J and is denoted by ‘int J’. We exclude from ‘int J’ the points actually lying on J. Similarly, the part of the plane lying outside J is called the exterior of J and is denoted by ‘ext J’.

Example:



Arc connecting point x in int J with point y in ext J.

Theorem

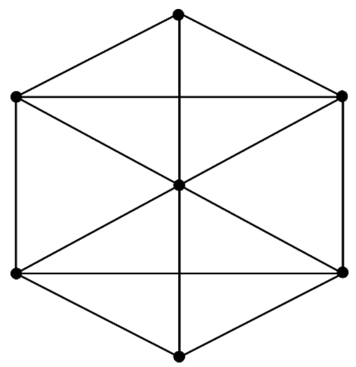
Let J be a Jordan Curve, if x is a point in int J and y is a point in ext J then any line joining x to y must meet J at some point, ie: must cross J. this is called Jordan Curve Theorem.

Boundary

The set of edges that bound a region is called its boundary.

Definition

A graph which is not planar is known as non-planar graph or a graph that cannot be drawn in the plane without any edge crossing is known as non-planar graph.



Theorem

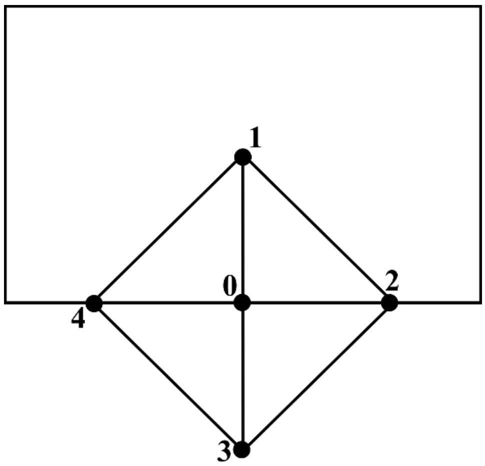
K_5 is nonplanar:

Every drawing of the complex graph K_5 in the plane (or sphere) contains at least one edge crossing.

Proof:

Label the vertices 0, 1, 2, 3, 4. By the Jordan Curve theorem any drawing of the cycle (1, 2, 3, 4, 1) separates the plane into two regions. Consider the region with

vertex 0 in its interior as the ‘inside’ of the circle. By the Jordan Curve theorem, the edges joining vertex 0 to each of its vertices 1, 2, 3 and 4 must also lie entirely inside the cycle, as illustrated below.



Drawing most of the K_5 in the plane

Moreover, each of the 3-cycles $\{0, 1, 2, 0\}$, $\{0, 2, 3, 0\}$, $\{0, 3, 4, 0\}$ and $\{0, 4, 1, 0\}$ also separates the plane and hence the edges $(2, 4)$ must also lie to the exterior of the cycle $\{1, 2, 3, 4\}$ as shown. It follows that the cycle formed by edges $(2, 4)$, $(4, 0)$ and $(0, 2)$ separates the vertices 1 and 3, again by Jordan Curve theorem. Thus, it is impossible to draw edge $(1, 3)$ without crossing an edge of that cycle. So, it is proven that the drawing of the K_5 in the plane contains at least one edge-crossing.

Theorem

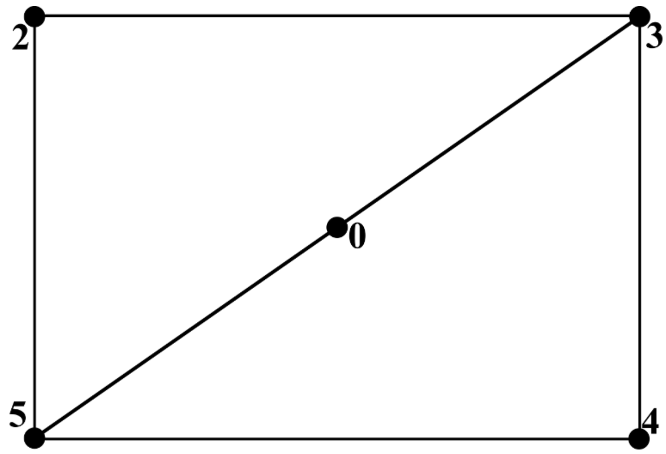
K_{33} is nonplanar:

Every drawing of the complete bipartite graph K_{33} in the plane (or sphere) contains at least one edge crossing.

Proof:

Label the vertices of one partite set 0, 2, 4 and of the order 1, 3, 5. By the Jordan Curve theorem, cycle $\{2, 3, 4, 5, 2\}$ separates the plane into two regions,

and as in the previous proof (K_5), we regard the region containing the vertex 0 as the 'inside' of the cycle. By the Jordan Curve theorem, the edges joining vertex 0 to each of the vertices 3 and 5 lie entirely inside that cycle, and each of the cycle $\{0, 3, 2, 5, 0\}$ and $\{0, 3. 4, 5, 0\}$ separates the plane, as illustrated below.



Drawing most of the K_{33} in the plane

Thus, there are 3 regions: the exterior of cycles $\{2, 3, 4, 5, 2\}$ and the inside of each of the other two cycles. It follows that no matter which region contains vertex 1, there must be some even numbered vertex that is not in that region, and hence the edge from vertex 1 to that even-numbered vertex would have to cross some cycle edge.

Corollary

Subgraph of a planar graph is planar.

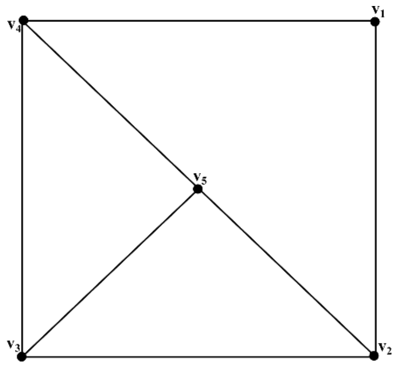
Definition

A plane graph partitions the plane into number of regions called faces.

Let G be plane graph. If x is a point on the plane which is not in G, ie: x is not a vertex of G or a point on any edge of G, then we define the faces of G containing x to be the set of all points on the plane which can be reached from x by a line which does not cross any edge of G or go through any vertex of G.

The number of faces of a plane graph G denoted by $f(a)$ or simply f.

Each plane graph has exactly one unbounded face called the exterior face.



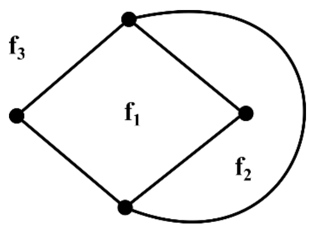
Here $f(G) = 4$

Degree of faces

The degree $d(f)$ of a face f is the number of edges with which it is incident, that is the number of edges in the boundary of a face.

Cut edge being counted twice.

Eg:



$d(f_2) = 3$

$d(f_1) = 4$

$d(f_3) = 3$

Theorem

A graph is planar if and only if each of its blocks is planar.

Proof:

If G is planar, then each of its blocks is planar since a subgraph of planar graph is planar.

Conversely, suppose that each block of G is planar. We now use induction on the number of blocks of G to prove the result. Without loss of generality, we

assume that G is connected. If G has only one block, then G itself is a block, and hence G is planar.

Now suppose G has k planar blocks and that the result has been proved for all connected graph having (k-1) planar blocks. Choose any end block B_0 of G and delete from G all the vertices of B_0 except the unique cut vertex, say v_0 of G in B_0. The resulting connected graph G\` of G contains (k-1) planar blocks. Hence, by the induction hypothesis G\` is planar. Let G˜\` be plane embedded of G\` such that v_0 belongs to the boundary of unbounded face, say f \`. Let $B_0$˜ be a plane embedding of $B_0$ in $f$ \`, so that v_0 is in the exterior face of B_0˜. Then G˜\` and B_0˜ is a plane embedding of G.

Chapter 3

EULER'S FORMULA

Theorems

Euler Formula:

For a connected plain graph G, $n - m + f = 2$ where n, m, and f denote the number of vertices, edges and faces of G respectively.

Proof:

We apply the induction on f.

If $f = 1$ the G is a tree and $m = n - 1$.

Hence $n - m + f = 2$ and suppose that G has f faces.

Since $f \geq 2$, G is not a tree and hence contains a cycle C. Let e be an edge of C. Then e belongs to exactly 2 faces, say $f_1$and f_2 and the deletion of e from G results in the formation of a single face from $f_1$and f_2. Also, since e is not a cut edge of G. $G - e$ is connected.

Further the number of faces of $G - e$ is $f - 1$, number of edges in $G - e$ is $m - 1$ and number of vertices in $G - e$ is n. So, applying induction to $G - e$, we get $n - (m - 1) + (f - 1) = 2$ and this implies that $n - m + f = 2$. This completes the proof of theorem.

Corollary 1

All plane embedding of a planar graph have the same number of faces.

Proof:

Since $f = m - n + 2$ the number of faces depends only on n and m and not on the particular embedding.

Corollary 2

If G is a simple planar graph with at least 3 vertices, then $m \leq 3n - 6$.

Proof:

Without the generality we can assume that G is a simple connected plane graph. Since G is simple and $n \geq 3$, each face of G has degree at least 3. Hence if f denote the set of faces of G $\sum_{f \epsilon F} d(f) \geq 3f$. But $\sum_{f \epsilon F} d(f) = 2m$.

Consequently $2m \geq 3f$ so that $f \leq \frac{2m}{3}$.

By the Euler formula $m = n + f - 2$ now $f \leq \frac{2m}{3}$ implies $\mathrm{m} \leq \mathrm{n} + \left(\frac{2\mathrm{m}}{3}\right) - 2$. This gives. $m \leq 3n - 6$.

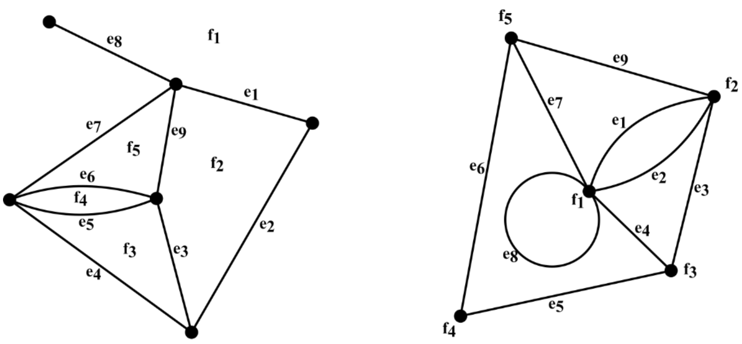
Chapter 4

DUAL OF A PLANE GRAPH

Definition

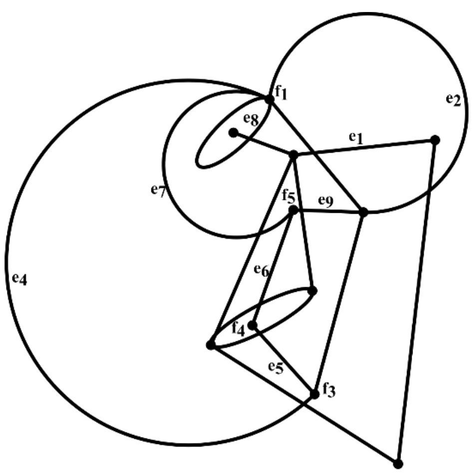
Let G be a plane graph. One can form out of G a new graph H in the following way corresponding to each face f(g), take the vertex f* and corresponding to each edge e(g), take an edge e*. Then edge e* joins vertices f* and g* in H iff edge e is common to the boundaries of faces f and g in G. The graph H is then called dual of G.

Example:



(a) (b)

Plane graph and its Dual



(c)

CONCLUSION

In this project we discussed the topic planar graph in graph theory.

We discussed about Euler formula and verified that some graphs are planar, and some are non-planar. A related important property of planar graphs, maps and triangulations is that they can be enumerated very nicely.

We also discussed about duality of a graph.in mathematical discipline of graph theory, the dual graph of a plane graph G is a graph that has a vertex of each face of G .it has many applications in mathematical and computational study.

In fact, graph theory is being used in our so many routine activities. For eg; using GPS or google maps to determine a route based on used settings.

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1) R. Balakrishnan and K. Ranganathan, A textbook of Graph Theory, University Text

2) J. A. Bondy and U. S. R. Murty, Graph Theory with Great Britain: The Macmillian Press Ltd. 1976

PLANARITY IN GRAPH THEORY

Project report submitted to
The Kannur University
for the award of the degree
of

Bachelor of Science

by

KEERTHANA M

DB18CMSR05

Under the guidance of

Mrs. PRIJA V

Department of Mathematics
Don Bosco Arts and Science College
Angadikkadavu
June 2021

CERTIFICATE

Certified that this project **'Planar Graph'** is a bona fide project of **Keerthana M** carried out the project work under my supervision.

Mrs. Riya Baby	**Mrs. Prija V**
Head of Department	**Supervisor**

Department of Mathematics

Don Bosco Arts and Science College

Angadikkadavu

DECLARATION

I **Keerthana M** hereby declare that the project **'Planar Graph'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Mrs. Prija V, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu and has not submitted by me elsewhere for the award of my degree, diploma, title, or recognition, before.

Keerthana M

DBI8CMSR05

Department of Mathematics

Don Bosco Arts and Science College,

Angadikkadavu

ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express my deepest gratitude to people along the way.

No words can adequately express the sense of gratitude; still, I try to express my heartfelt thanks through words. At the outset, I am deeply indebted to my project supervisor Mrs. Prija V, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu, for the invaluable guidance, loving encouragement, and meticulous care towards me throughout my career. I express my deep sense of gratitude to all the faculty members of the Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu.

I can never forget the support and encouragement rendered by the principal and the staff of Don Bosco Arts and Science College, Angadikkadavu.

I could not name many who sincerely supported and helped for the successful completion of this project. It is my pleasure and duty to thank each and every one of them who walked with me.

My greatest debt is always, to God Almighty.

CONTENTS

INTRODUCTION

In recent years, Graph Theory has established itself as an important mathematical tool in a wide variety of subjects, ranging from Operational Research and Chemistry to Genetics and Linguistics, and from Electrical Engineering and Geography to Sociology and Architecture. At the same time, it has also emerged as a worthwhile mathematical discipline in its own right.

A great mathematician, Euler become the Father of Graph Theory, when in 1736, he solved a famous unsolved problem of his days, called Konigsberg Bridge Problem. This is today, called as the First Problem of the Graph theory. This problem leads to the concept of the planar graph as well as Eulerian Graphs, while planar graphs were introduced for practical reasons, they pose many remarkable mathematical properties. In 1936, the psychologist Lewin used planar graphs to represent the life space of an individual.

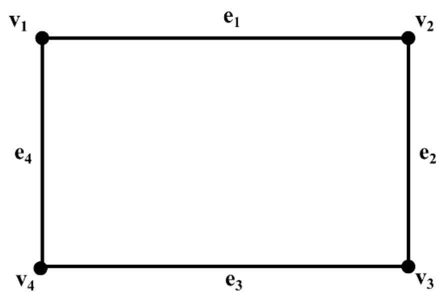
Chapter 1

BASIC CONCEPTS

Graph

A **graph** is an ordered triple $G = \{V(G), E(G), I_G\}$ where $V(G)$ is a non-empty set, $E(G)$ is a set disjoint from $V(G)$ and $I(G)$ is an incidence map that associates each element of $E(G)$ and unordered pair of elements of $V(G)$. The elements of $V(G)$ are called vertices (or nodes or points) of G and the element of $E(G)$ are called edges or lines of G.

Example:



Here $V(G) = \{v_1, v_2, v_3, v_4\}$

$E(G) = \{e_1, e_2, e_3, e_4\}$

$I_G(e_1) = \{v_1, v_2\}$ or $\{v_2, v_1\}$

$\mathrm{I}_G(e_2) = \{v_2, v_3\}$ or $\{v_3, v_2\}$

$I_G(e_3) = \{v_3, v_4\}$ or $\{v_4, v_3\}$

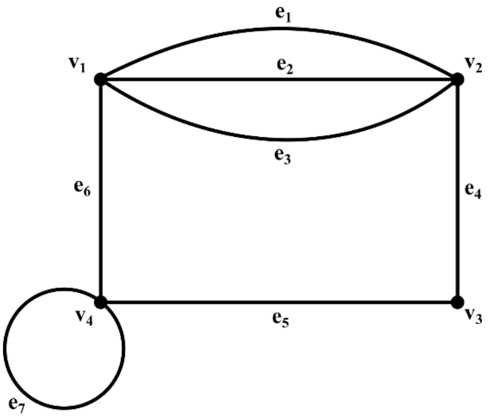
$I_G(e_4) = \{v_4, v_1\}$ or $\{v_1, v_4\}$

Multiple edges

A set of two or more edges of a graph G is called multiple edges or parallel edges if they have the same end vertices.

Loop

An edge for which the two end vertices are same is called a loop.



Here $\{e_1, e_2, e_3, e_4\}$ form the parallel edges.

e_7 is the Loop.

Simple Graph

A graph is simple if it has no loops and no multiple edges.

Finite & Infinite Graphs

A graph is called finite if both $V(G)$ & $E(G)$ are finite. A graph that is not finite is called infinite graph.

Adjacent Vertices

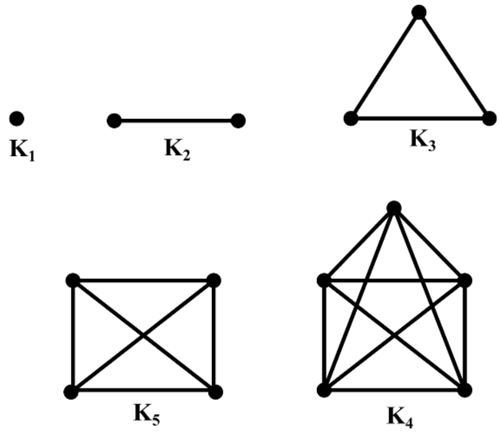
Two vertices u and v are said to be adjacent vertices if and only if there is an edge with u and v as its end vertices.

Adjacent Edges

Two distinct edges are said to be adjacent edges if and only if they have a continuous end vertex.

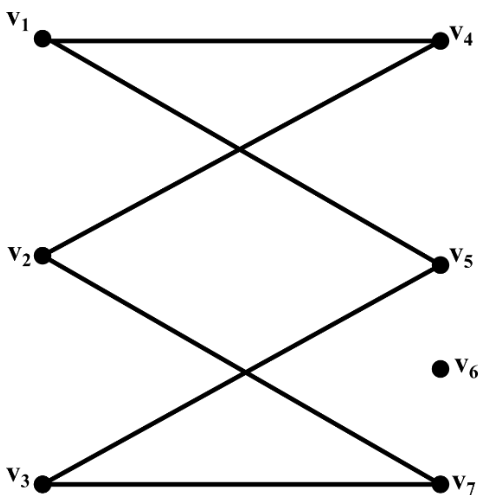
Complete Graph

A simple graph G is said to be complete if every pair of distinct vertices of G are adjacent in G. A complete graph with n vertices is denoted by K_n.



Bipartite Graph

A graph is bipartite if its vertex set can be partitioned into two non-empty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a bipartition of the bipartite graph G. The bipartite graph G with bipartition (X, Y) denoted by $G\ (X, Y)$.



Here $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$

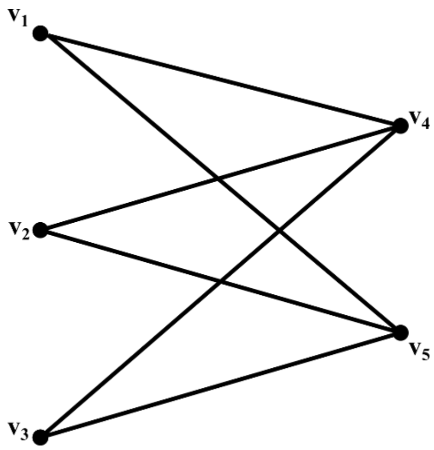
The Bipartition is

$X = \{v_1, v_2, v_3\}$

$Y = \{v_4, v_5, v_6, v_7\}$

Complete Bipartite Graph

A simple bipartite graph $G\ (X, Y)$ is complete if each vertex X is adjacent to all the vertices of Y.

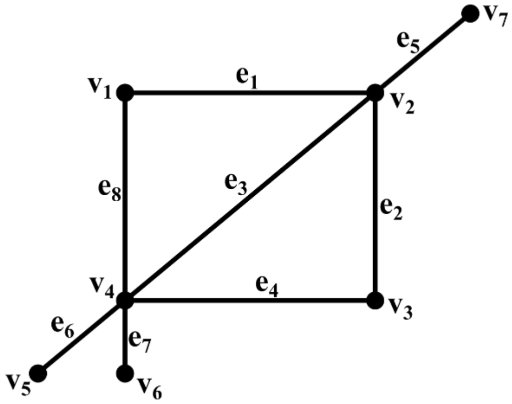


Here $X = \{v_1, v_2, v_3\}$

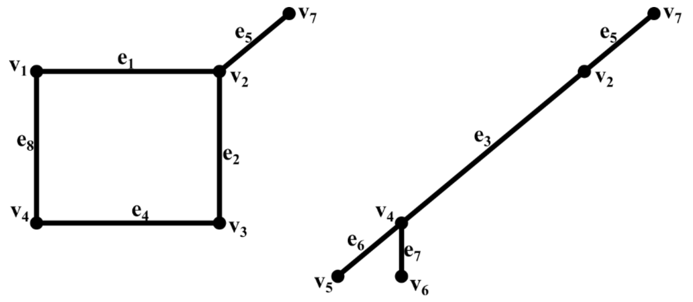
$Y = \{v_4, v_5\}$

Subgraph

A graph H is called subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and I_H is the restriction of I_G to $E(H)$ [ie, $I_H(e) = I_G(e)$ whenever $e \in E(H)$.



G



Subgraphs

Degrees of Vertices

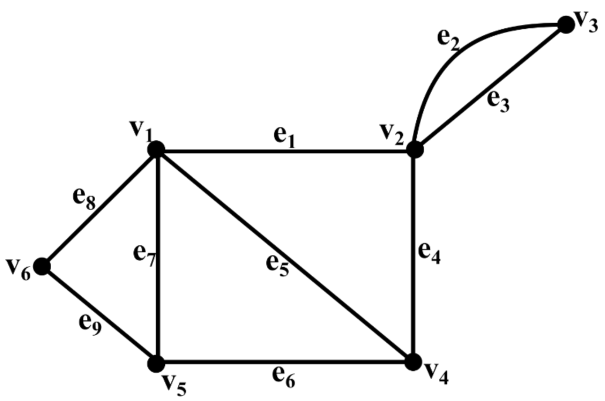
The number of edges incident with vertex V is called degree of a vertex or valency of a vertex and it is denoted by $d(v)$.

Isomorphism of Graph

A graph isomorphism from a graph G to a graph H is a pair (ϕ, θ), where $\phi : V(G) \rightarrow V(H)$ and $\theta : E(G) \rightarrow E(H)$ are bijection with a property that $I_G(e) = \{u, v\}$ and $I_H(\theta(e)) = \{\phi(u), \phi(v)\}$.

Walk

A walk in a graph G is an alternative sequence $W = v_0 v_1 e_1 v_2 e_2 \dots v_n e_n$ vertices and edges, beginning and ending with vertices where v_0 is the origin and v_n is the terminus of W.



$W = v_6 e_8 v_1 e_1 v_2 e_2 v_3 e_3 v_2 e_1 v_1$

Closed Walk

A walk to begin and ends at the same vertex is called a closed walk. That is, the walk W is closed if $v_0 = v_n$.

Open Walk

If the origin of the walk and terminus of the walk are different vertices, then it is called an open walk.

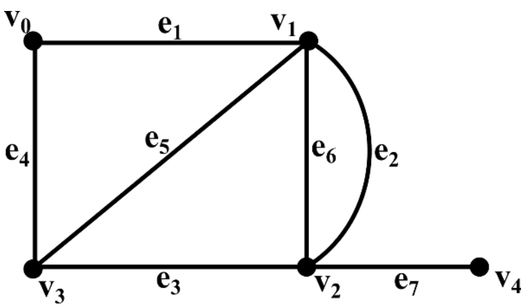
Trail

A walk is called a trail if all the edges in the walk are distinct.

Path

A walk is called a path if all the vertices are distinct.

Example:



$v_0 e_1 v_1 e_2 v_2 e_6 v_1 \rightarrow$ A trail

$v_0 e_1 v_1 e_2 v_2 e_3 v_3 \rightarrow$ A path

$v_0 e_1 v_1 e_2 v_2 e_3 v_3 e_5 v_1 \rightarrow$ A trail, but not a path

Euler's Theorem

The sum of the degrees of the vertices of a graph is equal to the twice the number of edges.

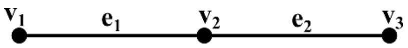
ie: $\sum_{i=1}^{n} d(v_i) = 2m$

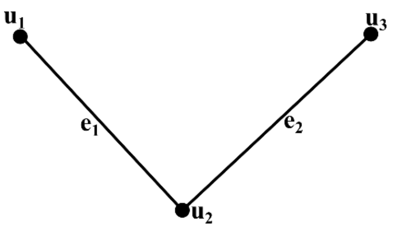
Isomorphic Graph

$I_H(\theta(e)) = \{\phi(u), \phi(v)\}$

A graph $G_1 = (V_1, E_1)$ is said to be isomorphic to graph $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and v_1 in G_1 then the corresponding edge e_2 in G_2 has its end vertices u_2 and v_2 in G_2. This correspondence is called a graph isomorphism.

Example:

$G =$ 

$H =$ 

ie: G and H are isomorphic.

Components

A connected component of a graph is a maximal connected subgraph. The term is also used for maximal subgraph or subset of a graph 's vertices that have some higher order of connectivity, including bi-connected components, tri-connected components and strongly connected components.

Tree

A connected graph without cycles is called a tree.

Vertex Cut

Let G be a connected graph. The set V' subset of $V(G)$ is called a Vertex cut of G, if $G - V'$ is a disconnected graph.

Cut Vertex

If $V' = \{v\}$ is a Vertex cut of the connected Graph G, then the vertex $\boldsymbol{v}$ is called a Cut vertex.

Edge Cut

Let G be a non-trivial connected graph with vertex set V and let S be a non-empty subset of V and $\bar{S} = V - S$. Let $E' = [S, \bar{S}]$ denote the set of all edges of G that have one end vertex is S and the other is $\bar{S}$. Then $G - E'$ is a disconnected graph and $E' = [S, \bar{S}]$ is called an edge cut of G.

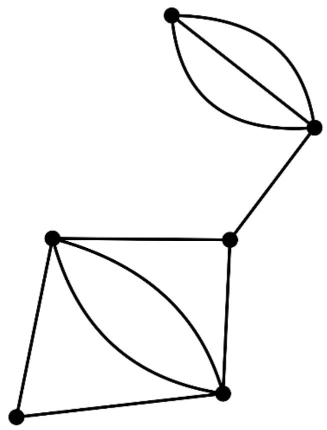
Cut Edge

If $E' = \{e\}$ is an edge cut of G then e is called a cut edge of G.

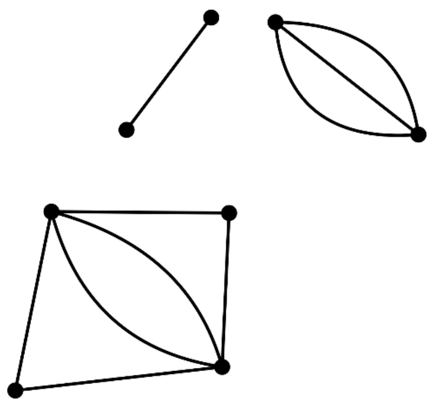
Block

A block is a Connected graph without any cut vertices.

Eg:



Graph G



Blocks of G

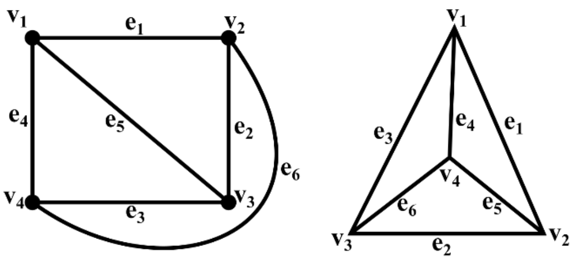
Chapter 2

PLANAR GRAPHS

Plane Graph

A plane graph is a graph drawn in the plane, such a way that any pair of edges meet only at their end vertices.

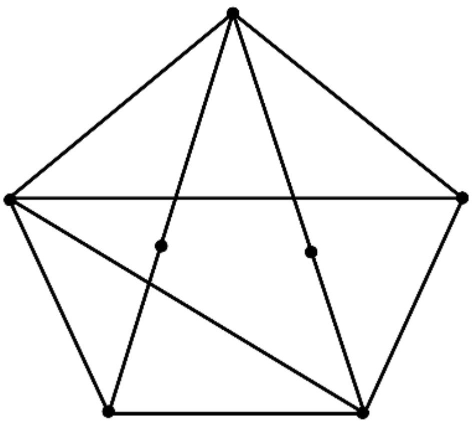
Example:



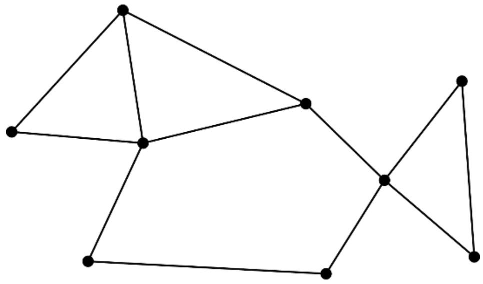
Planar Graph

A planar graph is a graph which is isomorphic to a plane graph, ie: it can be drawn as a plane graph.

A plane graph is a graph that can be drawn in the plane without any edge crossing.



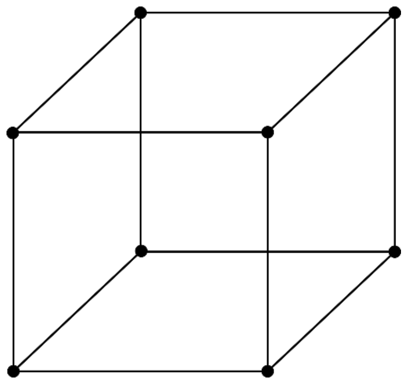
Example of Planar graph:



Planar Representation

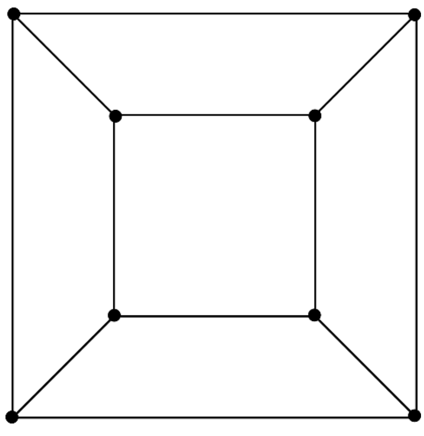
The pictorial representation of a planar graph as a plane graph is called a planar representation.

Eg: Is Q_3 shown below, planar?



The graph Q_3

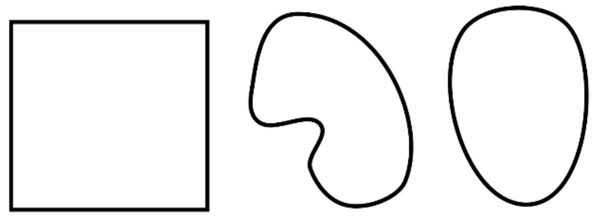
Planar representation of Q_3 is:



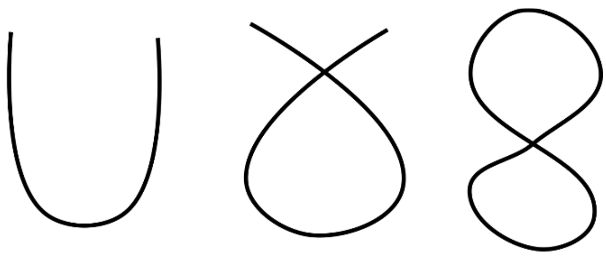
Jordan Curve

A Jordan Curve in the plane is a continuous non-self-intersecting curve where Origin and Terminals coincide.

Example:



Jordan Curves

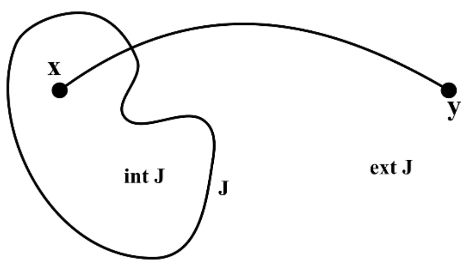


Non-Jordan Curves

Remark

If J is a Jordan Curve in the plane, then the part of the plane enclosed by J is called interior of J and is denoted by 'int J'. We exclude from 'int J' the points actually lying on J. Similarly, the part of the plane lying outside J is called the exterior of J and is denoted by 'ext J'.

Example:



Arc connecting point x in int J with point y in ext J.

Theorem

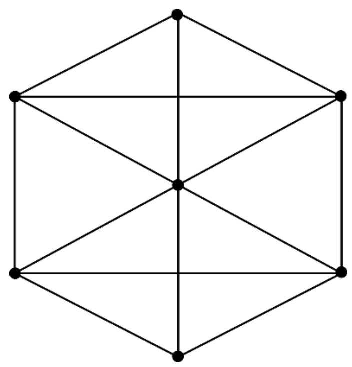
Let J be a Jordan Curve, if x is a point in int J and y is a point in ext J then any line joining x to y must meet J at some point, ie: must cross J. this is called Jordan Curve Theorem.

Boundary

The set of edges that bound a region is called its boundary.

Definition

A graph which is not planar is known as non-planar graph or a graph that cannot be drawn in the plane without any edge crossing is known as non-planar graph.



Theorem

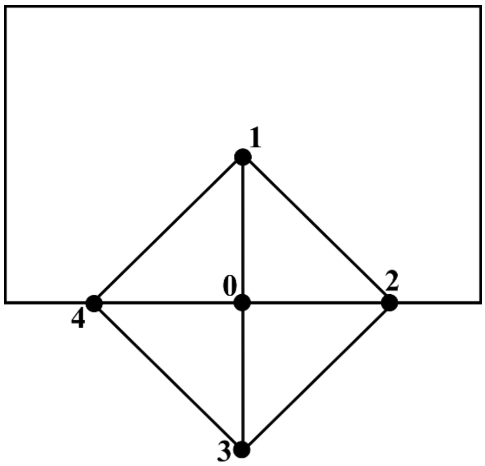
K_5 is nonplanar:

Every drawing of the complex graph K_5 in the plane (or sphere) contains at least one edge crossing.

Proof:

Label the vertices 0, 1, 2, 3, 4. By the Jordan Curve theorem any drawing of the cycle (1, 2, 3, 4, 1) separates the plane into two regions. Consider the region with

vertex 0 in its interior as the 'inside' of the circle. By the Jordan Curve theorem, the edges joining vertex 0 to each of its vertices 1, 2, 3 and 4 must also lie entirely inside the cycle, as illustrated below.



Drawing most of the K_5 in the plane

Moreover, each of the 3-cycles $\{0,1,2,0\}$, $\{0,2,3,0\}$, $\{0,3,4,0\}$ and $\{0,4,1,0\}$ also separates the plane and hence the edges $(2,4)$ must also lie to the exterior of the cycle $\{1,2,3,4\}$ as shown. It follows that the cycle formed by edges $(2,4)$, $(4,0)$ and $(0,2)$ separates the vertices 1 and 3, again by Jordan Curve theorem. Thus, it is impossible to draw edge $(1,3)$ without crossing an edge of that cycle. So, it is proven that the drawing of the K_5 in the plane contains at least one edge-crossing.

Theorem

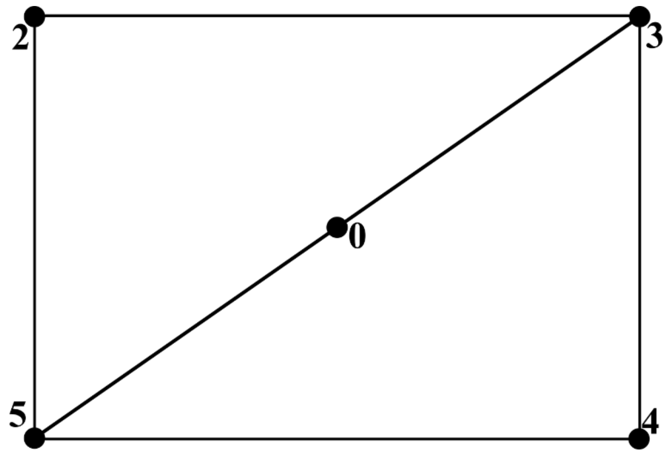
K_{33} is nonplanar:

Every drawing of the complete bipartite graph K_{33} in the plane (or sphere) contains at least one edge crossing.

Proof:

Label the vertices of one partite set 0, 2, 4 and of the order 1, 3, 5. By the Jordan Curve theorem, cycle $\{2,3,4,5,2\}$ separates the plane into two regions,

and as in the previous proof (K_5), we regard the region containing the vertex 0 as the 'inside' of the cycle. By the Jordan Curve theorem, the edges joining vertex 0 to each of the vertices 3 and 5 lie entirely inside that cycle, and each of the cycle $\{0, 3, 2, 5, 0\}$ and $\{0, 3. 4, 5, 0\}$ separates the plane, as illustrated below.



Drawing most of the K_{33} in the plane

Thus, there are 3 regions: the exterior of cycles $\{2, 3, 4, 5, 2\}$ and the inside of each of the other two cycles. It follows that no matter which region contains vertex 1, there must be some even numbered vertex that is not in that region, and hence the edge from vertex 1 to that even-numbered vertex would have to cross some cycle edge.

Corollary

Subgraph of a planar graph is planar.

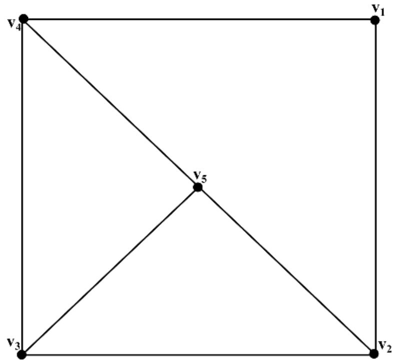
Definition

A plane graph partitions the plane into number of regions called faces.

Let G be plane graph. If x is a point on the plane which is not in G, ie: x is not a vertex of G or a point on any edge of G, then we define the faces of G containing x to be the set of all points on the plane which can be reached from x by a line which does not cross any edge of G or go through any vertex of G.

The number of faces of a plane graph G denoted by $f(a)$ or simply f.

Each plane graph has exactly one unbounded face called the exterior face.



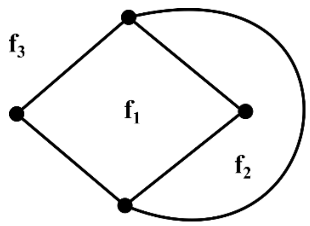
Here $f(G) = 4$

Degree of faces

The degree $d(f)$ of a face f is the number of edges with which it is incident, that is the number of edges in the boundary of a face.

Cut edge being counted twice.

Eg:



$d(f_2) = 3$

$d(f_1) = 4$

$d(f_3) = 3$

Theorem

A graph is planar if and only if each of its blocks is planar.

Proof:

If G is planar, then each of its blocks is planar since a subgraph of planar graph is planar.

Conversely, suppose that each block of G is planar. We now use induction on the number of blocks of G to prove the result. Without loss of generality, we

assume that G is connected. If G has only one block, then G itself is a block, and hence G is planar.

Now suppose G has k planar blocks and that the result has been proved for all connected graph having (k-1) planar blocks. Choose any end block B_0 of G and delete from G all the vertices of B_0 except the unique cut vertex, say v_0 of G in B_0. The resulting connected graph G` of G contains (k-1) planar blocks. Hence, by the induction hypothesis G` is planar. Let G˜` be plane embedded of G` such that v_0 belongs to the boundary of unbounded face, say f `. Let $B_0^{\sim}$ be a plane embedding of $B_0$ in $f$ `, so that v_0 is in the exterior face of $B_0^{\sim}$. Then G˜` and $B_0^{\sim}$ is a plane embedding of G.

Chapter 3

EULER'S FORMULA

Theorems

Euler Formula:

For a connected plain graph G, $n - m + f = 2$ where n, m, and f denote the number of vertices, edges and faces of G respectively.

Proof:

We apply the induction on f.

If $f = 1$ the G is a tree and $m = n - 1$.

Hence $n - m + f = 2$ and suppose that G has f faces.

Since $f \geq 2$, G is not a tree and hence contains a cycle C. Let e be an edge of C. Then e belongs to exactly 2 faces, say $f_1$and f_2 and the deletion of e from G results in the formation of a single face from $f_1$and f_2. Also, since e is not a cut edge of G. $G - e$ is connected.

Further the number of faces of $G - e$ is $f - 1$, number of edges in $G - e$ is $m - 1$ and number of vertices in $G - e$ is n. So, applying induction to $G - e$, we get $n - (m - 1) + (f - 1) = 2$ and this implies that $n - m + f = 2$. This completes the proof of theorem.

Corollary 1

All plane embedding of a planar graph have the same number of faces.

Proof:

Since $f = m - n + 2$ the number of faces depends only on n and m and not on the particular embedding.

Corollary 2

If G is a simple planar graph with at least 3 vertices, then $m \leq 3n - 6$.

Proof:

Without the generality we can assume that G is a simple connected plane graph. Since G is simple and $n \geq 3$, each face of G has degree at least 3. Hence if f denote the set of faces of G $\sum_{f \epsilon F} d(f) \geq 3f$. But $\sum_{f \epsilon F} d(f) = 2m$.

Consequently $2m \geq 3f$ so that $f \leq \frac{2m}{3}$.

By the Euler formula $m = n + f - 2$ now $f \leq \frac{2m}{3}$ implies $\mathrm{m} \leq \mathrm{n} + \left(\frac{2\mathrm{m}}{3}\right) - 2$. This gives. $m \leq 3n - 6$.

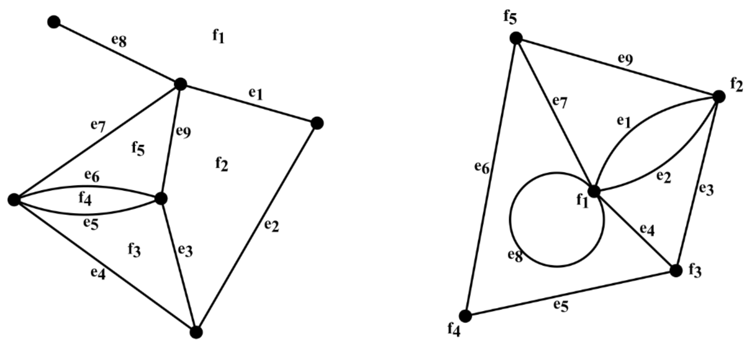
Chapter 4

DUAL OF A PLANE GRAPH

Definition

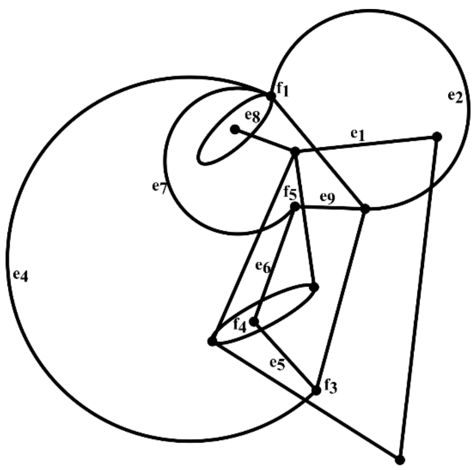
Let G be a plane graph. One can form out of G a new graph H in the following way corresponding to each face f(g), take the vertex f* and corresponding to each edge e(g), take an edge e*. Then edge e* joins vertices f* and g* in H iff edge e is common to the boundaries of faces f and g in G. The graph H is then called dual of G.

Example:



(a) (b)

Plane graph and its Dual



(c)

CONCLUSION

In this project we discussed the topic planar graph in graph theory.

We discussed about Euler formula and verified that some graphs are planar, and some are non-planar. A related important property of planar graphs, maps and triangulations is that they can be enumerated very nicely.

We also discussed about duality of a graph.in mathematical discipline of graph theory, the dual graph of a plane graph G is a graph that has a vertex of each face of G .it has many applications in mathematical and computational study.

In fact, graph theory is being used in our so many routine activities. For eg; using GPS or google maps to determine a route based on used settings.

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2) J. A. Bondy and U. S. R. Murty, Graph Theory with Great Britain: The Macmillian Press Ltd. 1976

DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS

Project report submitted to

The Kannur University

for the award of the degree

of

Bachelor of Science

by

MEGHA A

DB18CMSR06

Under the guidance of

MS. Sneha P Sebastian

Department of Mathematics
Don Bosco Arts and Science College
Angadikadavu

March 2021

Examiners 1: Examiner 2:

CERTIFICATE

It is to certify that this project report '**DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS**' is the bonafide project of **MEGHA A** who carried out the project under my supervision.

Mrs.Riya Baby	**Ms.Sneha P Sebastian**
Head of Department	**Supervisor**

Department of Mathematics

Don Bosco Arts and Science College

Angadikadavu

DECLARATION

I, **MEGHA A**, hereby declare that this project report entitled '**DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS**' is an original record of studies and bonafide project carried out by me during the period from November 2019 to March 2020, under the guidance of **Ms.Sneha P Sebastian**, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not been submitted by me elsewhere for the award of any degree, diploma, title or recognition, before.

MEGHA A

Department of Mathematics

Don Bosco Arts and Science College

Angadikadavu

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I sincerely express my deep sense of gratitude to all who have been of great help to me during the course of my dissertation. First and foremost I thank the almighty, for his blessing and protection during the period of this work. I express my thanks to Dr. Fr. Francis Karakkatt, Principal, for support in the completion of this dissertation. I express my gratitude to Ms.Sneha P Sebastian, my project supervisor, for the constant encouragement, valuable guidance and timely corrections, which made this work a success.

I am also indebted to all my classmates and friends who supported me throughout the study. I would like to express my thanks to my parents and dear ones for their constant encouragement and support. I also thank all those who helped me directly or indirectly to complete this project.

MEGHA A

CONTENTS

INTRODUCTION

In mathematics, a group is a set equipped with a binary operation that combines any two elements to form a third element in such a way that the three conditions called group axioms are satisfied , namely associativity , identity and invertability.

Let us take a moment to review our present stockpile of groups. Starting with finite groups, we have the cyclic group $\mathbb{Z}_n$,the symmetric group S_n , and the alternating group A_n for each positive integer n. We also have the dihedral group D_n and klein 4-group . Of course we know that subgroups of these groups exists. Turning to infinite groups , we have $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ under addition , and their non zero elements under multiplication we also have the group S_A of all permutation of an infinite set A , as well as various groups formed from matrices .

One purpose of this section is to show a way to use known groups as building blocks to form more groups. Given two groups G and H, it is possible to construct a new group from the cartesian product of G and H . Conversely , given a large group , it is sometimes possible to decompose the group ; that is , a group is sometimes isomorphic to the direct product of two smaller groups. Rather than studying a large group , it is often easier to study the component group of that group.

PRELIMINARY

Groups : A non empty set G together with an operation $*$ is said to be a group , denote by $(G\ ,*)$, if it satisfy the following axioms.

- Closure property
- Associative property
- Existence of identity
- Existence of inverse

Abelian group

A group $(G\ ,*)$ is said to be abelian if it satisfies commutative law .

Finite group

If the underlying set G of the group $(G\ ,*)$ consist of finite number of elements , then the group is finite group .

Infinite group

A group that is not finite is an infinite group .

Order of a group : The number of elements in a finite group is called the order of the group , denoted by $O(G)$.

Example

Show that the set of integers $\mathbb{Z}$ is a group with respect to the operation of addition of integers.

$$\mathbb{Z} = \{\dots\dots\dots.-3,-2,-1,0,1,2,3,\dots\dots\dots\}$$

Since the addition of two integers gives an integer , it satisfy closure property .

If $a, b, c \epsilon \mathbb{Z}$ then the $(a + b) + c = a + (b + c)$, hence associativity holds .

There is a number $0 \epsilon \mathbb{Z}$ such that $0 + a = a + 0$, hence identity exists

If $a \epsilon \mathbb{Z}$ then there exists $-a \epsilon \mathbb{Z}$, such that $-a + a = 0 = a + -a$

Therefore inverse exist .

Therefore $\mathbb{Z}$ is a group under addition .

Subgroup

A subset H of G is said to be a subgroup of G if H itself is a group under the same operation in G.

There are two different types of group structure of order 4 .

$\mathbb{Z}_4 = \{ 0,1,2,3\}$

Klein 4 – group , $V = \{e, a, b, c\}$

Cyclic group

A group G is cyclic if there is some element 'a' in G that generate G. And the element 'a' is called generator of G.

Group Homomorphism

A function $\Psi: G \rightarrow G'$ is a group homomorphism (or simply homomorphism).

If $\Psi(ab) = \Psi(a)\, \Psi(b)$ hold for all $a , b \in G$, is called homomorphism property .

Isomorphism

A one to one and onto homomorphism $\Psi: G \rightarrow G'$ is called an isomorphism .

CHAPTER – 1

DIRECT PRODUCT OF GROUPS

Definition

The Cartesian product of sets $S, S_2, \ldots \ldots, S_n$ is the set of all ordered n-tuples $(a_1, a_2, \ldots \ldots, a_n)$, where $a_i \in S_i$ for $i = 1,2,3, \ldots \ldots, n$. The Cartesian product is denoted by either

$S_1 \times S_2 \times \ldots \ldots \times S_n$ or by $\Pi_{i=1}^{n} S_i$.

Let $G_1, G_2, \ldots \ldots, G_n$ be groups and let us use multiplicative notation for all the group operations.

If we consider G_i as a set , $i = 1,2, \ldots \ldots n$. we have the products $G_1 \times G_2 \times \ldots \ldots, \times G_n$ we denote it by $\Pi_{i=1}^{n} G_i$. This product is called direct-product of groups. We can make $\Pi_{i=1}^{n} G_i$ into a group by means of a binary operation of multiplication by components.

Theorem

Let $G_1, G_2, \ldots \ldots, G_n$ be groups. For $(a_1, a_2, \ldots \ldots, a_n)$ and $(b_1, b_2, \ldots \ldots, b_n)$ in $\Pi_{i=1}^{n} G_i$ define ;

$$(a_1, a_2, \ldots \ldots, a_n)(b_1, b_2, \ldots \ldots, b_n) = (a_1 b_1, a_2 b_2, \ldots \ldots, a_n b_n)$$

Then $\Pi_{i=1}^{n} G_i$ is a group.

Proof

We have ,

$$\Pi_{i=1}^{n} G_i = \{(a_1, a_2, \dots \dots ., a_n) : a_i \in G_i\}$$

(1) **Closure property**

Let $(a_1, a_2, \dots \dots ., a_n), (b_1, b_2, \dots \dots ., b_n) \in \Pi_{i=1}^{n} G_i$

And we have ,

$$(a_1, a_2, \dots \dots ., a_n)(b_1, b_2, \dots \dots ., b_n) = (a_1 b_1, a_2 b_2, \dots \dots ., a_n b_n)$$

Here $a_i \in G_i$ and $b_i \in G_i$ for $i = 1,2, \dots \dots ., n$

$\because$ G_i is a group , $a_i b_i \in G_i$ for $i = 1,2, \dots \dots ., n$

$\Rightarrow (a_1 b_1, a_2 b_2, \dots \dots ., a_n b_n) \in \Pi_{i=1}^{n} G_i$

i.e. $\Pi_{i=1}^{n} G_i$ is closed under the binary operation.

(2) **Associativity**

Let $(a_1, a_2, \dots \dots ., a_n), (b_1, b_2, \dots \dots ., b_n), (c_1, c_2, \dots \dots ., c_n) \in \Pi_{i=1}^{n} G_i$

We have,

$$(a_1, a_2, \dots \dots ., a_n)(b_1, b_2, \dots \dots ., b_n)(c_1, c_2, \dots \dots ., c_n)$$

$$= (a_1 b_1 c_1 , a_2 b_2 c_2, \dots \dots ., a_n b_n c_n) \in \Pi_{i=1}^{n} G_i$$

$$[(a_1, a_2, \dots \dots ., a_n)(b_1, b_2, \dots \dots ., b_n)](c_1, c_2, \dots \dots ., c_n)$$

$$\begin{aligned}
&= [a_1 b_1, a_2 b_2, \dots \dots ., a_n b_n](c_1, c_2, \dots \dots ., c_n) \\
&= [(a_1 b_1) c_1, (a_2 b_2) c_2, \dots \dots ., (a_n b_n) c_n] \\
&= [a_1 (b_1 c_1), a_2 (b_2 c_2), \dots \dots ., a_n (b_n c_n)] \\
&= (a_1, a_2, \dots \dots ., a_n)[b_1 c_1, b_2 c_2, \dots \dots ., b_n c_n] \\
&= (a_1, a_2, \dots \dots ., a_n)[(b_1, b_2, \dots \dots ., b_n)(c_1, c_2, \dots \dots ., c_n)]
\end{aligned}$$

Hence associativity holds.

(3) **Existence of identity**

If e_i is the identity element in G_i.

Then,

$$(e_1, e_2, \dots \dots ., e_n) \in \Pi_{i=1}^{n} G_i$$

Also for ,

$$(a_1, a_2, \dots \dots ., a_n) \in \Pi_{i=1}^{n} G_i,$$

$$\begin{aligned}(a_1, a_2, \dots \dots ., a_n)(e_1, e_2, \dots \dots ., e_n) &= (a_1 e_1, a_2 e_2, \dots \dots ., a_n e_n) \\ &= (a_1, a_2, \dots \dots ., a_n)\end{aligned}$$

$\therefore$ $(e_1, e_2, \dots \dots ., e_n)$ is the identity element 'e' in $\Pi_{i=1}^{n} G_i$

(4) **Existence of inverse**

Let $(a_1, a_2, \dots \dots ., a_n) \in \Pi_{i=1}^{n} G_i$

Here $a_i \in G_i$ for $i = 1,2, \dots \dots ., n$.

Since G_i is a group ,

$\exists$ an inverse element a_i^{-1} in G_i : $a_i a_i^{-1} = e_i$ $\qquad$ $i = 1,2, \dots \dots ., n$

Clearly, $\qquad$ $(a_1^{-1}, a_2^{-1}, \dots \dots ., a_n^{-1}) \in \Pi_{i=1}^{n} G_i$ &

$$(a_1, a_2, \dots \dots ., a_n)(a_1^{-1}, a_2^{-1}, \dots \dots ., a_n^{-1}) = (e_1, e_2, \dots \dots ., e_n)$$

Hence $\Pi_{i=1}^{n} G_i$ is a group.

Note

If the operation of each G_i is a commutative. We sometimes use additive notation in $\Pi_{i=1}^{n} G_i$ and refer to $\Pi_{i=1}^{n} G_i$ as the direct sum of the group G_i. The notation $\oplus_{i=1}^{n} G_i$, especially with abelian groups with operation $+$.

The direct sum of abelian groups $G_1, G_2, \dots \dots, G_n$ may be written $G_1 \oplus G_2 \oplus \dots \dots \oplus G_n$

- Direct product of abelian group is abelian

Example

Q. Check whether $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic or not.

$$\mathbb{Z}_2 = \{0,1\}$$

$$\mathbb{Z}_3 = \{0,1,2\}$$

$$\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$$

Consider,

$1(1,1) = (1,1)$

$2(1,1) = (1,1) + (1,1) = (0,2)$

$3(1,1) = (1,1) + (1,1) + (1,1) = (1,0)$

$4(1,1) = 3(1,1) + (1,1) = (1,0) + (1,1) = (0,1)$

$5(1,1) = 4(1,1) + (1,1) = (0,1) + (1,1) = (1,2)$

$6(1,1) = 5(1,1) + (1,1) = (1,2) + (1,1) = (0,0)$

$\therefore$ $(1,1)$ is a generator of $\mathbb{Z}_2 \times \mathbb{Z}_3$

$\therefore$ $\mathbb{Z}_2 \times \mathbb{Z}_3$ is a cyclic group generated by $(1,1)$.

Q. Check whether $\mathbb{Z}_3 \times \mathbb{Z}_3$ is cyclic or not.

$$\mathbb{Z}_3 = \{0,1,2\}$$

$$\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$$

$1(0,1) = (0,1)$

$2(0,1) = (0,2)$

$3(0,1) = (0,3) = (0,0)$ $\quad\quad\therefore$ order $(0,1) = 3$

$1(0,2) = (0,2)$

$2(0,2) = (0,4) = (0,1)$

$3(0,2) = (0,6) = (0,0)$ $\quad\quad\therefore$ order $(0,2) = 3$

Every element added to itself three times gives the identity. Thus no element can generate the group. Hence $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not cyclic.

similarly $\mathbb{Z}_m \times \mathbb{Z}_m$ is not cyclic for any m.

Theorem

The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime, that is, the gcd of m and n is 1.

Proof

Suppose $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and isomorphic to $\mathbb{Z}_{mn}$.

To show that m and n are relatively prime.

Suppose not, let d be the gcd of m and n.

So that $d > 1$

Consider $\frac{mn}{d}$, which is an integer since $d|m$ and $d|n$

Let (r, s) be an arbitrary element of $\mathbb{Z}_m \times \mathbb{Z}_n$, add (r, s) repeatedly $\frac{mn}{d}$ times

$$\left.(r, s) + (r, s) +, \dots \dots ., +(r, s)\right\} \ \frac{mn}{d} times = (0,0)$$

$\therefore$ no element of $\mathbb{Z}_m \times \mathbb{Z}_n$ having order mn. $\therefore$ no element of $\mathbb{Z}_m \times \mathbb{Z}_n$ can generate $\mathbb{Z}_m \times \mathbb{Z}_n$ which is not possible. $\because$ $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic. Hence $gcd(m, n) = 1$.

i.e. m and n are relatively prime.

Conversely, suppose m and n are relatively prime, i.e. $\gcd(m, n) = 1$

To show that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic.

If $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic, then it is isomorphic to $\mathbb{Z}_{mn}$, $\because$ $\mathbb{Z}_m \times \mathbb{Z}_n$ has mn elements.

Consider the cyclic subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$ generated by the element $(1,1)$.The order of this cyclic subgroup is the smallest power of $(1,1)$,that gives the identity $(0,0)$. Here taking a power of $(1,1)$ in our additive notation will involve adding $(1,1)$ to itself repeatedly.

Consider $(1,1) + (1,1) +, \dots \dots ., +(1,1)$

If we add first coordinates m times , we get zero.

$\therefore$ order of first coordinate $= m$.

Similarly , Order of second coordinate $= n$.

The two coordinates together become zero. If we add them $lcm(m, n)$ times.

$\because$ $\gcd(m, n) = 1$, We get the $lcm = mn$.

i.e. $(1,1)$ generates a cyclic subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$ of order mn , which is the order of the whole group.

$$\Rightarrow \mathbb{Z}_m \times \mathbb{Z}_n = < (1,1) >$$

$\Rightarrow \mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic.

Corollary

The group $\prod_{i=1}^{n} \mathbb{Z}_{m_i}$ is cyclic and isomorphic to $\mathbb{Z}_{m_1 m_2 \ldots\ldots m_n}$ if and only if the numbers m_i for $i = 1,2, \ldots \ldots, n$ are such that the gcd of any two of them is 1.

Example

If n is written as a product of powers of distinct prime numbers , as in ,

$$n = (p_1)^{n_1} . (p_2)^{n_2} \ldots \ldots . (p_n)^{n_r}$$

Then $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{(p_1)^{n_1}} \times \mathbb{Z}_{(p_2)^{n_2}} \times \ldots \ldots . \times \mathbb{Z}_{(p_r)^{n_r}}$.

In particular , $\mathbb{Z}_{72}$ is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_9$.

Consider set of integers $\mathbb{Z}$, cyclic subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$, $n \in \mathbb{Z}$. Consider $2\mathbb{Z}$ and $3\mathbb{Z}$, then $< 2 > \cap < 3 > = < 6 >$

$\therefore$ if we take $r\mathbb{Z}$, $s\mathbb{Z}$ of $\mathbb{Z}$, then the $lcm(r, s)$ =generator of $< r > \cap < s >$

Using this we can define the lcm of the positive integers.

Definition

Let $r_1, r_2, \ldots \ldots, r_n$ be positive integers. Their least common multiple (abbreviated lcm) is the positive generator of the cyclic group of all common multiples of the r_i , that is the cyclic group of all integers divisible by each r_i for $i = 1,2, \ldots \ldots, n$.

Theorem

Let $(a_1, a_2, \dots \dots., a_n) \in \prod_{i=1}^{n} G_i$.

If a_i is of finite order r_i in G_i , then the order of $(a_1, a_2, \dots \dots., a_n)$ in $\prod_{i=1}^{n} G_i$ is equal to the least common multiple of all the r_i.

Proof

Given,

order of $a_1 = r_1 \Rightarrow a_1^{r_1} = e_1$ in G_1

order of $a_2 = r_2 \Rightarrow a_2^{r_2} = e_2$ in G_2

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order of $a_n = r_n \Rightarrow a_n^{r_n} = e_n$ in G_n.

We have to find a power k for $(a_1, a_2, \dots \dots., a_n)$.

So that $(a_1, a_2, \dots \dots., a_n)^k = (e_1, e_2, \dots \dots., e_n)$.

The power must simultaneously be a multiple of r_1 , multiple of r_2 and so on. But k is the least positive integers having the above property.

$$\therefore k = lcm(r_1, r_2, \dots \dots., r_n).$$

Q. Find the order of $(8,4,10)$ in the group $\mathbb{Z}_{12} \times \mathbb{Z}_{60} \times \mathbb{Z}_{24}$.

$O(8) = 3$ in Z_{12}

$O(4) = 15$ in Z_{60}

$O(10) = 12$ in Z_{24}

$O(8,4,10) = lcm(3,15,12) = 60$

Q. Find a generator of $\mathbb{Z} \times \mathbb{Z}_2$

$$\mathbb{Z} \times \mathbb{Z}_2 = \{(n, 0), (n, 1): n \in \mathbb{Z}\}$$
$$(n, 0) = n(1,0)$$
$$(n, 1) = (n, 0) + (0,1) = n(1,0) + (0,1)$$

$\therefore$ $\mathbb{Z} \times \mathbb{Z}_2$ is generated by $\{(1,0), (0,1)\}$

In general , $\mathbb{Z} \times \mathbb{Z}_n$ is generated by ,

$$\{(1,0,0, \dots \dots .,0), (0,1,0, \dots \dots .,0), \dots \dots ., (0,0, \dots \dots .,1)\}$$

Q. Find the order of (3,10,9) in $(\mathbb{Z}_4, \mathbb{Z}_{12}, \mathbb{Z}_{15})$

$O(3) = 4$ in $\mathbb{Z}_4$

$O(10) = 6$ in $\mathbb{Z}_{12}$

$O(9) = 5$ in $\mathbb{Z}_{15}$

$\therefore O(3,10,9) = lcm(4,6,5)$
$= 60$

$\therefore$ order of (3,10,9) in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$ is 60.

CHAPTER-2

FUNDAMENTAL THEOREM OF FINITELY GENERATED ABELIAN GROUPS

Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups in the form,

$$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \ldots \ldots . \times \mathbb{Z}_{(p_n)^{r_n}} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \ldots \ldots . \times \mathbb{Z}$$

Where the p_i are primes , not necessarily distinct and the r_i are positive integers.

Remark

- The direct product is unique except for possible rearrangement of the factors.
- The number of factors $\mathbb{Z}$ is unique and this number is called Betti number.

Example

Find all abelian groups , upto isomorphism of order

$1)8$, $2)16$, $3)360$

(1) Order 8

$$8 = 1 \times 8$$

$$8 = 2 \times 4 = 2 \times 2^2$$

$$8 = 2 \times 2 \times 2$$

3 non-isomorphic groups are $\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

(2) Order 16

$16 = 1 \times 16 = 1 \times 2^4$

$16 = 2 \times 8 \quad = 2 \times 2^3$

$16 = 4 \times 4 \quad = 2^2 \times 2^2$

$16 = 2 \times 2 \times 2 \times 2$

$16 = 2 \times 2 \times 2^2$

$\mathbb{Z}_{16}, \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$

(3) Order 360

$360 = 2^2 \cdot 3^2 \cdot 5$

Possibilities are,

1) $\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
2) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
3) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
4) $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
5) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
6) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

Definition

A group G is decomposable if it is isomorphic to a direct product of two proper non-trivial subgroups , otherwise G is indecomposable.

Example

$\mathbb{Z}_6$ is decomposable while $\mathbb{Z}_5$ is indecomposable.

$\mathbb{Z}_6$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$

$\mathbb{Z}_{mn}$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$, if m and n are prime.

Theorem

The finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.

Proof

Let G be a finite indecomposable abelian group $\because G$ is finitely generated , we can apply fundamental theorem of finitely generated abelian groups.

$\therefore G \;\cong \mathbb{Z}_{(p)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \dots \dots . \times \mathbb{Z}_{(p_n)^{r_n}}$

$\because G$ is indecomposable and $\mathbb{Z}_{(p_i)^{r_i}}$'s are proper subgroups we get in the above , there is only one factor say $\mathbb{Z}_{(p_i)^{r_i}}$ which is cyclic group with order a prime power.

Theorem

If m divides the order of a finite abelian group , then G has a subgroup of order m.

Proof

Given G is a finite abelian group.

$\therefore$ we can apply Fundamental Theorem ,

Hence,

$$G \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \dots \dots . \times \mathbb{Z}_{p_n^{r_n}}$$

Here all primes need not be distinct.

Then,

$$O(G) = p_1^{r_1} . p_2^{r_2} \dots \dots p_n^{r_n}$$

Let m is a $+ve$ integer which divides $O(G)$.

$0 \le s_i \le r_i$ By theorem , " let G be a cyclic group with n elements and generated by a. Let $b \in G$, $b = a^s$, then ' b' generates a cyclic subgroup H of G containing $\frac{n}{d}$ elements , where $d = \gcd(n, s)$."

$p_i^{r_i - s_i}$ generates a cyclic subgroup of $\mathbb{Z}_{p_i^{r_i}}$ having order $\frac{p_i^{r_i}}{gcd(p_i^{r_i}, p_i^{r_i - s_i})}$

$$= \frac{p_i^{r_i}}{p_i^{r_i - s_i}} = p_i^{s_i}$$

$$\therefore O(< p_i^{r_i - s_i} >) = p_i^{s_i}$$

i.e. $< p_1^{r_1 - s_1} >$ is a subgroup of $\mathbb{Z}_{p_1^{r_1}}$ having order $p_1^{s_1}$.

$< p_2^{r_2 - s_2} >$ is a subgroup of $\mathbb{Z}_{p_2^{r_2}}$ having order $p_2^{s_2}$.

………………………………………………………………

$< p_n^{r_n - s_n} >$ is a subgroup of $\mathbb{Z}_{p_n^{r_n}}$ having order $p_n^{s_n}$.

$\therefore < p_1^{r_1 - s_1} > \times < p_2^{r_2 - s_2} > \times \dots \dots . \times < p_n^{r_n - s_n} >$ is a subgroup of $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \dots \dots . \times \mathbb{Z}_{p_n^{r_n}}$ having order $p_1^{s_1} \cdot p_2^{s_2} \cdots\cdots p_n^{s_n} = m$.

Theorem

If m is a square free integer , that is m is not divisible by the square of any prime . Then every abelian group of order m is cyclic.

Proof

Let m be a square free integer , then $p^i \nmid m$ for every i greater than 1 for a prime p.

Given G is a finite abelian group having order m , by fundamental theorem , then

$$G \cong \mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \dots \dots . \times \mathbb{Z}_{(p_n)^{r_n}}$$

Then,

$$O(G) = p_1^{r_1} \cdot p_2^{r_2} \cdots\cdots p_n^{r_n}$$

$\because$ $O(G)$ is a square free integer , the only possibility

$$r_1 = r_2 = \dots \dots .. = r_n = 1$$

Then,

$$G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \dots . \times \mathbb{Z}_{p_n}$$

$$\cong \mathbb{Z}_{p_1, p_2, \dots\dots, p_n}$$, which is cyclic.

Example

15 is a square free integer. So an abelian group of order 15 is cyclic.

CONCLUSION

Direct product of groups is the product $G_1 \times G_2, \dots \dots . G_n$,where each G_i is a set. We have discussed about definition and some properties related to the direct product of groups. The fundamental theorem of finitely generated abelian group helped us to get a deeper understanding about the topic. The theorems gives us complete structural information about abelian group, in particular finite abelian group. We have also discussed some examples in order to develope more intrest in algebra.

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POWER SERIES SOLUTIONS AND SPECIAL FUNCTIONS

Project report submitted to
The Kannur University
for the award of the degree
of

Bachelor of Science

by

MERIN K JOHN

DB18CMSR23

Under the guidance of

Ms. Athulya P

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CERTIFICATE

Certified that this project **'Power Series'** is a bona fide project of **MERIN K JOHN** carried out the project work under my supervision.

Mrs. Riya Baby
Head of Department

Ms. Athulya P
Supervisor

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DECLARATION

I **MERIN K JOHN** hereby declare that the project **'Power Series'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Athulya P, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

Name

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ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express my deepest gratitude to people along the way.

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My greatest debt is always, to God Almighty.

CONTENTS

INTRODUCTION

A power series is a type of series with terms involving a variable. Power series are often used by calculators and computers to evaluate trigonometric, hyperbolic, exponential and logarithm functions. So any application of these kind of functions is a possible application of power series. Many interesting and important differential equations can be found in power series.

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PRELIMINERY

A. An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{1}$$

is called a *power series in x*. The series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

is a *power series in* $x - x_0$.

B. The series (1) is said to *converge* at a point *x* if the limit

$$\lim_{m \to \infty} \sum_{n=0}^{m} a_n x^n$$

exists, and in this case the *sum* of the series is the value of this limit.

Radius of convergence: Series in x has a radius of convergence R, where $0 \le R \le \infty$, with the property that the series converges if $|x| < R$ and diverges if $|x| > R$. It should be noted that if $R = 0$ then no x satisfies $|x| < R$, and if $R = \infty$ then no x satisfies $|x| > R$

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \text{, if the limit exists.}$$

C. Suppose that (1) converges for $|x| < R$ with $R > 0$, and denote its sum by $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Then $f(x)$ is automatically continuous and has derivatives of all orders for $|x| < R$.

D. Let $f(x)$ be a continuous function that has derivatives of all orders for $|x| < R$ with $R > 0$. $f(x)$ be represented as power series using *Taylor's formula*:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x)$$

where the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(\bar{x})}{(n+1)!} x^{n+1}$$

for some point $\bar{x}$ between 0 and x.

E. A function $f(x)$ with the property that a power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is valid in some neighbourhood of the point x_0 is said to be *analytic* at x_0. In this case the a_n are necessarily given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

and is called the *Taylor series* of $f(x)$ at x_0.

Analytic functions: A function f defined on some open subset U of R or C is called analytic if it is locally given by a convergent power series. This means that every a ∈ U has an open neighbourhood V ⊆ U, such that there exists a power series with centre a that converges to f(x) for every x ∈ V.

CHAPTER 1

SERIES SOLUTION OF FIRST ORDER EQUATION

We have studied to solve linear equations with constants coefficient but with variable coefficient only specific cases are discussed. Now we turn to these latter cases and try to find a general method to solve this. The idea is to assume that the unknown function y can be explained into a power series. Our purpose in this section is to explain the procedures by showing how it works in the case of first order equation that are easy to solve by elementary methods.

Example 1: we consider the equation

$$y' = y$$

Consider the above equation as (1). Assume that y has a power series solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Which converges for $|x| < R$, $R > 0$

That is we assume that $y' = y$ has a solution that is analytic at origin. We have

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots\cdots$$

then

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots \ldots \ldots$$

$$\therefore (1) \Rightarrow a_1 + 2a_2 x + 3a_3 x^2 \cdots$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$\Rightarrow a_1 = a_0$

$2a_2 = a_1 \Rightarrow \qquad a_2 = \dfrac{a_1}{2} = \dfrac{a_0}{2}$

$3a_3 = a_2 \Rightarrow \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3} = \frac{a_0}{3!}$

$4a_4 = a_3 \Rightarrow \qquad a_4 = \frac{a_3}{4} = \frac{a_0}{2 \cdot 3 \cdot 4} = \frac{a_0}{4!}$

$\therefore$ we get

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{3!} x^3 + \frac{a_0}{4!} x^4 + \cdots$$

$$= a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right)$$

$$y = a_0 e^x$$

To find the actual function we have $y' = y$

$$\text{i.e.,} \quad \frac{dy}{dx} = y \quad \Rightarrow \quad \frac{dy}{y} = dx$$

integrating

$$\log\, y = x + c$$

$$\text{i.e.,} \quad y = e^{x+c} = e^x \cdot e^c$$

$y = a_0 e^x$, where $a_0 = e^c$, a constant.

Example 2: solve $y' = 2xy$. Also find its actual solution.

<u>Solution:</u> $\qquad y' = 2xy \qquad (1)$

Assume that y has a power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Which converges for $|x| < R, R > 0$

We have

$$y = \sum_{n=0}^{\infty} a_n \, x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= a_1 + 2a_2x + 3a_3x^2 + \cdots$$

Then (1) $\Rightarrow a_1 + 2a_2x + 3a_3x^2 + \cdots = 2x(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)$

$$= 2xa_0 + 2xa_1x + 2xa_2x^2 + 2xa_3x^3 + \cdots$$

$$= 2xa_0 + 2a_1x^2 + 2a_2x^3 + 2a_3x^4 + \cdots \ldots..$$

$\Rightarrow a_1 = 0 \qquad 2a_2 = 2a_0 \Rightarrow a_2 = \frac{2a_0}{z} = a_0$

$$3.a_3 = 2a_1 \Rightarrow a_3 = \frac{2a_1}{3} = 0$$

$$4a_4 = 2a_2 \Rightarrow a_4 = \frac{2a_2}{42} = \frac{a_0}{2}$$

$$5a_5 = 2a_3 = 0 \Rightarrow a_5 = 0$$

$$6a_6 = 2a_4 \Rightarrow a_6 = \frac{2a_4}{6} = \frac{a_4}{3} = \frac{a_0}{2\cdot 3} = \frac{a_0}{3!}$$

We get,

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ &= a_0 + 0 + a_0x^2 + 0x^3 + \frac{a_0}{2}x^4 + \cdots \\ &= a_0 + a_0x^2 + \frac{a_0}{2}x^4 + \cdots \\ &= a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) \end{aligned}$$

$$y = a_0e^{x^2}$$

To find an actual solution

$$y' = 2xy$$

$$\frac{dy}{dx} = 2xy$$

$$\Rightarrow \qquad \frac{dy}{y} = 2x \cdot dx$$

$$\log y = x^2 + c$$

$$y = e^{x^2} + c$$

$$\Rightarrow y = a_0e^{x^2}, \text{ where } a_0 = e^c$$

Example 3: Consider $y = (1+x)^p$ where p is an arbitrary constant. Construct a differential equation from this and then find the solution using power series method.

Solution

First, we construct a differential equation

i.e. $y = (1+x)^p$

$$y' = p(1+x)^{p-1} = \frac{p(1+x)^p}{1+x} = \frac{py}{1+x}$$

$$\therefore (1+x)y' = py, \quad y(0) = r$$

Assume that y has a power series solution of the form,

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + \dot{a}_2 x^2 + \cdots \ldots \ldots$$

Which converges for $|x| < \dot{R}, \quad R > 0$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots \ldots \ldots ..$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Then $(1+x)y' = py$

$$\Rightarrow (1+x)a_1 + 2a_2 x + 3a_3 x^2 + \cdots = p(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$\Rightarrow (a_1 + 2a_2 x + 3a_3 x^2 + \cdots) + (a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots)$$

$$= a_0 p + a_1 p x + a_2 p x^2 + \cdots$$

Equating the coefficients of $x, x^2, \ldots$

$a_1 = a_0 p$ i.e. $a_1 = p$, (since $a_0 = 1$)

$$\Rightarrow 2a_2 = a_1(\mathrm{p} - 1)$$

$$a_2 = \frac{a_1(\mathrm{p}-1)}{2} = \frac{a_0 P(p-1)}{2}$$

$$3a_3 + 2a_2 = a_2 p$$
$$sa_3 = a_2 p - 2a_2$$
$$= a_2(p-2)$$
$$a_3 = \frac{a_2(p-2)}{3} = \frac{a_0 p(p-1)(p-2)}{2\cdot 3}$$

$$4a_4 + 3a_3 = a_3 p$$
$$4a_4 = a_3 p - 3a_3$$
$$= a_3(p-3)$$
$$a_4 = \frac{a_3(p-3)}{4} = \frac{a_0 p(p-1)(p-2)(p-3)}{2\cdot 3\cdot 4}$$

∴ we get,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= a_0 + a_0 px + \frac{a_0 p(p-1)}{2}x^2 + \frac{a_0 p(p-1)(p-2)}{2\cdot 3}x^3 + \cdots \ldots$$
$$= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 +$$
$$\frac{p(p-1)(p-2)(p-3)}{4!}x^4 + \cdots + \frac{p(p-1)(p-2)(p-(n-1))}{n!}x^n$$

Since the initial problem y(0) = 1 has one solution the series converges for |x|<1

So this is a power solution,

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!}x^n$$

Which is binomial series.

Example 4: Solve the equation $y' = x - y,\ y\,(0) = 0$

Solution: Assume that y has a power series solution of the form

$$y = \sum_{n=0}^{\infty} \text{an } x^n$$

which converges for $|x| < R, R > 0$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Now $y' = x - y$

$$(a_1 + 2a_2x + 3a_3x^2 + \cdots) = x - (a_0 + a_1x + a_2x^2 + \cdots)$$

Equating the coefficients of x, x^2,

$$\begin{aligned} a_1 &= -a_0 = 0, \quad \text{Since} \quad y(0) = 0 \\ 2a_2 &= 1 - a_1 \\ &= 1 - 0 \end{aligned}$$

$$\Rightarrow a_2 = \frac{1}{2}$$

$$3a_3 = -a_2$$

$$a_3 = \frac{-a_2}{3} = -\frac{1}{2 \cdot 3}$$

$$4a_4 = -a_3$$

$$\Rightarrow a_4 = \frac{1}{2 \cdot 3 \cdot 4}$$

$$\therefore y = 0 + 0 + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \dots \dots$$

$$= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) + x - 1$$

$$= e^{-x} + x - 1$$

By direct method

$$y' = x - y$$

$$\frac{dy}{dx} = x - y \;\Rightarrow \frac{dy}{dx} + y = x \qquad \left(\frac{dy}{dx} + py = Q\ form\right)$$

here $P(x) = 1$, integrating factor

$$= e^{\int p(x) \cdot dx}$$

$$= e^x$$

$$\therefore ye^x = \int xe^x \cdot dx$$

$$ye^x = x \cdot e^x - \int e^x \cdot dx$$

$$= xe^x - e^x$$

$$ye^x = e^x(x - 1) + c$$

$$y = \frac{e^x(x - 1) + c}{dx} = x - 1 + \frac{c}{e^x} = ce^{-x} + (x - 1)$$

$$\therefore y = (x - 1) + ce^{-x}$$

CHAPTER 2

SECOND ORDER LINEAR EQUATION, ORDINARY POINTS

Consider the general homogeneous second order linear equation,

$$y'' + P(x)y' + Q(x)y = 0 \qquad (1)$$

As we know, it is occasionally possible to solve such an equation in terms of familiar elementary functions. This is true, for instance, when $P(x)$ and $Q(x)$ are constants, and in a few other cases as well. For the most part, however, the equations of this type having the greatest significance in both pure and applied mathematics are beyond the reach of elementary methods, and can only be solved by means of power series.

P(x) and Q(x) are called coefficients of the equation. The behaviour of its solutions near a point x_0 depends on the behaviour of its coefficient functions $P(x)$ and $Q(x)$ near this point. we confine ourselves to the case in which $P(x)$ and $Q(x)$ are well behaved in the sense of being analytic at x0, which means that each has a power series expansion valid in some neighbourhood of this point. In this case x0 is called an *ordinary point* of equation (1). Any point that is not an ordinary point of (1) is called a *singular point.*

Consider the equation,

$$y'' + y = 0 \qquad (2)$$

the coefficient functions are $P(x) = 0$ and $Q(x) = 1$, These functions are analytic at all points, so we seek a solution of the form,

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \qquad (3)$$

Differentiating (3) we get,

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots \qquad (4)$$

And

$$y'' = 2a_2 + 2\cdot 3a_3x + 3\cdot 4a_4x^2 + \cdots + (n+1)(n+2)a_{n+2}x^n + \cdots \quad (5)$$

If we substitute (5) and (3) into (2) and add the two series term by term, we get

$$y'' + y = \begin{matrix} (2a_2 + a_0) + (2\cdot 3a_3 + a_1)x + (3\cdot 4a_4 + a_2)x^2 + \\ (4\cdot 5a_5 + a_3)x^3 \ + \cdots + [(n+1)(n+2)a_{n+2} + a_n]x^n + \cdots \end{matrix} = 0$$

and equating to zero the coefficients of successive powers of *x* gives

$$2a_2 + a_0 = 0, \qquad 2\cdot 3a_3 + a_1 = 0, \qquad 3\cdot 4a_4 + a_2 = 0$$

$$4\cdot 5a_5 + a_3 = 0, \dots\dots, \qquad (n+1)(n+2)a_{n+2} + a_n = 0, \dots$$

By means of these equations we can express a_n in terms of a_0 or a_0, according as *n* is even or odd:

$$a_2 = -\frac{a_0}{2}, \qquad a_3 = -\frac{a_1}{2\cdot 3}, \qquad a_4 = -\frac{a_2}{3\cdot 4} = \frac{a_0}{2\cdot 3\cdot 4}$$

$$a_5 = -\frac{a_3}{4\cdot 5} = \frac{a_1}{2\cdot 3\cdot 4\cdot 5}, \cdots$$

With these coefficients, (3) becomes

$$y = a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{2\cdot 3}x^3 + \frac{a_0}{2\cdot 3\cdot 4}x^4 + \frac{a_1}{2\cdot 3\cdot 4\cdot 5}x^5 - \cdots$$

$$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \qquad (6)$$

$i.e,\qquad y = a_0\cos\, x + a_1\sin\, x$

Since each of the series in the parenthesis converges for all *x*. This implies the series (2) for all *x*.

Solve the legenders equation,

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0$$

Solution

Consider $(1 - x^2)y'' - 2xy' + p(p+1)y = 0$ as equation (1)

Assume that y has a power series solution of the form

$$y = \sum a_n x^n$$

Which converges $|x| < R, R > 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

put $n = n + 2$ (Since y'' is not x^n form)

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+2-1) a_{n+2} x^{n+2-2}$$

$$\therefore y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

Now (1) $\Rightarrow$ $\quad y'' - x^2 y'' - 2xy' + p(p+1)y = 0$

$\Rightarrow \sum (n+1)(n+2) a_{n+2} x^n - \sum n(n-1) a_n x^n - \sum 2n a_n x^n + \sum p(p+1) a_n x^n = 0$

$$\Rightarrow \sum_{n=0}^{\infty} [((n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + p(p+1) a_n) x^n] = 0$$

for n = 0,1,2,3…….

$\Rightarrow (n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + p(p+1) a_n = 0$

$$\Rightarrow a_{n+2} = \frac{[n(n-1) + 2n - p(p+1)]}{(n+1)(n+2)} a_n$$

$$= \frac{(n^2 - n + 2n - p^2 - p) a_n}{(n+1)(n+2)}$$

$$= \frac{(n^2 + n - p^2 - p) a_n}{(n+1)(n+2)}$$

$$\therefore a_{n+2} = \frac{-(p-n)(p+n+1)}{(n+1)(n+2)} a_n , \qquad n = 0,1,2 \ldots$$

This is an Recursion formula

put $n = 0$, $\quad a_2 = \dfrac{-p(p+1)}{1 \cdot 2} a_0$

$n = 1$, $\quad a_3 = \dfrac{-(p-1)(p+2)}{2 \cdot 3} \cdot a_1$

$n = 2$, $\quad a_4 = \dfrac{-(p-2)(p+3)}{3i4} a_2$

$$= \frac{p(p-2)(p+1)(p+3)}{4!} a_0$$

$n = 3$, $\quad a_5 = \dfrac{-(p-3)[p+4)}{4 \cdot 5} a_3$

$$= \frac{(p-1)(p-3)(p+2)(p+4)}{5!} a_1$$

$n = 4$, $\quad a_6 = \dfrac{-(p-4)(p+5)}{5 \cdot 6} a_4$

$$= \frac{-p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} a_0$$

$n = 5$, $\quad a_7 = -\dfrac{(p-5)(p+6)}{6 \cdot 7} a_5$

$$= -\frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} a_1$$

$$y = a_0 \left[1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p-2)(p+1)(p+3)}{4!} x^4 - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} x^6 + \cdots \right]$$
$$+ a_1 \left[x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!} x^5 - \frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} x^7 + \cdots \right]$$

Find the general solution of $(1 + x^2)y'' + 2xy' - 2y = 0$ in terms of power series in x. Can you express this solution by means of elementary functions?

Solution

Consider the equation $(1 + x^2)y'' + 2xy' - 2y = 0$ as equation (1)

Assume that y has a power series solution of the form

$$y = \sum a_n x^n$$

Which converges $|x| < R, R > 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1 + x^2)y'' = y'' + x^2 y''$$

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$\text{Now } y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

put $n = n + 2$

$$\Rightarrow \quad \sum_{n=0}^{\infty} (n+2)(n+2-1)a_n + 2x^{n+2=2}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$(1) \Rightarrow \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x_n$$
$$+ \sum_{n=1}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\Rightarrow \sum[((n+1)(n+2)a_{n+2} + n(n-1)a_n + 2na_n - 2a_n)x^n] = 0$$

$$\Rightarrow (n+1)(n+2)a_{n+2} + n(n-1)a_n + 2na_n - 2a_n = 0$$

$$a_{n+2} = \frac{[-n(n-1) - 2n + 2]}{(n+1)(n+2)} a_n$$
$$= \frac{(-n^2 + n - 2n + 2)}{(n+1)(n+2)} a_n$$

put $n = 0$, $\quad a_2 = \frac{2}{1 \cdot 2} a_0 = \frac{2a_0}{2!} = a_0$

$n = 1$, $\quad a_3 = \frac{(1 - 1 - 2 + 2)}{2 \cdot 3} a_1 = 0$

$n = 2$, $\quad a_4 = \frac{2 - 4 - 4 + 2}{3 \cdot 4} a_2 \quad = \frac{-4}{3 \cdot 4} a_0 = \frac{-a_0}{3}$

$n = 3$, $\quad a_5 = \frac{3 - 9 - 16 + 2}{4.5} a_3 \quad = 0$

$n = 4$, $\quad a_6 = \frac{4 - 16 - 8 + 2}{5.6} a_4 \quad = \frac{-3}{5} a_4 \quad = \frac{3a_0}{3.5} = \frac{a_0}{5}$

$$\therefore y = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$= a_0 + a_1 x + a_0 x^2 - \frac{a_0}{3} x^4 + \frac{a_0}{5} x^6 \ldots.$$
$$= a_0 \left[1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \cdots \right] + a_1 x$$
$$= a_0 \left[1 + x\left(x - \frac{x^3}{3} + \frac{x^5}{5} \cdots \right)\right] + a_1 x$$
$$= a_0(1 + x\tan^{-1} x) + a_1 x$$

Consider the equation $y'' + xy' + y = 0$

(a) Find its general solution $y = \sum a_n x^n$ in the form $y = a_0 y_1(x) + a_1 y_2(x)$ where $y_1(x)$ and $y_2(x)$ are power series

(b) use the ratio test to verify that the two series $y_1(x)$ and $y_2(x)$ converges for all x.

Solution:

Given $y'' + xy' + y = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$

Assume that y has a power series solution the form $\sum a_n x^n$ which converges for $|x| \quad R > 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n$$

$$(1) \Rightarrow \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum \left[\left((n+1)(n+2) a_{n+2} + n a_n + a_n\right) x^n\right] = 0$$

$$\Rightarrow (n+1)(n+2) a_{n+2} + n a_n + a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{(-n-1) a_n}{(n+1)(n+2)} = \frac{-a_n}{n+2}$$

put $n = 0, a_2 = -\frac{a_0}{2}$

$n = 1, a_3 = \frac{-2a_1}{2 \cdot 3} = \frac{-a_1}{3}$

$$n = 2, \quad a_4 = \frac{-3a_2}{3 \cdot 4} = \frac{-a_2}{4} = \frac{a_0}{8}$$
$$n = 3, \quad a_5 = \frac{-4a_3}{4 \cdot 5} = \frac{a_1}{15}$$
$$n = 4, \quad a_6 = \frac{-5a_4}{5 \cdot 6} = \frac{-a_0}{48}$$

$\therefore$ we get $$y = a_0 + a_1 x + -\frac{a_0}{2}x^2 - \frac{a_1}{3}x^3 + \frac{a_0}{8}x^4 + \frac{a_1}{15}x^5 - \frac{a_0}{48}x^6 + \cdots$$

$$= a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots\right] + a_1\left[x - \frac{x^3}{3} + \frac{x^5}{3.5} + \cdots\right]$$

where $$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{\dot{x}^2}{2 \cdot 4 \cdot 6} +$$

$$y_2(x) = x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \cdots$$

(b) $$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(-1)^n}{2 \cdot 4 \cdot (2n)} \Big/ \frac{(-1)^{n+1}}{2 \cdot 4 \cdot\cdot (2n+2)}\right|$$
$$= \lim_{n\to\infty}\left|\frac{2(n+1)}{-1}\right|$$
$$= \lim_{n\to\infty}\left|-2n(1 + \frac{1}{n})\right| = \infty$$

$\therefore y_1(x)$ converges for all x

$$y_2(x) = x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} - \cdots$$

$$R = \lim_{n\to\infty} \left| \frac{(-1)^n}{3\cdot 5 \cdots (2n+1)} \Big/ \frac{(-1)^{n+1}}{3\cdot 5\cdot \cdots (2n+3)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)\cdot 3\cdot 5 \cdots (2n+1)(2n+3)}{3\cdot 5 \cdot\cdots\cdot (2n+1)} \right|$$

$$= \lim_{n\to\infty} |(-1)n(2+3/n)| = \infty$$

$\therefore y_2(x)$ converges for all x

REGULAR SINGULAR POINTS

A singular point x_0 of equation

$$y'' + P(x)y' + Q(x)y = 0$$

is said to be regular if the functions $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic, and irregular otherwise. Roughly speaking, this means that the singularity in $P(x)$ cannot be worse than $1/(x - x_0)$, and that in $Q(x)$ cannot be worse than $1/(x - x_0)^2$.

If we consider Legendre's equation in the form

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$$

it is clear that $x = 1$ and $x = -1$ are singular points. The first is regular because

$$(x-1)P(x) = \frac{2x}{x+1} \text{ and } (x-1)^2Q(x) = -\frac{(x-1)p(p+1)}{x+1}$$

are analytic at $x = 1$, and the second is also regular for similar reasons.

Example: *Bessel's equation of order p*, where *p* is a nonnegative constant:

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

If this is written in the form

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0,$$

it is apparent that the origin is a regular singular point because $xP(x) = 1$ and $x^2Q(x) = x^2 - p^2$ are analytic at $x = 0$.

CONCLUSION

The purpose of this project gives a simple account of series solution of first order equation, second order linear equation, ordinary points. The study of these topics given excellent introduction to the subject called ‘POWER SERIES’

we used application of power series extensively throughout this project. We take it for granted that most readers are reasonably well acquainted with these series from an earlier course in calculus. Nevertheless, for the benefit of those whose familiarity with this topic may have faded slightly, we presented a brief review of the main facts of power series.

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NORMED LINEAR SPACES

Project report submitted to
The Kannur University
for the award of the degree
of

Bachelor of Science

by

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DB18CMSR07

Under the guidance of

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CERTIFICATE

It is to certify that this project report **'Normed Linear Spaces'** is the bonafide project of **Nivya Joseph** carried out the project work under my supervision.

Mrs. Riya Baby
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DECLARATION

I **Nivya Joseph** hereby declare that the project **'Normed Linear Space'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Athulya P , Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

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CONTENTS

INTRODUCTION

This chapter gives an introduction to the theory of normed linear spaces. A skeptical reader may wonder why this topic in pure mathematics is useful in applied mathematics. The reason is quite simple: Many problems of applied mathematics can be formulated as a search for a certain function, such as the function that solves a given differential equation. Usually the function sought must belong to a definite family of acceptable functions that share some useful properties. For example, perhaps it must possess two continuous derivatives. The families that arise naturally in formulating problems are often linear spaces. This means that any linear combination of functions in the family will be another member of the family. It is common, in addition, that there is an appropriate means of measuring the “distance” between two functions in the family. This concept comes into play when the exact solution to a problem is inaccessible, while approximate solutions can be computed. We often measure how far apart the exact and approximate solutions are by using a norm. In this process we are led to a normed linear space, presumably one appropriate to the problem at hand. Some normed linear spaces occur over and over again in applied mathematics, and these, at least, should be familiar to the practitioner. Examples are the space of continuous functions on a given domain and the space of functions whose squares have a finite integral on a given domain.

PRELIMINARIES

1) LINEAR SPACES

We introduce an algebraic structure on a set $X$and study functions on X which are well behaved with respect to this structure. From now onwards , $\boldsymbol{K}$ will denote either $\boldsymbol{R}$, the set of all real numbers or $\boldsymbol{C}$**,** the set of all complex numbers. For $k \in \boldsymbol{C}$, *Re k* and *Im k* will denote the real and imaginary part of k.

A linear space(or a vector space) over $\boldsymbol{K}$ is a non-empty set X along with a function $+ \ : X \times X \to X$, called addition and a function $\cdot$: $\boldsymbol{K} \times X \to X$ called scalar multiplication, such that for all $x\, ,\ y\, ,\ z \in X$and $k\, ,\ l \in \boldsymbol{K}$, we have

$$x + y = y + x$$

$$x + (y + z) = (x + y) + z$$

$$\exists 0 \in X \ such\ that\ x + 0 = x,$$

$$\exists - x \in X\ such\ that\ x + (- x) = 0\, ,$$

$$k \cdot (x + y) = k \cdot x + k \cdot y,$$

$$(k + l)\cdot x = k \cdot x + l \cdot x,$$

$$(kl)\cdot x = k \cdot (l \cdot x),$$

$$1\cdot x = x.$$

We shall write kx in place of $k \cdot x$. We shall also adopt the following notations. For $x, y \in X, k \in \boldsymbol{K}$ and subsets $E, F$of X ,

$$x + F = \{x + y : y \in F\},$$

$$E + F = \{x + y : x \in E, y \in F\},`$$

$$kE = \{kx : x \in E\}.$$

2) BASIS

A nonempty subset E of X is said to be a subspace of X if $kx + ly \in E$ whenever $x, y \in E$ and $k, l \in \boldsymbol{K}$. If $\emptyset \neq E \subset X$, then the smallest subspace of X containing E is

$$spanE = \left\{k_1x_1 + \cdots + k_nx_n : x_1, ..., x_n \in E\ ,\ k_1, ..., k_n \in K\right\}$$

It is called the span of E. If span $E = X$, we say that E spans X. A subset E of X is said to be linearly independent if for all $x_1, ..., x_n \in E$ and $k_1, ..., k_n \in \boldsymbol{K}$, the equation $k_1x_1 + \cdots + k_nx_n = 0$implies that $k_1 = \cdots = k_n = 0$. It is called linearly dependent if it is not linearly independent, that is, if there exist $x_1, ..., x_n \in E$ and $k_1, ..., k_n \in \boldsymbol{K}$ such that $k_1x_1 + \cdots + k_nx_n = 0$, where at least one $k_j$is nonzero.

A subset E of X is called a Hamel basis or simply basis for X if *span of* $E = X$ and E is linearly independent .

3) DIMENSION

If a linear space X has a basis consisting of a finite number of elements , then X is called finite dimensional and the number of elements in a basis for X is called the dimension of X, denoted as $dimX$. Every basis for a finite dimensional linear space has the same (finite) number of elements and hence the dimension is well-defined. The space $\{0\}$ is said to have zero dimension. Note that it has no basis !

If a linear space contains an infinite linearly independent subset, then it is said to be infinite dimensional.

4)METRIC SPACE

We introduce a distance structure on a set X and study functions on X which are well-behaved with respect to this structure.

A metric d on a nonempty set X is a function $d: X \times X \to \boldsymbol{R}$
such that for all $x, y, z \in X$

$$d(x, y) \geq 0 \text{ and } d(x , y) = 0 \text{ iff } x=y$$

$$d(y , x) = d(x , y)$$

$$d(x , y) \leq d(x , z) + d(z , y) .$$

The last condition is known as the triangle inequality. A metric space is a nonempty set X along with a metric on it.

5)CONTINUOUS FUNCTIONS

Roughly speaking, a function from a metric space to a metric space is continuous if it sends 'nearby' points to 'nearby' points. If X and Y are metric spaces with metrics d and e respectively, then a function $F: X \to Y$ is said to be continuous at $x_0 \in X$ if for every $\epsilon \rangle 0$, there is some $\delta > 0$ (possibly depending on ϵ and x_0) such that $e\left(F(x), F\left(x_0\right)\right) < \epsilon$ for all $x \in X$ satisfying $d\left(x, x_0\right) < \delta$. Further, F is said to be continuous on X if it is continuous at every point of X. It is easy to see that F is continuous on X if and only if the set $F^{-1}(E)$ is open in X whenever the set E is open in Y. Also , this happens iff $F(x_n) \to F(x)$ in Y whenever $x_n \to x$ in X.

6) UNIFORM CONTINUITY

We note that a continuous function $F: T \to S$ is, in fact, uniformly continuous, that is, for every $\epsilon > 0$, there exists some $\delta > 0$ such that $e(F(t), F(u)) < \epsilon$ whenever $d\left(t, u\ \right) < \delta$. This can be seen as follows. Let $t \in T$. By the continuity of F at $t \in T$, there is some δ_t, such that $e(F(t), F(u)) < \frac{\varepsilon}{2}$ whenever $d(t, u) < \delta_t$.

7) FIELD

A ring is a set R together with two binary operations + and ·(which we call addition and multiplication) such that the following axioms are satisfied .

- R is an abelian group with respect to addition
- Multiplication is associative
- $\forall a\ , b, c\ \in\ R$the left distributive law $a(b + c) = (a \cdot b) + (a \cdot c)$ and the right distributive law $(a + b)c = (a \cdot c) + (b \cdot c)$, hold .

A field is a commutative division ring

CHAPTER 1

NORMED LINEAR SPACE

Let X be a linear space over $\boldsymbol{K}$. A norm on X is the function $|| \ ||$ from X to $\boldsymbol{R}$ such that $\forall$ $x,y \in X$ *and* $k \in \boldsymbol{K}$,

$$||x|| \geq 0 \text{ and } ||x|| = 0 \text{ if and only if } x = 0 \ ,$$

$$||x + y|| \leq ||x|| + ||y|| \ ,$$

$$||kx|| = |k| \ ||x|| \ .$$

A norm is the formalization and generalization to real vector spaces of the intuitive notion of " length" in the real world .

A normed space is a linear space with norm on it .

For x and y in X , let

$$d(x,y) = ||x - y||$$

Then d is a metric on X so that (X,d) is a metric space , thus every normed space is a metric space

- *Every normed linear space is a metric space . But converse may not be true .*

Example :

$$d(x,y) = \frac{|x-y|}{1+|x-y|} \ , \forall x , y \in X$$

$$\Rightarrow \ ||x - y|| = \frac{|x-y|}{1+|x-y|}$$

$$\Rightarrow \ ||z|| = \frac{|z|}{1+|z|} \ , z = x - y \in X$$

$$||\alpha z|| = \frac{|\alpha z|}{1+|\alpha z|}$$

$$= \frac{|\alpha|\,|z|}{1+|\alpha|\,|z|}$$

$$= |\alpha| \left(\frac{|z|}{1+|\alpha|\,|z|} \right)$$

$$\neq |\alpha|\,||z|| .$$

- *<u>Result</u>*

Let X be a normed linear space . Then ,

$$\big|\, ||x|| - ||y|| \,\big| \leq ||x - y|| , \ \forall x , y \in X$$

Proof :

$$||x|| = ||(x - y) + y|| \leq ||x - y|| + ||y||$$

$$\Rightarrow ||x|| - ||y|| \leq ||x - y|| \rightarrow (1)$$

$x \leftrightarrow y$

$$||y|| - ||x|| \leq ||y - x||$$

$$\Rightarrow -(||x|| - ||y||) \leq ||x - y|| \rightarrow (2)$$

From (1) and (2)

$$\big|\,||x|| - ||y||\,\big| \leq ||x - y||$$

- *<u>Norm is a continuous function</u>*

Let $x_n \rightarrow x$, as $n \rightarrow \infty$

$\Rightarrow x_n - x \to 0 \text{ as } n \to \infty$

$\Rightarrow ||x_n - x|| \to 0 \text{ as } n \to \infty$

$\big|\, ||x_n|| - ||x|| \,\big| \leq ||x_n - x|| \to 0 \text{ as } n \to \infty$

$\Rightarrow ||x_n|| - ||x|| \to 0 \text{, as } n \to \infty$

$\Rightarrow ||x||$ is continuous

- *<u>Norm is a uniformly continuous function</u>*

We have , $|| \; || : X \to \boldsymbol{R}$. Let $x, y \in X$ and $\varepsilon > 0$

Then $\quad ||x|| = ||x - y + y||$

$\leq ||x - y|| + ||y||$

$\Rightarrow ||x|| - ||y|| \leq ||x - y|| \to (1)$

Interchanging x and y ,

$||y|| - ||x|| \leq ||y - x||$

$\Rightarrow -(\, ||x|| - ||y|| \,) \leq ||x - y||$

$\Rightarrow \quad ||x|| - ||y|| \geq -||x - y|| \to (2)$

Combining (1) and (2)

$-||x - y|| \leq ||x|| - ||y|| \leq ||x - y||$

That is ,

$\big|\, ||x|| - ||y|| \,\big| \leq ||x - y||$

Take $\delta = \varepsilon$, then whenever $||x - y|| < \delta$, $\big|\, ||x|| - ||y|| \,\big| < \varepsilon$

Therefore $|| \ ||$ is a uniformly continuous function .

➢ ***Continuity of addition and scalar multiplication***

To show that $+ : X \times X \to X$ and $\cdot : \boldsymbol{K} \times X \to X$ are continuous functions.

Let $(x,y) \in X \times X$. To show that + is continuous at (x,y) , that is , to show that for each $(x,y) \in X \times X$ *if* $x_n \to x$ and $y_n \to y$ in X , then

$$+(x_n , y_n) \to +(x , y) ;$$

That is ,

$$x_n + y_n \to x + y .$$

Consider

$$||(x_n + y_n) - (x + y)|| = ||x_n - x + y_n - y||$$

$$\leq ||x_n - x|| + ||y_n - y||$$

Given $x_n \to x$ *and* $y_n \to y$ *, for each* $\epsilon > 0$, $\exists N_1 \ni$

$$||x_n - x|| < \frac{\varepsilon}{2} \forall n \geq N_1 , \quad \text{and } \exists N_2 \ni$$

$$||y_n - y|| < \frac{\varepsilon}{2} \quad \forall n \geq N_2$$

\

Take $N = max \{ N_1, N_2 \}$

Then $||x_n - x|| < \frac{\varepsilon}{2}$ and $||y_n - y|| < \frac{\varepsilon}{2} \forall n \geq N$

Therefore $||(x_n + y_n) - (x + y)|| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \forall n \geq N$

That is , $x_n + y_n \to x + y$

Now to show that $\cdot : \boldsymbol{K} \times X \to X$ is continuous

Let $(k , x) \in \boldsymbol{K} \times X$

To show that if $k_n \to k$ and $x_n \to x$, then $k_n x_n \to kx$

Since $k_n \to k$, $\forall\ \varepsilon > 0\ \exists\ N_1 \ni |k_n - k| < \frac{\varepsilon}{2}\ \ \forall\ n \geq N_1$

Since $x_n \to x$, $\forall\ \varepsilon > 0\ \exists\ N_2 \ni ||x_n - x|| < \frac{\varepsilon}{2}\ \ \forall\ n \geq N_2$

Consider $||k_n x_n - kx|| = ||k_n x_n - kx + x_n k - x_n k||$

$$= ||x_n (k_n - k) + k(x_n - x)||$$

$$\leq ||x_n(k_n - k)|| + ||k(x_n - x)||$$

$$= ||x_n||\ |k_n - k| + |k|\ ||x_n - x||$$

$$\leq ||x_n||\frac{\varepsilon}{2} + |k|\frac{\varepsilon}{2}$$

$\therefore\ \ k_n x_n \to kx$

➢ <u>*Examples of normed space*</u>

1) Spaces $\boldsymbol{K}^n$ ($\boldsymbol{K} = \boldsymbol{R}$ *or* $\boldsymbol{C}$)

For $n = 1$, the absolute value of function | | is a norm on $\boldsymbol{K}$, since $\forall k \in \boldsymbol{K}$

We have ,

$||k|| = ||k \cdot 1|| = |k|\ ||1||$, by definition .

But $||1||$ is a positive scalar .

$\therefore\ ||k||$ is a positive scalar multiple of the absolute value function .

$\therefore$ any norm on $\boldsymbol{K}$ is a positive scalar multiple of the absolute value function

For $n > 1$, let $p \geq 1$ be a real number

$$K^n = \{ (x(1), x(2), ..., x(n)) : x(i) \in K, i = 1, 2, ..., n \}$$

For $x \in K^n$, that is, $x = (x(1), x(2), ..., x(n))$, define

$$||x||_p = (|x(1)|^p + ... + |x(n)|^p)^{1/p}$$

Then $|| \, ||_p$ is a norm on K^n

When $p = 1$,

Then, $||x||_1 = |x(1)| + |x(2)| + ... + |x(n)|$

Since $|x(i)| \geq 0 \ \forall \, i = 1, 2, ..., n$, $||x||_1 \geq 0$

And $||x||_1 = 0 \Leftrightarrow |x(1)| + ... + |x(n)| = 0$

$\Leftrightarrow |x(i)| = 0 \ \forall \, i$

$\Leftrightarrow x(i) = 0 \ \forall \, i$

$\Leftrightarrow x = (x(1), ..., x(n)) = 0$

Now $||kx||_1 = |kx(1)| + |kx(2)| + ... + |kx(n)|$

$= |k| \, |x(1)| + ... + |k| \, |x(n)|$

$= |k| \, (|x(1)| + ... + |x(n)|)$

$= |k| \, ||x||_1$

$||x + y||_1 = |(x + y)(1)| + ... + |(x + y)(n)|$

$= |x(1) + y(1)| + ... + |x(n) + y(n)|$

$\leq |x(1)| + |y(1)| + ... + |x(n)| + |y(n)|$

$= |x(1)| + ... + |x(n)| + |y(1)| + ... + |y(n)|$

$= ||x||_1 + ||y||_1$

Consider $1<p<\infty$

Now , $||x||_p = (|x(1)|^p +. . . + |x(n)|^p)^{1/p}$

Since $|x(i)|^p \geq 0 \ \forall i$, we have $||x||_p \geq 0$

And $||x||_p = 0 \Leftrightarrow (|x(1)|^p +. . . + |x(n)|^p)^{1/p} = 0$

$\Leftrightarrow |x(i)|^p = 0 \ \forall i$

$\Leftrightarrow |x(i)| = 0 \ \forall i$

$\Leftrightarrow x(i) = 0 \ \forall i$

$\Leftrightarrow x = (x(1) , . . . , x(n)) = 0$.

Now

$$||kx||_p = (|kx(1)|^p +. . . + |kx(n)|^p)^{1/p}$$

$$= (|k|^p |x(1)|^p +. . . + |k|^p |x(n)|^p)^{1/p}$$

$$= |k| (|x(1)|^p +. . . + |x(n)|^p)^{1/p}$$

$$= |k| \, ||x||_p .$$

$$||x + y||_p = (|x(1) + y(1)|^p +. . . + |x(n) + y(n)|^p)^{1/p}$$

We have by Minkowski's inequality ,

$$\left(\sum_{i=1}^{n} |x(i) + y(i)|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y(i)|^p\right)^{1/p}$$

Then

$$||x + y||_p \leq \left(|x(1)|^p + . . . + |x(n)|^p \right)^{1/p} + \left(|y(1)|^p + . . . + |y(n)|^p \right)^{1/p}$$

$$= ||x||_p + ||y||_p$$

Then , for $1 \leq p < \infty$, $|| \; ||_p$ is a norm on $\boldsymbol{K^n}$

When $p = \infty$, define $||x||_\infty = max \{ |x(1)| , |x(2)| ,. . . , |x(n)| \}$

Then it is a norm on $\boldsymbol{K^n}$

$||x||_p \geq 0$ since each values $|x(i)| \geq 0$

So that

$$max \{|x(i)| , i=1, . . . , n\} \geq 0$$

$$||x||_\infty = 0 \Leftrightarrow max \{ |x(i)| : i = 1,. . . , n \} = 0$$

$$\Leftrightarrow |x(i)| = 0 \;\; \forall i$$

$$\Leftrightarrow x(i) = 0 , \forall i$$

$$\Leftrightarrow x = 0$$

$$||kx||_\infty = max \{ |kx(1)| ,. . . , |kx(n)| \}$$

$$= max \{ |k| \, |x(1)| , . . . , |k| \, |x(n)|\}$$

$$= |k| \, max \{|x(1)| , . . . , |x(n)|\}$$

$$= |k| \, ||x||_\infty$$

$$||x + y||_\infty = max \{ |x(1) + y(1)| , . . . , |x(n) + y(n)| \}$$

$$\leq max \{ |x(1)| + |y(1)| , . . . , |x(n)| + |y(n)| \}$$

$\leq max\ \{\ |x(1)|\ , \ldots ,\ |x(n)|\ \} + max\ \{\ |y(1)|\ , \ldots ,\ |y(n)|\ \}$

$= ||x||_\infty + ||y||_\infty$

2) Sequence space

Let $1 \leq p < \infty$, $l^p = \{\ x = (\ x(1)\ ,\ x(2)\ , \ldots)\ ;\ \ x(i) \in \boldsymbol{K}\ and\ \sum_{j=1}^{\infty} |x(j)|^p < \infty \}$, that is , l^p is the space of p-summable scalar sequences in $\boldsymbol{K}$. For $x = (x(1)\ ,\ x(2)\ , \ldots) \in l^p$,

let $||x||_p = (\ |x(1)|^p + |x(2)|^p + \ldots)^{1/p}$. Then it is a norm on l^p.

That is , $||\ ||_p$ is a function from l^p to $\boldsymbol{R}$.

If $p = 1$, then l^1 is a linear space and $||x||_1 = (\ |x(1)| + |x(2)| + \ldots)$ is a norm on l^1

Let $p = \infty$. Then l^∞ is the linear space of all bounded scalar sequences . And ,

$$||x||_\infty = sup\ \{\ |x(j)| : j = 1, 2, 3, \ldots\ \}$$

Then $||\ ||_\infty$ is a norm on l^∞

CHAPTER 2

THEOREMS ON NORMED SPACES

a) Let Y be a subspace of a normed space X, then Y and its closure $\overline{Y}$ are normed spaces with the induced norm.

b) Let Y be a closed subspace of a normed space X, for $x+Y$ in the quotient space X/Y, let $|||x+Y||| = inf\{ ||x+y|| : y \in Y\}$. Then $|||\ \ |||$ is a norm on X/Y, called the quotient norm.

A sequence $(x_n + Y)$ converges to $x + Y$ in X/Y iff there is a sequence (y_n) in Y, $(x_n + y_n)$ converges to x in X.

c) Let $||\ \ ||_p$ be a norm on the linear space X_p, $j = 1,2,\ldots$. Fix p such that $1 \le p \le \infty$

For $x = (x(1), x(2), \ldots, x(m))$ that is the product space $X = X_1 \times X_2 \times \ldots \times X_m$,

Let $||x||_p = \left(||x(1)||_1^p + ||x(2)||_2^p + \ldots + ||x(m)||_m^p \right)^{1/p}$, if $1 \le p < \infty$

$||x||_p = max\{ ||x(1)||_1, \ldots, ||x(m)||_m\}$, if $p = \infty$.

Then $||\ \ ||_p$ is a norm on X.

A sequence (x_n) converges to x in X $\Leftrightarrow$ $(x_n(j))$ converges to $x(j)$ in X_j $\forall j=1,2,\ldots,m.$

Proof:

a) Since X is a normed space, there is a norm on X to Y. Since Y is a subspace of X,

$||\ \ ||_y : Y \to \boldsymbol{R}$ is a function. To show that $||\ \ ||_y$ is a norm on Y.

For $y \in Y$, $||y||_Y = ||y||$, then

$$||y||_Y \geq 0 \quad (\because ||y|| \geq 0) \quad and \quad ||y||_Y = 0 \Leftrightarrow y = 0$$

$$||ky||_Y = ||ky|| = |k|\ ||y|| = |k|\ ||y||_y .$$

Let $y_1, y_2 \in Y$. Then ,

$$||y_1 + y_2||_y = ||y_1 + y_2|| \le ||y_1|| + ||y_2|| = ||y_1||_y + ||y_2||_y$$

Now the continuity of addition and scalar multiplication shows that $\overline{Y}$ is a subspace of X, since if $x_n \to x$ and $y_n \to y$, $x_n, y_n \in \overline{Y}$, then

$x_n + y_n \to x + y$ (by continuity of addition) and

$kx_n \to kx$ (by continuity of scalar X^n) .

Since $\overline{Y}$ is closed , $x + y \in \overline{Y}$ and $kx \in \overline{Y}$. Therefore $\overline{Y} \leq X$.

$\therefore$ norm on X induces a norm on Y and $\overline{Y}$

b) X/Y , the quotient space equals $X/Y=\{ x + Y : x \in X \}$.

$$|||x + y||| = inf \{ ||x + y|| : y \in Y \}$$

Claim: $|||\ \ |||$ is a norm on X/Y , called quotient norm

- Let $x \in X$,

$$|||x + Y||| = inf \{ ||x + y|| : y \in Y \} \geq 0.$$

$\therefore$ $|||x + Y||| \geq 0$.

If $|||x + y||| = 0$ (0 in X/Y is Y) , then there is a sequence (y_n) in Y $\ni$

$$||x + y_n || \to 0$$

$$\Rightarrow \qquad x + y_n \to 0$$

$$\Rightarrow \qquad y_n \to -x$$

Since $y_n \in Y$ and Y is closed

$-x \in Y \quad \Leftrightarrow x \in Y$ ($\because$ Y is a subspace)

$\Leftrightarrow x + Y = Y$, zero in X/Y.

- For $k \in \boldsymbol{K}$,

$$\begin{aligned} |||k(x + Y)||| &= |||kx + Y||| \\ &= inf \{ ||k(x + y)|| : y \in Y\} \\ &= inf \{ |k|\ ||x + y|| : y \in Y\} \\ &= |k|\ inf \{ ||x + y|| : y \in Y\} \\ &= |k|\ |||x + Y||| . \end{aligned}$$

- Let x_1 , $x_2 \in X$. Then

$|||x_1 + Y||| = inf \{ ||x_1 + y|| : y \in Y \}$. Then $\exists$ $y_1 \in Y \ni$

$$|||x_1 + Y||| + \frac{\varepsilon}{2} > ||x_1 + y_1|| , \ and$$

$|||x_2 + Y||| = inf\{ ||x_2 + y|| : y \in Y\}$, *Then* $\exists y_2 \in Y \ni$

$$|||x_2 + Y||| + \frac{\varepsilon}{2} > ||x_2 + y_2|| .$$

$$||x_1 + y_1 + x_2 + y_2|| \leq ||x_1 + y_1|| + ||x_2 + y_2||$$

$$\leq |||x_1 + Y||| + \frac{\varepsilon}{2} + |||x_2 + Y||| + \frac{\varepsilon}{2}$$

Let $y = y_1 + y_2 \in Y$. Then ,

$$||(x_1+x_2) + y|| \leq |||x_1 + Y||| + |||x_2 + Y||| + \mathcal{E} \quad —(1)$$

Now , $|||(x_1 + Y) + (x_2 + Y)||| = |||x_1 + x_2 + Y|||$

$$=inf\{ ||x_1 + x_2 + y|| : y \in Y \}$$

$$< ||x_1 + x_2 + y||$$

$$\leq |||x_1 + Y||| + |||x_2 + Y||| + \mathcal{E} \qquad \text{(by (1))}$$

since $\mathcal{E}$ is arbitrary , we have

$$|||(x_1 + Y) + (x_2 + Y)||| \leq |||x_1 + Y||| + |||x_2 + Y|||$$

$\therefore$ ||| ||| is a norm on X/Y.

Let $(x_n + Y)$ be a sequence in X/Y . Assume that (y_n) is a sequence in $Y \ni (x_n + y_n)$ converges to x in X.

That is , $(x_n - x + y_n)$ converges to 0 . —(1)

Claim: $(x_n + Y)$ converges to $x + Y$.

Consider

$$|||x_n + Y - (x+Y)||| = |||(x_n - x) + Y|||$$

$$= inf\{ ||x_n - x + y_n|| : y \in Y \}$$

$$\leq ||x_n - x + y_n|| \quad \forall y_n \in Y .$$

Then by (1) , $x_n + Y$ converges to $x + Y$ in X/Y.

Conversely assume that the sequence $(x_n + Y) \to x + Y$ in X/Y.

Consider $|||x_n + Y - (x + Y)||| = |||x_n - x + Y|||$

$$= inf\{ ||x_n -x + y|| : y \in Y \}$$

Then we can choose $y_n \in Y \ni$

$$||x_n - x + y_n|| < |||(x_n - x) + Y||| + \frac{1}{n} \ , \ n=1,2,3,....$$

Since $x_n+Y \to x+Y$, we get

$(x_n - x + y_n)$ converges to zero as $n \to\infty$

That is , $(x_n + y_n)$ converges to x in X as $n \to\infty$

c) Consider $1\leq p < \infty$

Given that

$$||x||_p = (||x(1)||_1^p + ||x(2)||_2^p +. . . + ||x(m)||_m^p)^{1/p}$$

Clearly , $||x||_p \geq 0$.

Since each $||x(i)||_i^p \geq 0$.

$$||x||_p = 0 \Leftrightarrow |x(j)|_j^p = 0 \quad \forall\ j = 1, . . . ,m$$

$$\Leftrightarrow x(j) = 0 \quad \forall\ j.$$

$$\Leftrightarrow x = (x(1), . . . ,x(m)) = 0$$

$$||kx||_p = \left(||kx(1)||_1^p +. . .+ ||kx(m)||_m^p \right)^{1/p}$$

$$= \left(|k|^p||x(1)||_1^p +. . .+ |k|^p||x(m)||_m^p \right)^{1/p}$$

$$= |k| \left(||x(1)||_1^p +. . .+ ||x(m)||_m^p \right)^{1/p}$$

$$= |k|\ ||x||_p \quad , k\in \boldsymbol{K} \text{ and } x\in X$$

Now, $||x + y||_p = \left(||x(1) + y(1)||_1^p +. . .+ ||x(m) + y(m)||_m^p \right)^{1/p}$

(by Minkowski's inequality)

$$\leq \left(\left(||x(1)||_1 + ||y(1)||_1\right)^p +. . . + \left(||x(m)||_m + ||y(m)||_m\right)^p\right)^{1/p}$$

$$\leq \left(\sum_{j=1}^{m} ||x(j)||_j^p\right)^{1/p} + \left(\sum_{j=1}^{m} ||y(j)||_j^p\right)^{1/p} \qquad \text{(Minkowski's inequality)}$$

$$= \left(||x(1)||_1^p + . . . + ||x(m)||_m^p \right)^{1/p}$$

$$= ||x||_p + ||y||_p$$

Now suppose $p = \infty$

$$||x||_\infty = max \{ ||x(1)||_1 , . . . , ||x(m)||_m \}$$

$||x||_\infty \geq 0$ Since $||x(j)|| \geq 0$, $\forall j$

$$||x||_\infty = 0 \Leftrightarrow ||x(m)|| = 0 \quad \forall m$$

$$\Leftrightarrow x(m) = 0 \quad \forall m$$

$$\Leftrightarrow x = 0$$

$$||kx||_\infty = max \{ ||kx(1)||_1 , . . . , ||kx(m)||_m \}$$

$$= |k| \, max \{ ||x(1)||_1 , . . . , ||x(m)||_m \}$$

$$= |k| \, ||x||_\infty$$

$$||x + y||_\infty = max \{ ||x(1) + y(1)||_1, . . . , ||x(m) + y(m)||_m \}$$

$$\leq max \{ ||x(1)||_1 + ||y(1)||_1 , . . . , ||x(m)||_m + ||y(m)||_m \}$$

$$= max \{ ||x(1)||_1 , . . . , ||x(m)||_m \} + max \{ ||y(1)||_1 , . . . , ||y(m)||_m \}$$

$$= ||x||_\infty + ||y||_\infty$$

We now consider ,

$$||x_n - x(1)||_p = \left(||x_n(1) - x(1)||_1^p + . . . + ||x_n(m) - x(m)||_m^p \right)^{1/p}$$

Then

$$x_n \to x \text{ in } X \Leftrightarrow ||x_n - x||_p \to 0$$

$$\Leftrightarrow ||x_n(j) - x(j)||_j^p \to 0$$

$$\Leftrightarrow x_n(j) - x(j) \to 0$$

$$\Leftrightarrow x_n(j) \to x(j) \text{ in } X \, \forall j .$$

RIESZ LEMMA

Let X be a normed space . Y be a closed subspace of X and $X \neq Y$. Let r be a real number such that $0 < r < 1$. Then there exist some $x_r \in X$ such that $||x_r|| = 1$ and

$r < dist\,(x_r , Y) \leq 1$

Proof :

We have ,

$$dist\,(x , Y) = inf\{ d(x , y) : y \in Y\}$$

$$= inf\{ ||x - y|| : y \in Y\}$$

Since $Y \neq X$, consider $x \in X \ni x \notin Y$.

If $dist(x , Y) = 0$, then $||x - y|| = 0 \Rightarrow x \in \overline{Y} = Y$ ($\because$ Y is closed)

Therefore ,

$$dist\,(x , Y) \neq 0$$

That is ,

$$dist\,(x , Y) > 0$$

Since $0 < r < 1$, $\frac{1}{r} > 1$

$$\Rightarrow \frac{dist\,(x , Y)}{r} > dist\,(x , Y)$$

That is , $\frac{dist\,(x , Y)}{r}$ is not a lower bound of $\{ ||x - y|| : y \in Y \}$

Then $\exists$ $y_0 \in Y \ni ||x - y_0|| < \frac{dist(x , Y)}{r}$ $\rightarrow$(1)

Let $x_r = \frac{x - y_0}{||x - y_0||}$. Then $x_r \in X$

($\because y_0 \in Y$, $x \notin Y \Rightarrow x - y_0 \in X$ and $||x - y_0|| \neq 0$)

Then $||x_r|| = \left|\left|\frac{x - y_0}{||x - y_0||}\right|\right| = \frac{||x - y_0||}{||x - y_0||} = 1$

Now to prove $r < dist(x_r, Y) \leq 1$

We have $dist(x_r , Y) = inf\{ ||x_r - y|| : y \in Y \}$

$$\leq ||x_r - y|| \quad \forall y \in Y$$

In particular, $0 \in Y$, so that $dist(x_r , Y) \leq ||x_r - 0|| = 1$

That is ,

$$dist (x_r , Y) \leq 1$$

Now ,

$$dist (x_r , Y) = dist \left(\frac{x - y_0}{||x-y_0||}, Y\right)$$

$$= \frac{1}{||x-y_0||} dist (x - y_0 , Y)$$

$$= \frac{1}{||x-y_0||} inf\{ ||x - y_0 - y|| : y \in Y\}$$

$$= \frac{1}{||x-y_0||} inf\{||x - (y_0 + y)|| : y_0 + y \in Y\}$$

$$= \frac{1}{||x-y_0||} dist (x , Y)$$

$$> \frac{r}{dist (x , Y)} dist (x , Y) \quad by (1)$$

$$\Rightarrow dist (x_r , Y) > r$$

That is ,

$$r < dist (x_r , Y) \leq 1$$

CONCLUSION

This project discusses the concept of normed linear space that is fundamental to functional analysis . A normed linear space is a vector space over a real or complex numbers ,on which the norm is defined . A norm is a formalization and generalization to real vector spaces of the intuitive notion of “length” in real world

In this project , the concept of a norm on a linear space is introduced and thus illustrated . It mostly includes the properties of normed linear spaces and different proofs related to the topic.

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POWER SERIES SOLUTIONS AND SPECIAL FUNCTIONS

Project report submitted to
The Kannur University
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of

Bachelor of Science

by

P K NIVEDITHA

DB18CMSR24

Under the guidance of

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CERTIFICATE

Certified that this project **'Power Series'** is a bona fide project of **P K NIVEDITHA** carried out the project work under my supervision.

Mrs. Riya Baby
Head of Department

Ms. Athulya P
Supervisor

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DECLARATION

I **P K NIVEDITHA** hereby declare that the project **'Power Series'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Athulya P, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

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ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express my deepest gratitude to people along the way.

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My greatest debt is always, to God Almighty.

CONTENTS

INTRODUCTION

A power series is a type of series with terms involving a variable. Power series are often used by calculators and computers to evaluate trigonometric, hyperbolic, exponential and logarithm functions. So any application of these kind of functions is a possible application of power series. Many interesting and important differential equations can be found in power series.

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PRELIMINERY

A. An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad (1)$$

is called a *power series in x*. The series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

is a *power series in* $x - x_0$.

B. The series (1) is said to *converge* at a point x if the limit

$$\lim_{m \to \infty} \sum_{n=0}^{m} a_n x^n$$

exists, and in this case the *sum* of the series is the value of this limit.

Radius of convergence: Series in x has a radius of convergence R, where $0 \leq R \leq \infty$, with the property that the series converges if $|x| < R$ and diverges if $|x| > R$. It should be noted that if $R = 0$ then no x satisfies $|x| < R$, and if $R = \infty$ then no x satisfies $|x| > R$

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \text{, if the limit exists.}$$

C. Suppose that (1) converges for $|x| < R$ with $R > 0$, and denote its sum by $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Then $f(x)$ is automatically continuous and has derivatives of all orders for $|x| < R$.

D. Let $f(x)$ be a continuous function that has derivatives of all orders for $|x| < R$ with $R > 0$. $f(x)$ be represented as power series using *Taylor's formula*:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x)$$

where the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(\bar{x})}{(n+1)!} x^{n+1}$$

for some point $\bar{x}$ between 0 and x.

E. A function $f(x)$ with the property that a power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is valid in some neighbourhood of the point x_0 is said to be *analytic* at x_0. In this case the a_n are necessarily given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

and is called the *Taylor series* of $f(x)$ at x_0.

Analytic functions: A function f defined on some open subset U of R or C is called analytic if it is locally given by a convergent power series. This means that every a ∈ U has an open neighbourhood V ⊆ U, such that there exists a power series with centre a that converges to f(x) for every x ∈ V.

CHAPTER 1

SERIES SOLUTION OF FIRST ORDER EQUATION

We have studied to solve linear equations with constants coefficient but with variable coefficient only specific cases are discussed. Now we turn to these latter cases and try to find a general method to solve this. The idea is to assume that the unknown function y can be explained into a power series. Our purpose in this section is to explain the procedures by showing how it works in the case of first order equation that are easy to solve by elementary methods.

Example 1: we consider the equation

$$y^{'} = y$$

Consider the above equation as (1). Assume that y has a power series solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Which converges for $|x| < R$, $R > 0$

That is we assume that $y^{'} = y$ has a solution that is analytic at origin. We have

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots \cdots$$

then

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots \ldots \ldots$$

$$\therefore (1) \Rightarrow a_1 + 2a_2 x + 3a_3 x^2 \cdots$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$\Rightarrow a_1 = a_0$

$2a_2 = a_1 \Rightarrow \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2}$

$3a_3 = a_2 \Rightarrow \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3} = \frac{a_0}{3!}$

$4a_4 = a_3 \Rightarrow \qquad a_4 = \frac{a_3}{4} = \frac{a_0}{2 \cdot 3 \cdot 4} = \frac{a_0}{4!}$

∴ we get

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{3!} x^3 + \frac{a_0}{4!} x^4 + \cdots$$

$$= a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right)$$

$$y = a_0 e^x$$

To find the actual function we have $y' = y$

$$\text{i.e.,} \quad \frac{dy}{dx} = y \quad \Rightarrow \quad \frac{dy}{y} = dx$$

integrating

$$\log\, y = x + c$$

$$\text{i.e.,} \quad y = e^{x+c} = e^x \cdot e^c$$

$y = a_0 e^x$, where $a_0 = e^c$, a constant.

Example 2: solve $y' = 2xy$. Also find its actual solution.

<u>Solution:</u> $\qquad y' = 2xy \qquad (1)$

Assume that y has a power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Which converges for $|x| < R, R > 0$

We have

$$y = \sum_{n=0}^{\infty} a_n \ x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= a_1 + 2a_2x + 3a_3x^2 + \cdots$$

Then (1) $\Rightarrow a_1 + 2a_2x + 3a_3x^2 + \cdots = 2x(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)$

$$= 2xa_0 + 2xa_1x + 2xa_2x^2 + 2xa_3x^3 + \cdots$$

$$= 2xa_0 + 2a_1x^2 + 2a_2x^3 + 2a_3x^4 + \cdots \ldots..$$

$\Rightarrow a_1 = 0 \qquad 2a_2 = 2a_0 \Rightarrow a_2 = \frac{2a_0}{z} = a_0$

$$3.a_3 = 2a_1 \Rightarrow a_3 = \frac{2a_1}{3} = 0$$

$$4a_4 = 2a_2 \Rightarrow a_4 = \frac{2a_2}{42} = \frac{a_0}{2}$$

$$5a_5 = 2a_3 = 0 \Rightarrow a_5 = 0$$

$$6a_6 = 2a_4 \Rightarrow a_6 = \frac{2a_4}{6} = \frac{a_4}{3} = \frac{a_0}{2\cdot 3} = \frac{a_0}{3!}$$

We get,

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ &= a_0 + 0 + a_0x^2 + 0x^3 + \frac{a_0}{2}x^4 + \cdots \\ &= a_0 + a_0x^2 + \frac{a_0}{2}x^4 + \cdots \\ &= a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) \end{aligned}$$

$$y = a_0e^{x^2}$$

To find an actual solution

$$y' = 2xy$$

$$\frac{dy}{dx} = 2xy$$

$$\Rightarrow \qquad \frac{dy}{y} = 2x \cdot dx$$

$$\log y = x^2 + c$$

$$y = e^{x^2} + c$$

$\Rightarrow y = a_0e^{x^2}$, where $a_0 = e^c$

Example 3: Consider $y = (1 + x)^p$ where p is an arbitrary constant. Construct a differential equation from this and then find the solution using power series method.

Solution

First, we construct a differential equation

i.e. $y = (1 + x)^p$

$$y' = p(1 + x)^{p-1} = \frac{p(1+x)^p}{1+x} = \frac{py}{1+x}$$

$$\therefore (1 + x)y' = py, \quad y(0) = r$$

Assume that y has a power series solution of the form,

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + \dot{a}_2 x^2 + \cdots \ldots \ldots$$

Which converges for $|x| < \dot{R}, \quad R > 0$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots \ldots \ldots ..$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Then $(1 + x)y' = py$

$$\Rightarrow (1 + x)a_1 + 2a_2 x + 3a_3 x^2 + \cdots = p(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$\Rightarrow (a_1 + 2a_2 x + 3a_3 x^2 + \cdots) + (a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots)$$

$$= a_0 p + a_1 p x + a_2 p x^2 + \cdots$$

Equating the coefficients of $x, x^2, \ldots$

$$a_1 = a_0 p \text{ i.e. } a_1 = p, (\text{since } a_0 = 1)$$

$$\Rightarrow 2a_2 = a_1(\mathrm{p} - 1)$$

$$a_2 = \frac{a_1(\mathrm{p} - 1)}{2} = \frac{a_0 P(p - 1)}{2}$$

$$3a_3 + 2a_2 = a_2 p$$
$$sa_3 = a_2 p - 2a_2$$
$$= a_2(p-2)$$
$$a_3 = \frac{a_2(p-2)}{3} = \frac{a_0 p(p-1)(p-2)}{2 \cdot 3}$$

$$4a_4 + 3a_3 = a_3 p$$
$$4a_4 = a_3 p - 3a_3$$
$$= a_3(p-3)$$
$$a_4 = \frac{a_3(p-3)}{4} = \frac{a_0 p(p-1)(p-2)(p-3)}{2 \cdot 3 \cdot 4}$$

∴ we get,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= a_0 + a_0 p x + \frac{a_0 p(p-1)}{2} x^2 + \frac{a_0 p(p-1)(p-2)}{2 \cdot 3} x^3 + \cdots \ldots$$
$$= 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 +$$
$$\frac{p(p-1)(p-2)(p-3)}{4!} x^4 + \cdots + \frac{p(p-1)(p-2)(p-(n-1))}{n!} x^n$$

Since the initial problem y(0) = 1 has one solution the series converges for |x|<1

So this is a power solution,

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n$$

Which is binomial series.

Example 4: Solve the equation $y' = x - y,\ y\,(0) = 0$

Solution: Assume that y has a power series solution of the form

$$y = \sum_{n=0}^{\infty} \text{an}\ x^n$$

which converges for $|x| < R, R > 0$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Now $y' = x - y$

$$(a_1 + 2a_2x + 3a_3x^2 + \cdots) = x - (a_0 + a_1x + a_2x^2 + \cdots)$$

Equating the coefficients of x, x^2,

$$\begin{aligned} a_1 &= -a_0 = 0, \quad \text{Since} \quad y(0) = 0 \\ 2a_2 &= 1 - a_1 \\ &= 1 - 0 \end{aligned}$$

$$\Rightarrow a_2 = \frac{1}{2}$$

$$3a_3 = -a_2$$

$$a_3 = \frac{-a_2}{3} = -\frac{1}{2 \cdot 3}$$

$$4a_4 = -a_3$$

$$\Rightarrow a_4 = \frac{1}{2 \cdot 3 \cdot 4}$$

$$\therefore y = 0 + 0 + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \ldots \ldots$$

$$= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) + x - 1$$

$$= e^{-x} + x - 1$$

By direct method

$$y' = x - y$$

$$\frac{dy}{dx} = x - y \Rightarrow \frac{dy}{dx} + y = x \qquad \left(\frac{dy}{dx} + py = Q\ form\right)$$

here $P(x) = 1$, integrating factor

$$= e^{\int p(x) \cdot dx}$$

$$= e^x$$

$$\therefore ye^x = \int xe^x \cdot dx$$

$$ye^x = x \cdot e^x - \int e^x \cdot dx$$

$$= xe^x - e^x$$

$$ye^x = e^x(x-1) + c$$

$$y = \frac{e^x(x-1) + c}{dx} = x - 1 + \frac{c}{e^x} = ce^{-x} + (x-1)$$

$$\therefore y = (x-1) + ce^{-x}$$

CHAPTER 2

SECOND ORDER LINEAR EQUATION, ORDINARY POINTS

Consider the general homogeneous second order linear equation,

$$y'' + P(x)y' + Q(x)y = 0 \qquad (1)$$

As we know, it is occasionally possible to solve such an equation in terms of familiar elementary functions. This is true, for instance, when $P(x)$ and $Q(x)$ are constants, and in a few other cases as well. For the most part, however, the equations of this type having the greatest significance in both pure and applied mathematics are beyond the reach of elementary methods, and can only be solved by means of power series.

P(x) and Q(x) are called coefficients of the equation. The behaviour of its solutions near a point x_0 depends on the behaviour of its coefficient functions $P(x)$ and $Q(x)$ near this point. we confine ourselves to the case in which $P(x)$ and $Q(x)$ are well behaved in the sense of being analytic at x0, which means that each has a power series expansion valid in some neighbourhood of this point. In this case x0 is called an *ordinary point* of equation (1). Any point that is not an ordinary point of (1) is called a *singular point*.

Consider the equation,

$$y'' + y = 0 \qquad (2)$$

the coefficient functions are $P(x) = 0$ and $Q(x) = 1$, These functions are analytic at all points, so we seek a solution of the form,

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \qquad (3)$$

Differentiating (3) we get,

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots \qquad (4)$$

And

$$y'' = 2a_2 + 2\cdot 3a_3x + 3\cdot 4a_4x^2 + \cdots + (n+1)(n+2)a_{n+2}x^n + \cdots \quad (5)$$

If we substitute (5) and (3) into (2) and add the two series term by term, we get

$$y'' + y = \begin{matrix}(2a_2 + a_0) + (2\cdot 3a_3 + a_1)x + (3\cdot 4a_4 + a_2)x^2 + \\ (4\cdot 5a_5 + a_3)x^3 + \cdots + [(n+1)(n+2)a_{n+2} + a_n]x^n + \cdots\end{matrix} = 0$$

and equating to zero the coefficients of successive powers of *x* gives

$$2a_2 + a_0 = 0, \qquad 2\cdot 3a_3 + a_1 = 0, \qquad 3\cdot 4a_4 + a_2 = 0$$

$$4\cdot 5a_5 + a_3 = 0, \dots\dots, \qquad (n+1)(n+2)a_{n+2} + a_n = 0, \dots$$

By means of these equations we can express a_n in terms of a_0 or a_0, according as *n* is even or odd:

$$a_2 = -\frac{a_0}{2}, \qquad a_3 = -\frac{a_1}{2\cdot 3}, \qquad a_4 = -\frac{a_2}{3\cdot 4} = \frac{a_0}{2\cdot 3\cdot 4}$$

$$a_5 = -\frac{a_3}{4\cdot 5} = \frac{a_1}{2\cdot 3\cdot 4\cdot 5}, \cdots$$

With these coefficients, (3) becomes

$$y = a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{2\cdot 3}x^3 + \frac{a_0}{2\cdot 3\cdot 4}x^4 + \frac{a_1}{2\cdot 3\cdot 4\cdot 5}x^5 - \cdots$$

$$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \qquad (6)$$

$i.e,$ $\qquad y = a_0\cos\ x + a_1\sin\ x$

Since each of the series in the parenthesis converges for all *x*. This implies the series (2) for all *x*.

Solve the legenders equation,

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0$$

Solution

Consider $(1 - x^2)y'' - 2xy' + p(p+1)y = 0$ as equation (1)

Assume that y has a power series solution of the form

$$y = \sum a_n x^n$$

Which converges $|x| < R, R > 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

put $n = n + 2$ (Since y'' is not x^n form)

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+2-1) a_{n+2} x^{n+2-2}$$

$$\therefore y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

Now (1) $\Rightarrow$ $\quad y'' - x^2 y'' - 2xy' + p(p+1)y = 0$

$$\Rightarrow \sum (n+1)(n+2) a_{n+2} x^n - \sum n(n-1) a_n x^n - \sum 2n a_n x^n + \sum p(p+1) a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [((n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + p(p+1) a_n) x^n] = 0$$

for n = 0,1,2,3…….

$$\Rightarrow (n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + p(p+1) a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{[n(n-1) + 2n - p(p+1)]}{(n+1)(n+2)} a_n$$

$$= \frac{(n^2 - n + 2n - p^2 - p) a_n}{(n+1)(n+2)}$$

$$= \frac{(n^2 + n - p^2 - p) a_n}{(n+1)(n+2)}$$

$$\therefore a_{n+2} = \frac{-(p-n)(p+n+1)}{(n+1)(n+2)} a_n\,, \qquad n = 0,1,2\,...$$

This is an Recursion formula

$$\text{put } n = 0, \qquad a_2 = \frac{-p(p+1)}{1 \cdot 2} a_0$$

$$n = 1, \qquad a_3 = \frac{-(p-1)(p+2)}{2 \cdot 3} \cdot a_1$$

$$n = 2, \qquad a_4 = \frac{-(p-2)(p+3)}{3i4} a_2$$

$$= \frac{p(p-2)(p+1)(p+3)}{4!} a_0$$

$$n = 3, \qquad a_5 = \frac{-(p-3)[p+4)}{4 \cdot 5} a_3$$

$$= \frac{(p-1)(p-3)(p+2)(p+4)}{5!} a_1$$

$$n = 4, \qquad a_6 = \frac{-(p-4)(p+5)}{5 \cdot 6} a_4$$

$$= \frac{-p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} a_0$$

$$n = 5, \qquad a_7 = -\frac{(p-5)(p+6)}{6 \cdot 7} a_5$$

$$= -\frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} a_1$$

$$y = a_0 \left[1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p-2)(p+1)(p+3)}{4!} x^4 - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} x^6 + \cdots \right]$$
$$+ a_1 \left[x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!} x^5 - \frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} x^7 + \cdots \right]$$

Find the general solution of $(1+x^2)y'' + 2xy' - 2y = 0$ in terms of power series in x. Can you express this solution by means of elementary functions?

Solution

Consider the equation $(1+x^2)y'' + 2xy' - 2y = 0$ as equation (1)

Assume that y has a power series solution of the form

$$y = \sum a_n x^n$$

Which converges $|x| < R, R > 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1+x^2)y'' = y'' + x^2 y''$$

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$\text{Now } y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

put $n = n+2$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+2-1)a_n + 2x^{n+2=2}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n$$

$$(1) \Rightarrow \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x_n + \sum_{n=1}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\Rightarrow \sum[((n+1)(n+2)a_{n+2} + n(n-1)a_n + 2na_n - 2a_n)x^n] = 0$$

$$\Rightarrow (n+1)(n+2)a_{n+2} + n(n-1)a_n + 2na_n - 2a_n = 0$$

$$\begin{aligned} a_{n+2} &= \frac{[-n(n-1) - 2n + 2]}{(n+1)(n+2)} a_n \\ &= \frac{(-n^2 + n - 2n + 2)}{(n+1)(n+2)} a_n \end{aligned}$$

$$\begin{aligned}
&\text{put } n = 0, \quad && a_2 = \frac{2}{1 \cdot 2} a_0 = \frac{2a_0}{2!} = a_0 \\
&n = 1, && a_3 = \frac{(1 - 1 - 2 + 2)}{2 \cdot 3} a_1 = 0 \\
&n = 2, && a_4 = \frac{2 - 4 - 4 + 2}{3 \cdot 4} a_2 \quad = \frac{-4}{3 \cdot 4} a_0 = \frac{-a_0}{3} \\
&n = 3, && a_5 = \frac{3 - 9 - 16 + 2}{4.5} a_3 \quad = 0 \\
&n = 4, && a_6 = \frac{4 - 16 - 8 + 2}{5.6} a_4 \quad = \frac{-3}{5} a_4 \quad = \frac{3a_0}{3.5} = \frac{a_0}{5}
\end{aligned}$$

$$\begin{aligned} \therefore y &= a_0 + a_1 x + a_2 x^2 + \cdots \\ &= a_0 + a_1 x + a_0 x^2 - \frac{a_0}{3} x^4 + \frac{a_0}{5} x^6 \ldots . \\ &= a_0 \left[1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \cdots \right] + a_1 x \\ &= a_0 \left[1 + x \left(x - \frac{x^3}{3} + \frac{x^5}{5} \cdots \right) \right] + a_1 x \\ &= a_0 (1 + x \tan^{-1} x) + a_1 x \end{aligned}$$

Consider the equation $y'' + xy' + y = 0$

(a) Find its general solution $y = \sum a_n x^n$ in the form
$y = a_0 y_1(x) + a_1 y_2(x)$ where $y_1(x)$ and $y_2(x)$ are power series

(b) use the ratio test to verify that the two series $y_1(x)$ and $y_2(x)$ converges for all x.

Solution:

Given $y'' + xy' + y = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$

Assume that y has a power series solution the form $\sum a_n x^n$ which converges for $|x| \quad R > 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n$$

$$(1) \Rightarrow \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum \left[\left((n+1)(n+2) a_{n+2} + n a_n + a_n \right) x^n \right] = 0$$

$$\Rightarrow (n+1)(n+2) a_{n+2} + n a_n + a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{(-n-1) a_n}{(n+1)(n+2)} = \frac{-a_n}{n+2}$$

put $\quad n = 0, a_2 = -\frac{a_0}{2}$

$$n = 1, a_3 = \frac{-2a_1}{2 \cdot 3} = \frac{-a_1}{3}$$

$$n = 2, \quad a_4 = \frac{-3a_2}{3 \cdot 4} = \frac{-a_2}{4} = \frac{a_0}{8}$$
$$n = 3, \quad a_5 = \frac{-4a_3}{4 \cdot 5} = \frac{a_1}{15}$$
$$n = 4, \quad a_6 = \frac{-5a_4}{5 \cdot 6} = \frac{-a_0}{48}$$

$\therefore$ we get $\qquad y = a_0 + a_1 x + -\frac{a_0}{2}x^2 - \frac{a_1}{3}x^3 + \frac{a_0}{8}x^4 + \frac{a_1}{15}x^5 - \frac{a_0}{48}x^6 + \cdots$

$$= a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots\right] + a_1\left[x - \frac{x^3}{3} + \frac{x^5}{3.5} + \cdots\right]$$

where $\qquad y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{\dot{x}^2}{2 \cdot 4 \cdot 6} +$

$$y_2(x) = x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \cdots$$

(b) $\qquad y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(-1)^n}{2 \cdot 4 \cdot (2n)} \Big/ \frac{(-1)^{n+1}}{2 \cdot 4 \cdot\cdot (2n+2)}\right|$$
$$= \lim_{n\to\infty}\left|\frac{2(n+1)}{-1}\right|$$
$$= \lim_{n\to\infty}\left| -2n(1 + \frac{1}{n})\right| = \infty$$

$\therefore y_1(x)$ converges for all x

$$y_2(x) = x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} - \cdots$$

$$R = \lim_{n\to\infty}\left|\frac{(-1)^n}{3\cdot 5\cdots(2n+1)}\Big/\frac{(-1)^{n+1}}{3\cdot 5\cdot\cdots(2n+3)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)\cdot 3\cdot 5\cdots(2n+1)(2n+3)}{3\cdot 5\cdot\cdots\cdot(2n+1)}\right|$$

$$= \lim_{n\to\infty}|(-1)n(2+3/n)| = \infty$$

$\therefore\ y_2(x)$ converges for all x

REGULAR SINGULAR POINTS

A singular point x_0 of equation

$$y'' + P(x)y' + Q(x)y = 0$$

is said to be regular if the functions $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic, and irregular otherwise. Roughly speaking, this means that the singularity in $P(x)$ cannot be worse than $1/(x - x_0)$, and that in $Q(x)$ cannot be worse than $1/(x - x_0)^2$.

If we consider Legendre's equation in the form

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$$

it is clear that $x = 1$ and $x = -1$ are singular points. The first is regular because

$$(x-1)P(x) = \frac{2x}{x+1} \text{ and } (x-1)^2Q(x) = -\frac{(x-1)p(p+1)}{x+1}$$

are analytic at $x = 1$, and the second is also regular for similar reasons.

Example: *Bessel's equation of order p*, where p is a nonnegative constant:

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

If this is written in the form

$$y'' + \frac{1}{x}y' + \frac{x^2-p^2}{x^2}y = 0,$$

it is apparent that the origin is a regular singular point because $xP(x) = 1$ and $x^2Q(x) = x^2 - p^2$ are analytic at $x = 0$.

CONCLUSION

The purpose of this project gives a simple account of series solution of first order equation, second order linear equation, ordinary points. The study of these topics given excellent introduction to the subject called ‘POWER SERIES’

we used application of power series extensively throughout this project. We take it for granted that most readers are reasonably well acquainted with these series from an earlier course in calculus. Nevertheless, for the benefit of those whose familiarity with this topic may have faded slightly, we presented a brief review of the main facts of power series.

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NUMBER THEORETIC FUNCTION

Project report submitted to
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of

Bachelor of Science

by

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DB18CMSR16

Under the guidance of

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CERTIFICATE

Certified that this project **‘Number Theoretic Function’** is a bona fide project of **RON M JOJO** carried out the project work under my supervision.

Mrs. Riya Baby | **Ms. Ajeena joseph**

Head of Department | **Supervisor**

Department of Mathematics

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DECLARATION

I **RON M JOJO** hereby declare that the project **'Number Theoretic Function'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Ajeena joseph, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

Name

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Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express my deepest gratitude to people along the way.

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My greatest debt is always, to God Almighty.

CONTENTS

INTRODUCTION

A Number Theoretic Function is a complex valued function defined for all positive integers. In Number Theory, there exist many number theoretic functions. This includes Divisor Function, Sigma Function, Euler's-Phi Function and Mobius Function. All these functions play a very important role in the field of Number Theory.

In the first chapter we will discuss about Arithmetic Function. In the second chapter we will introduce Euler's-Phi Function and Mobius Function.

PRELIMINARY

Let n be a fixed positive integer. Two integers a and b are said to be *congruent modulo n,* symbolized by

$$a \equiv b \pmod{n}$$

if n divides the difference $a - b$; that is, provided that $a - b = kn$ for some integer k.

Example:

To fix the idea, consider $n = 7$. It is routine to check that

$3 \equiv 24 \pmod 7$ $\quad -31 \equiv 11 \pmod 7$ $\quad -15 \equiv -64 \pmod 7$

Because $3 - 24 = (-3)7$, $-31 - 11 = (-6)7$ and $-15 - (-64) = 77$. When n does not divide $(a - b)$, we say that a is *incongruent to b modulo n*, and in this case we write

$a \not\equiv b \pmod{n}$. For a simple example: $25 \not\equiv 12 \pmod 7$, because 7 fails to divide $25 - 12 = 13$.

It is to be noted that any two integers are congruent modulo 1, whereas two integers are congruent modulo 2 when they are both even or both odd. In as much as congruence modulo 1 is not particularly interesting, the usual practice is to assume that $n > 1$.

Remark:

Given an integer a, let q and r be its quotient and remainder upon division by n, so that

$$a = qn + r \quad 0 \leq r < n$$

Then, by definition of congruence, $a \equiv r \pmod{n}$. Because there are n choices for r , we see that every integer is congruent modulo n to exactly one of the values $0, 1, 2, \ldots, n - 1$; in particular, $a \equiv 0 \pmod{n}$ if and only if $n \mid a$.

Fundamental Theorem of Arithmetic

is Every integer $n > 1$ can be represented as Product of prime factor in only one way, apart from the order of the factors.

Residue

If a is an integer and m is a positive integer then the residue class of a modulo m is denoted by $\hat{a}$ and is given by

$$\begin{aligned} \hat{a} \quad &= \{x : x \equiv a (mod m)\} \\ &= \{x : x = a + mk,\ k = 0, \pm 1, \pm 2, \cdots\} \end{aligned}$$

CHAPTER 1

ARITHMETIC FUNCTION

An arithmetic Function is a function defined on the positive integers which take values in the real or complex numbers. i.e., A function f: N→ C is called an arithmetic function.

An arithmetic function is called multiplicative if f(mn) = f(m)f(n) for all coprime natural numbers m and n.

Examples

a) Sum of divisors σ(n)

b) Number of divisors τ(n)

c) Euler's function ϕ(n)

d) Mobius function μ(n)

Definition 1.1

Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of n and $\sigma(n)$ denote the sum of positive divisors of n.

Example

Consider $n = 12$. Since 12 has the positive divisors 1, 2, 3, 4, 6, 12, we find that

$\tau(12) = 6$ and $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$

For the first few integers,

$\tau(1) = 1 \quad \tau(2) = 2 \quad \tau(3) = 2 \quad \tau(4) = 3 \quad \tau(5) = 2 \quad \tau(6) = 4, \ldots$

$\sigma(1) = 1, \quad \sigma(2) = 3, \quad \sigma(3) = 4, \quad \sigma(4) = 7, \quad \sigma(5) = 6, \quad \sigma(6) = 12, \ldots$

It is not difficult to see that $\tau(n) = 2$ if and only if n is a prime number; also, $\sigma(n) = n + 1$ if and only if n is a prime.

Theorem 1.1

If n = $p_1^{k_1} \dots \dots \dots \dots . p_r^{k_r}$ is the prime factorization of $n > 1$, then

(a) $\tau(n) = (k_1 + 1)(k_2 + 1) \cdot \cdot \cdot (k_r + 1)$, and

(b) $\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1} \dots \dots \dots \dots \dots \dots \dots . \frac{p_r^{k_r+1}-1}{p_r-1}$

Proof

The positive divisors of n are precisely those integers

$$d = p_1^{a_1}\, p_2^{a_2} \dots \dots . . p_r^{a_r}$$

where $0 \le a_i \le k_i$. There are $k_1 + 1$ choices for the exponent a_1; $k_2 + 1$ choices for a_2, . . . ; and $k_r + 1$ choices for a_r . Hence, there are

$(k_1 + 1)(k_2 + 1) \cdot \cdot \cdot (k_r + 1)$

possible divisors of n.

To evaluate σ(n), consider the product

$$\left(1 + p_1 + P_1^2 + \cdots \dots \dots \dots P_1^{K_1}\right)\left(1 + p_2 + P_2^2 + \cdots \dots \dots \dots P_2^{K_2}\right) \dots \dots \dots \dots \dots \dots$$

$$\dots \dots \dots \dots \left(1 + p_r + P_r^2 + \cdots \dots \dots \dots P_r^{K_r}\right)$$

Each positive divisor of n appears once and only once as a term in the expansion of this product, so that

$$\sigma(n) = \left(1 + p_1 + P_1^2 + \cdots \dots \dots \dots P_1^{K_1}\right)\left(1 + p_2 + P_2^2 + \cdots \dots \dots \dots P_2^{K_2}\right) \dots \dots \dots \dots \dots \dots$$

$$\dots \dots \dots \dots \left(1 + p_r + P_r^2 + \cdots \dots \dots \dots P_r^{K_r}\right)$$

Applying the formula for the sum of a finite geometric series to the ith factor on the right-hand side, we get

$$\left(1 + p_i + P_i^2 + \cdots \dots \dots \dots P_i^{K_i}\right) = \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

It follows that

$$\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1} \dots\dots\dots\dots\dots\dots\dots . \frac{p_r^{k_r+1}-1}{p_r-1} .$$

Corresponding to the $\sum$ notation for sums, the notation for products may be defined using $\prod$, the Greek capital letter pi. The restriction delimiting the numbers over which the product is to be made is usually put under the $\prod$ sign.

Examples

$$\prod_{1 \le d \le 5} f(d) = f(1)f(2)f(3)f(4)f(5)$$

$$\prod_{d \mid 9} f(d) = f(1)f(3)f(9)$$

$$\prod_{\substack{p \mid 30 \\ p \text{ prime}}} f(p) = f(2)f(3)f(5)$$

With this convention, the conclusion to Theorem 1.1 takes the compact form: if $n = p_1^{k_1} p_2^{k_2} \dots \dots . . p_r^{k_r}$ is the prime factorization of n > 1, then

$$\tau(n) = \prod_{1 \le i \le r} (k_i + 1)$$

and

$$\sigma(n) = \prod_{1 \le i \le r} \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

Theorem 1.2

The functions τ and σ are both multiplicative functions

Proof

Let m and n be relatively prime integers. Because the result is trivially true if either m or n is equal to 1, we may assume that m > 1 and n > 1. If

$$m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \qquad \text{and} \qquad n = q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$$

are the prime factorizations of m and n . It follows that the prime factorization of the product mn is given by

$$mn = p_1^{k_1} \cdots p_r^{k_r} q_1^{j_1} \cdots q_s^{j_s}$$

Applying to theorem 1.1, we obtain

$$\begin{aligned} \tau(mn) &= [(k_1+1)\cdots(k_r+1)][(j_1+1)\cdots(j_s+1)] \\ &= \tau(m)\tau(n) \end{aligned}$$

In a similar fashion, theorem 1.1 gives

$$\begin{aligned} \sigma(mn) &= \left[\frac{p_1^{k_1+1}-1}{p_1-1} \cdots \frac{p_r^{k_r+1}-1}{p_r-1}\right]\left[\frac{q_1^{j_1+1}-1}{q_1-1} \cdots \frac{q_s^{j_s+1}-1}{q_s-1}\right] \\ &= \sigma(m)\sigma(n) \end{aligned}$$

Thus, τ and σ are multiplicative functions.

Theorem 1.3

If f is a multiplicative function and F is defined by

$$F(n) = \sum_{d \mid n} f(d)$$

then F is also multiplicative.

Proof

Let m and n be relatively prime positive integers. Then

$$F(mn) = \sum_{d \mid mn} f(d)$$
$$= \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1 d_2)$$

because every divisor d of mn can be uniquely written as a product of a divisor d_1 of m and a divisor d_2 of n, where $\gcd(d_1, d_2) = 1$. By the definition of a multiplicative function,

$$f(d_1 d_2) = f(d_1)\, f(d_2)$$

It follows that

$$F(mn) = \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1) f(d_2)$$
$$= \left(\sum_{d_1 \mid m} f(d_1)\right)\left(\sum_{d_2 \mid n} f(d_2)\right)$$
$$= F(m)F(n)$$

It might be helpful to take time out and run through the proof of Theorem 1.3 in a concrete case. Letting m = 8 and n = 3, we have

$$F(8 \cdot 3) = \sum_{d \mid 24} f(d)$$

$= f(1) + f(2) + f(3) + f(4) + f(6) + f(8) + f(12) + f(24)$

$= f(1 \cdot 1) + f(2 \cdot 1) + f(1 \cdot 3) + f(4 \cdot 1) + f(2 \cdot 3) + f(8 \cdot 1) + f(4 \cdot 3) + f(8 \cdot 3)$

$= f(1)\, f(1) + f(2)\, f(1) + f(1)\, f(3) + f(4)\, f(1) + f(2)\, f(3) + f(8)\, f(1)$

$+ f(4) f(3) + f(8)\, f(3)$

$= [f(1) + f(2) + f(4) + f(8)][f(1) + f(3)]$

$$= \sum_{d \mid 8} f(d) \cdot \sum_{d \mid 3} f(d)$$

$= F(8)F(3)$

Theorem 1.3 provides a deceptively short way of drawing the conclusion that τ and σ are multiplicative

The Mangoldt function $\Lambda(n)$

Definition 1.2

For every integer $n \geq 1$ we define

$$\Lambda(n) = \begin{cases} \log\, p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Here is a short table of values of $\Lambda(n)$:

n:	1	2	3	4	5	6	7	8	9	10
$\Lambda(n)$:	0	log 2	log 3	log 2	log 5	0	log 7	log 2	log 3	0

The proof of the next theorem shows how this function arises naturally from the fundamental theorem of arithmetic.

Theorem 1.4

If $n \geq 1$ we have

$$\log n = \sum_{d|n} \Lambda(d) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Proof

The theorem is true if $n = 1$ since both members are 0 . Therefore, assume that $n > 1$ and write

$$n = \prod_{k=1}^{r} p_k{}^{a_k}$$

Taking logarithms we have

$$\log\, n = \sum_{k=1}^{r} a_k \log\, p_k$$

Now consider the sum on the right of (1). The only nonzero terms in the sum come from those divisors d of the form $p_k{}^{m}$ for $m = 1,2, \dots, a_k$ and $k = 1,2, \dots, r$. Hence

$$\sum_{d|n} \Lambda(d) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \Lambda(p_k^m) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \log\, p_k = \sum_{k=1}^{r} a_k \log\, p_k = \log\, n$$

which proves (1).

CHAPTER 2

EULER'S ϕ FUNCTION

Let n be positive integer. Let U_n denote the set of all positive integers less than n and coprime to it

For example,

$$U_6 = \{1,5\}$$

$$U_{10} = \{1,3,7,9\}$$

$$U_{18} = \{1,5,7,11,13,17\}$$

Definition 2.1

Euler's ϕ function is a function ϕ: N→N such that for any n ∈ N, ϕ (n) is the number of integers less than n and coprime to it

In other words

'Euler's ϕ function counts the number of elements in U_n'

For example,

$$\phi(1) = 1, \phi(2) = 1, \phi(3) = 2, \phi(4) = 2, \phi(5) = 4$$
$$\phi(6) = 2 \ldots.$$

Theorem 2.1

Let p be a prime. Then ϕ (p) = p-1

Proof:

By definition, any natural number strictly less than p is coprime to p, hence

$$\phi\ (p) = p-1$$

Theorem 2.2

If p is a prime and $k > 0$, then

$$\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1)$$

Proof:

Consider the successive p^k natural numbers not greater than p^k arranged in the following rectangular array of p columns and p^{k-1} rows

1	2	.	.	p
p+1	p+2	.	.	2p
.	.	.	.	.
.	.	.	.	.
p^k-p+1	p^k-p+2	.	.	p^k

among these numbers only the ones at the rightmost sides are not coprime to p^k and there are p^{k-1} members in that column. So

$$\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1).$$

For example, $\phi(8) = 2^3 - 2^2 = 4$ which counts the number of elements in the set $U_8 = \{1,3,5,7\}$

By the fundamental theorem of arithmetic, we can write any natural number n as

$$n = p_1^{k_1} \dots \dots \dots . . p_r^{k_r}$$

where P_i ‘s are distinct prime and $k_i \geq 1$ are integers. We already know how to find $\phi\left(p_i^{k_i}\right)$ we would lie to see how $\phi(n)$ is related to $\phi\left(p_i^{k_i}\right)$. This follows from a very important property of Euler’s ϕ Function

Multiplicativity of Euler’s ϕ Function

Theorem 2.3

$\phi(\mathrm{mn}) = \phi(\mathrm{m})\phi(\mathrm{n})$ if m and n are coprime natural numbers.

Proof:

Consider the array of natural numbers not greater than mn arranged in m columns and n rows in the following manner

$$\begin{array}{ccccccc} 1 & 2 & \cdots & r & \cdots & m \\ m+1 & m+2 & & m+r & & 2m \\ 2m+1 & 2m+2 & & 2m+r & & 3m \\ \vdots & \vdots & & \vdots & & \vdots \\ (n-1)m+1 & (n-1)m+2 & & (n-1)m+r & & nm \end{array}$$

Clearly each row of the above array has m distinct residues modulo m. Each column has n distinct residues modulo n: for $1 \leq i, i \leq n-1$

im +j $\equiv$ $im + j$ (mod n)

$\Rightarrow$ $im \equiv im$ (mod n)

$\Rightarrow$ $i \equiv i$ (mod n)   (as gcd(m,n) = 1)

$\Rightarrow$ $i \equiv i$

Each row has ϕ(m) residues coprime to m, and each column has ϕ(n) residues coprime to n. Hence in total ϕ(m)ϕ(n) elements in the above array which are coprime to both m and n, it follows that

$$\phi(\mathrm{mn}) = \phi(\mathrm{m})\phi(\mathrm{n})$$

Theorem 2.4

Let n be any natural numbers, then

$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$

Proof:

By fundamental theorem of arithmetic, we can write

$$n = P_1^{k_1}P_2^{k_2} \ldots \ldots . \ldots P_r^{k_r}$$

Where p_i are the distinct prime factor of n, and k_i are the non negative integers. By previous theorem and proposition,

$$\phi(n) = \phi\left(p_1^{k_1}\right) \cdot \ldots , \phi\left(p_r^{k_r}\right)$$

$$= P_1^{k_1-1}(P_1-1)\cdots P_r^{k_{r-1}}(P_r-1)$$

$$= p_1^{k_1}\left(1-\frac{1}{P_1}\right)\cdots P_r^{k_r}\left(1-\frac{1}{P_r}\right)$$

$$= n\left(1-\frac{1}{p_1}\right)\cdots\cdot\left(1-\frac{1}{p_r}\right)$$

Theorem 2.5

For $n > 2$, $\phi(n)$ is an even integer.

Proof:

First, assume that n is a power of 2, let us say that $n = 2^k$, with $k \geq 2$. By

theorem 2.2,

$$\phi(n) = \phi(2^k) = 2^k\left(1-\frac{1}{2}\right) = 2^{k-1}$$

an even integer. If n does not happen to be a power of 2, then it is divisible by an odd prime p; we therefore may write n as $n = p^k m$, where $k \geq 1$ and gcd $(p^k, m) = 1$. Exploiting the multiplicative nature of the phi-function, we obtain

$$\phi(n) = \phi(p^k)\phi(m) = p^{k-1}(p-1)\phi(m)$$

which again is even because $2 \mid p - 1$.

Theorem 2.6

For each positive integer n,

$$n = \sum_{d|n} \phi(d)$$

Proof:

Let us partition the set $\{1,2,\ldots\ldots,n\}$ into mutually disjoint subsets S_d for each d/n, where

$$S_d = \{1 \leq m \leq n \mid \text{gcd}\,(m,n) = d\}$$

$$= \{1 \leq \tfrac{m}{d} \leq \tfrac{n}{d} \mid \text{gcd}\,(\tfrac{m}{d},\tfrac{n}{d}) = 1\}$$

Then

$$\{1,2,\ldots\ldots,n\} = \sum_{d|n} S_d$$

$$\Rightarrow \qquad n = \sum_{d|n} \phi\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \phi(d)$$

As for each divisor of n, n/d is also a divisor of n

MOBIUS FUNCTION

Definition 2.2

The Mobius function $\mu: N \longrightarrow \{0, \pm 1\}$ is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2/n \text{ for some prime } p \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r, \text{ where } p_i \text{ are distinct primes} \end{cases}$$

For example,

$\mu(1) = 1$ $\qquad \mu(2) = -1$ $\qquad \mu(3) = -1$

$\mu(4) = 0$ $\qquad \mu(5) = -1$ $\qquad \mu(6) = 1$

If p is a prime number, it is clear that $\mu(p) = -1$; in addition, $\mu(p^e) = 0$ for $e \geq 2$.

Theorem 2.7

The Mobius function is a multiplicative function i.e.

$\mu(mn) = \mu(m)\mu(n)$, if m and n are relatively prime

Proof:

Let m and n be coprime integers, we can consider the following to cases

Case 1: let $\mu(mn) = 0$ then there is a prime p such that $p^2/_{mn}$. As m and n are coprime p cannot divide both m and n hence either $p^2/_m$ or $p^2/_n$. Therefore either $\mu(m) = 0$ or $\mu(n) = 0$ and we have $\mu(mn) = \mu(m)\mu(n)$

Case 2: suppose that $\mu(mn) \neq 0$ then mn is square free, hence so are m and n. let

$m = p_1 \dots\dots\dots p_r$ and $n = q_1 \dots\dots\dots q_s$ where p_i *and* q_j are all distinct primes then mn = $p_1 \dots\dots\dots p_r q_1 \dots\dots\dots q_s$ where all the primes occurring in the factorization of mn are distinct. Hence

$$\mu(mn) = (-1)^{r+s}$$

$$= (-1)^r(-1)^s$$

$$= \mu(m)\mu(n)$$

Theorem 2.8

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Where d runs through all the positive divisors of n.

Proof:

$$Let \quad F(n) = \sum_{d|n} \mu(d)$$

As μ is multiplicative, so is F(n) by the theorem (F be a multiplicative arithmetic function $F(n) = \sum_{d|n} f(d)$ then F is also a multiplicative arthmetic function)

Clearly

$$F(1) = \sum_{d|n} \mu(d)$$

= μ(1)

= 1

For integers which are prime power, i.e. of the form p^k for some $k \geq 1$

$$F(p^2) = \mu(1) + \mu(p) + \mu(p^2) + \cdots \dots\dots\dots.. + \mu(p^k)$$

= 1 + (-1) + 0…………….+ 0

= 0

Now consider any integer n, and consider its prime factorization. Then

$$n = p_1^{k_1} \dots\dots\dots\dots\dots. p_r^{k_r}, \qquad k_i \geq 1$$

$$\Rightarrow \quad F(n) = \prod F(p_i^{k_i})$$

$$= 0$$

Mobius inversion formula

The following theorem is known as Mobius inversion formula

Theorem 2.9

Let F and f be two function from the set N of natural number to the field complex number C such that

$$F(n) = \sum_{d|n} f(d)$$

Then we can express f(n) as

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d)$$

Proof:

First observe that if d is divisor of n so is n/d. Hence both the summation in the last line of the theorem are same. Now

$$\sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \left(\mu(d) \sum_{c|(n/d)} f(c)\right)$$

The crucial step in the proof is to observe that the set of S of pairs of integers (c,d) with d|n and c|n/d is the same as the set T of pairs (c,d) with c/n and d|n/c.

$$\sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \left(\mu(d) \sum_{c|(n/d)} f(c)\right)$$

$$= \sum_{d|n} \left(\sum_{c|(n/d)} \mu(d) f(c)\right)$$

$$= \sum_{(c,d)\in S} f(c)\mu(d)$$

$$= \sum_{(c,d)\in T} f(c)\mu(d)$$

$$= \sum_{c|n} \left(f(c) \sum_{d|(n/c)} \mu(d) \right)$$

= F(n)

$As \sum_{d|n} \mu(d) = 0 \quad unless\ {}^{n}/_{c} = 1,$ which happens when c = n

Let us demonstrate this with n = 15

$$\sum_{d|15} \mu(d)F\left(\frac{15}{d}\right) = \mu(1)[\, f\ (1) + \ f\ (3) + \ f\ (5) + \ f\ (15)] + \mu(3)[\, f\ (1) + \ f\ (5)]$$

$$+\mu(5)[\, f\ (1) + \ f\ (3)] + \mu(15)[\, f\ (1)]$$

= f (1)[μ(1) + μ(3) + μ(5) + μ(15)] + f (3)[μ(1) + μ(5)] + f (5)[μ(1) + μ (5)] + f(15) μ(1)

= f(1).0 + f(3).0 + f(5).0 + f(15)

= f(15)

The above theorem leads to the following interesting identities

1. we know that for any positive integer n,

$$\sum_{d|n} \phi(d) = n$$

Where $\phi(n)$ is Euler's ϕ function. Hence

$$\phi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d$$

For example,

$$\phi(10) = \mu(1)10 + \mu(2)5 + \mu(5)2 + \mu(10)1$$

$= 10 - 5 – 2 + 1$

$= 4$

2. similarly

$$\sigma(n) = \sum_{d|n} d$$
$$n = \sum_{d|n} \mu\left(\frac{n}{d}\right)\sigma(d)$$

For example,

With n = 10

$$\mu(10).1 + \mu(2)(1+5) + \mu(5)(1+3) + \mu(1)(1+3+5+10)$$

$= 1 – 1 – 5 – 1 – 3 + 1 + 3 +5 + 10$

$= 10$

We have seen before that if multiplicative so is $F(n) = \sum_{d|n} f(d)$. But we can now

Prove that converse applying the Mobius inversion formula

Theorem 2.10

If F is a multiplicative function and

$$F(n) = \sum_{d|n} f(d)$$

then f is also multiplicative.

Proof:

By the Mobius inversion formula we know that

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d)$$

Let m and n be relatively prime positive integers. We recall that any divisor

d of mn can be uniquely written as $d = d_1, d_2$, where $d_1 | m$, $d_2 | n$, and

$\gcd(d_1, d_2) = 1 = \gcd(\frac{m}{d_1}, \frac{n}{d_2})$.

Conversely if d_1/m and d_2/n then d_1d_2/mn thus,

$$\begin{aligned}
f(mn) &= \sum_{d|mn} \mu(d)F\left(\frac{mn}{d}\right) \\
&= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1d_2)F\left(\frac{mn}{d_1d_2}\right) \\
&= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1)\mu(d_2)F\left(\frac{m}{d_1}\right)F\left(\frac{n}{d_2}\right) \\
&= \sum_{d_1|m} \mu(d_1)F\left(\frac{m}{d_1}\right)\sum_{d_2|n} \mu(d_2)F\left(\frac{n}{d_2}\right) \\
&= f(m)f(n)
\end{aligned}$$

In view of the above theorem we can say that as N(n) = n is a multiplicative function so is $\phi(n)$ because

$$\sum_{d|n} \phi(d) = n = \mathrm{N(n)}$$

CONCLUSION

The purpose of this project gives a simple account of Arithmetic function, Euler's phi function and Mobius Function. The study of these topics given excellent introduction to the subject called 'NUMBER THEORETIC FUNCTION'

Number Theoretic Function demands a high standard of rigor. Thus, our presentation necessarily has its formal aspect with care taken to present clear and detailed argument. An understanding of the statement of the theorem, number theory proof is the important issue. In the first chapter we discuss about function τ and σ are both multiplicative function. If f is a multiplicative function and F is defined by

$F(n) = \sum_{d|n} f(d)$, then F is also multiplicative. In the second chapter 2 we discuss about that if p is prime the $\phi(p) = p - 1$, $\phi(mn) = \phi(m)\phi(n)$. The Mobius function is multiplicative function if f is multiplicative function and $F(n)$=$\sum_{d|n} f(d)$,

then F is also multiplicative.

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NUMBER THEORETIC FUNCTION

Project report submitted to
The Kannur University
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by

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CERTIFICATE

Certified that this project **'Number Theoretic Function'** is a bona fide project of **SHAMNA THASNI T** carried out the project work under my supervision.

Mrs. Riya Baby
Head of Department

Ms. Ajeena joseph
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DECLARATION

I **SHAMNA THASNI T** hereby declare that the project **'Number Theoretic Function'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Ajeena joseph, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

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Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express my deepest gratitude to people along the way.

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My greatest debt is always, to God Almighty.

CONTENTS

INTRODUCTION

A Number Theoretic Function is a complex valued function defined for all positive integers. In Number Theory, there exist many number theoretic functions. This includes Divisor Function, Sigma Function, Euler's-Phi Function and Mobius Function. All these functions play a very important role in the field of Number Theory.

In the first chapter we will discuss about Arithmetic Function. In the second chapter we will introduce Euler's-Phi Function and Mobius Function.

PRELIMINARY

Let n be a fixed positive integer. Two integers a and b are said to be *congruent modulo n,* symbolized by

$$a \equiv b \pmod{n}$$

if n divides the difference $a - b$; that is, provided that $a - b = kn$ for some integer k.

Example:

To fix the idea, consider $n = 7$. It is routine to check that

$3 \equiv 24 \pmod{7}$ $\quad -31 \equiv 11 \pmod{7}$ $\quad -15 \equiv -64 \pmod{7}$

Because $3 - 24 = (-3)7$, $-31 - 11 = (-6)7$ and $-15 - (-64) = 77$. When n does not divide $(a - b)$, we say that a is *incongruent to b modulo n*, and in this case we write

$a \not\equiv b \pmod{n}$. For a simple example: $25 \not\equiv 12 \pmod{7}$, because 7 fails to divide

$25 - 12 = 13$.

It is to be noted that any two integers are congruent modulo 1, whereas two integers are congruent modulo 2 when they are both even or both odd. In as much as congruence modulo 1 is not particularly interesting, the usual practice is to assume that $n > 1$.

Remark:

Given an integer a, let q and r be its quotient and remainder upon division by n, so that

$a = qn + r \quad 0 \leq r < n$

Then, by definition of congruence, $a \equiv r \pmod{n}$. Because there are n choices for r , we see that every integer is congruent modulo n to exactly one of the values $0, 1, 2, \ldots, n - 1$; in particular, $a \equiv 0 \pmod{n}$ if and only if $n \mid a$.

Fundamental Theorem of Arithmetic

is Every integer $n > 1$ can be represented as Product of prime factor in only one way, apart from the order of the factors.

Residue

If a is an integer and m is a positive integer then the residue class of a modulo m is denoted by $\hat{a}$ and is given by

$$\begin{aligned} \hat{a} \quad &= \{x : x \equiv a(mod m)\} \\ &= \{x : x = a + mk,\ k = 0, \pm 1, \pm 2, \cdots\} \end{aligned}$$

CHAPTER 1

ARITHMETIC FUNCTION

An arithmetic Function is a function defined on the positive integers which take values in the real or complex numbers. i.e., A function $f: N \to C$ is called an arithmetic function.

An arithmetic function is called multiplicative if $f(mn) = f(m)f(n)$ for all coprime natural numbers m and n.

Examples

a) Sum of divisors σ(n)

b) Number of divisors τ(n)

c) Euler's function ϕ(n)

d) Mobius function μ(n)

Definition 1.1

Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of n and $\sigma(n)$ denote the sum of positive divisors of n.

Example

Consider $n = 12$. Since 12 has the positive divisors 1, 2, 3, 4, 6, 12, we find that

$\tau(12) = 6$ and $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$

For the first few integers,

$\tau(1) = 1$ $\tau(2) = 2$ $\tau(3) = 2$ $\tau(4) = 3$ $\tau(5) = 2$ $\tau(6) = 4, \ldots$

$\sigma(1) = 1,$ $\sigma(2) = 3,$ $\sigma(3) = 4,$ $\sigma(4) = 7,$ $\sigma(5) = 6,$ $\sigma(6) = 12, \ldots$

It is not difficult to see that $\tau(n) = 2$ if and only if n is a prime number; also, $\sigma(n) = n + 1$ if and only if n is a prime.

Theorem 1.1

If n = $p_1^{k_1} \dots \dots \dots \dots . p_r^{k_r}$ is the prime factorization of $n > 1$, then

(a) $\tau(n) = (k_1 + 1)(k_2 + 1) \cdot \cdot \cdot (k_r + 1)$, and

(b) $\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1} \dots \dots \dots \dots \dots \dots \dots . \frac{p_r^{k_r+1}-1}{p_r-1}$

Proof

The positive divisors of n are precisely those integers

$$d = p_1^{a_1}\, p_2^{a_2} \dots \dots . . p_r^{a_r}$$

where $0 \le a_i \le k_i$. There are $k_1 + 1$ choices for the exponent a_1; $k_2 + 1$ choices for a_2, . . . ; and $k_r + 1$ choices for a_r . Hence, there are

$(k_1 + 1)(k_2 + 1) \cdot \cdot \cdot (k_r + 1)$

possible divisors of n.

To evaluate σ(n), consider the product

$$\left(1 + p_1 + P_1^2 + \cdots \dots \dots \dots P_1^{K_1}\right)\left(1 + p_2 + P_2^2 + \cdots \dots \dots \dots P_2^{K_2}\right) \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \left(1 + p_r + P_r^2 + \cdots \dots \dots \dots P_r^{K_r}\right)$$

Each positive divisor of n appears once and only once as a term in the expansion of this product, so that

$$\sigma(n) = \left(1 + p_1 + P_1^2 + \cdots \dots \dots \dots P_1^{K_1}\right)\left(1 + p_2 + P_2^2 + \cdots \dots \dots \dots P_2^{K_2}\right) \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \left(1 + p_r + P_r^2 + \cdots \dots \dots \dots P_r^{K_r}\right)$$

Applying the formula for the sum of a finite geometric series to the ith factor on the right-hand side, we get

$$\left(1 + p_i + P_i^2 + \cdots \dots \dots \dots P_i^{K_i}\right) = \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

It follows that

$$\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1} \dots\dots\dots\dots\dots\dots\dots \frac{p_r^{k_r+1}-1}{p_r-1} .$$

Corresponding to the $\sum$ notation for sums, the notation for products may be defined using $\prod$, the Greek capital letter pi. The restriction delimiting the numbers over which the product is to be made is usually put under the $\prod$ sign.

Examples

$$\prod_{1\le d\le 5} f(d) = f(1)f(2)f(3)f(4)f(5)$$

$$\prod_{d\,|\,9} f(d) = f(1)f(3)f(9)$$

$$\prod_{\substack{p\,|\,30\\ p\ \text{prime}}} f(p) = f(2)f(3)f(5)$$

With this convention, the conclusion to Theorem 1.1 takes the compact form: if $n = p_1^{k_1}\, p_2^{k_2} \dots \dots .\, p_r^{k_r}$ is the prime factorization of n > 1, then

$$\tau(n) = \prod_{1\le i\le r} (k_i + 1)$$

and

$$\sigma(n) = \prod_{1\le i\le r} \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

Theorem 1.2

The functions τ and σ are both multiplicative functions

Proof

Let m and n be relatively prime integers. Because the result is trivially true if either m or n is equal to 1, we may assume that m > 1 and n > 1. If

$$m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \qquad \text{and} \qquad n = q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$$

are the prime factorizations of m and n . It follows that the prime factorization of the product mn is given by

$$mn = p_1^{k_1} \cdots p_r^{k_r} q_1^{j_1} \cdots q_s^{j_s}$$

Applying to theorem 1.1, we obtain

$$\begin{aligned} \tau(mn) &= [(k_1 + 1) \cdots (k_r + 1)][(j_1 + 1) \cdots (j_s + 1)] \\ &= \tau(m)\tau(n) \end{aligned}$$

In a similar fashion, theorem 1.1 gives

$$\begin{aligned} \sigma(mn) &= \left[\frac{p_1^{k_1+1} - 1}{p_1 - 1} \cdots \frac{p_r^{k_r+1} - 1}{p_r - 1} \right] \left[\frac{q_1^{j_1+1} - 1}{q_1 - 1} \cdots \frac{q_s^{j_s+1} - 1}{q_s - 1} \right] \\ &= \sigma(m)\sigma(n) \end{aligned}$$

Thus, τ and σ are multiplicative functions.

Theorem 1.3

If f is a multiplicative function and F is defined by

$$F(n) = \sum_{d \mid n} f(d)$$

then F is also multiplicative.

Proof

Let m and n be relatively prime positive integers. Then

$$F(mn) = \sum_{d \mid mn} f(d)$$
$$= \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1 d_2)$$

because every divisor d of mn can be uniquely written as a product of a divisor d_1 of m and a divisor d_2 of n, where $\gcd(d_1, d_2) = 1$. By the definition of a multiplicative function,

$$f(d_1 d_2) = f(d_1)\, f(d_2)$$

It follows that

$$F(mn) = \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1) f(d_2)$$
$$= \left(\sum_{d_1 \mid m} f(d_1)\right)\left(\sum_{d_2 \mid n} f(d_2)\right)$$
$$= F(m)F(n)$$

It might be helpful to take time out and run through the proof of Theorem 1.3 in a concrete case. Letting m = 8 and n = 3, we have

$$F(8 \cdot 3) = \sum_{d \mid 24} f(d)$$

$= f(1) + f(2) + f(3) + f(4) + f(6) + f(8) + f(12) + f(24)$

$= f(1 \cdot 1) + f(2 \cdot 1) + f(1 \cdot 3) + f(4 \cdot 1) + f(2 \cdot 3) + f(8 \cdot 1) + f(4 \cdot 3) + f(8 \cdot 3)$

$= f(1)\, f(1) + f(2)\, f(1) + f(1)\, f(3) + f(4)\, f(1) + f(2)\, f(3) + f(8)\, f(1)$

$+ f(4)f(3) + f(8)\, f(3)$

$= [f(1) + f(2) + f(4) + f(8)][f(1) + f(3)]$

$$= \sum_{d \mid 8} f(d) \cdot \sum_{d \mid 3} f(d)$$

$= F(8)F(3)$

Theorem 1.3 provides a deceptively short way of drawing the conclusion that τ and σ are multiplicative

The Mangoldt function $\Lambda(n)$

Definition 1.2

For every integer $n \geq 1$ we define

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here is a short table of values of $\Lambda(n)$:

n:	1	2	3	4	5	6	7	8	9	10
$\Lambda(n)$:	0	log 2	log 3	log 2	log 5	0	log 7	log 2	log 3	0

The proof of the next theorem shows how this function arises naturally from the fundamental theorem of arithmetic.

Theorem 1.4

If $n \geq 1$ we have

$$\log n = \sum_{d|n} \Lambda(d) \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (1)$$

Proof

The theorem is true if $n = 1$ since both members are 0 . Therefore, assume that $n > 1$ and write

$$n = \prod_{k=1}^{r} p_k{}^{a_k}$$

Taking logarithms we have

$$\log\ n = \sum_{k=1}^{r} a_k \log\ p_k$$

Now consider the sum on the right of (1). The only nonzero terms in the sum come from those divisors d of the form $p_k{}^m$ for $m = 1,2,\dots,a_k$ and $k = 1,2,\dots,r$. Hence

$$\sum_{d|n} \Lambda(d) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \Lambda(p_k^m) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \log\ p_k = \sum_{k=1}^{r} a_k \log\ p_k = \log\ n$$

which proves (1).

CHAPTER 2

EULER'S ϕ FUNCTION

Let n be positive integer. Let U_n denote the set of all positive integers less than n and coprime to it

For example,

$$U_6 = \{1,5\}$$

$$U_{10} = \{1,3,7,9\}$$

$$U_{18} = \{1,5,7,11,13,17\}$$

Definition 2.1

Euler's ϕ function is a function ϕ: N→N such that for any n ∈ N, ϕ (n) is the number of integers less than n and coprime to it

In other words

'Euler's ϕ function counts the number of elements in U_n'

For example,

$$\phi(1) = 1, \phi(2) = 1, \phi(3) = 2, \phi(4) = 2, \phi(5) = 4$$
$$\phi(6) = 2 \dots.$$

Theorem 2.1

Let p be a prime. Then ϕ (p) = p-1

Proof:

By definition, any natural number strictly less than p is coprime to p, hence

$$\phi \text{ (p)} = \text{p-1}$$

Theorem 2.2

If p is a prime and $k > 0$, then

$$\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1)$$

Proof:

Consider the successive p^k natural numbers not greater than p^k arranged in the following rectangular array of p columns and p^{k-1} rows

1	2	.	.	p
p+1	p+2	.	.	2p
.	.	.	.	.
.	.	.	.	.
p^k-p+1	p^k-p+2	.	.	p^k

among these numbers only the ones at the rightmost sides are not coprime to p^k and there are p^{k-1} members in that column. So

$$\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1).$$

For example, $\phi(8) = 2^3 - 2^2 = 4$ which counts the number of elements in the set $U_8 = \{1,3,5,7\}$

By the fundamental theorem of arithmetic, we can write any natural number n as

$$n = p_1^{k_1} \ldots\ldots\ldots.. p_r^{k_r}$$

where P_i 's are distinct prime and $k_i \geq 1$ are integers. We already know how to find $\phi\left(p_i^{k_i}\right)$ we would lie to see how $\phi(n)$ is related to $\phi\left(p_i^{k_i}\right)$. This follows from a very important property of Euler's ϕ Function

Multiplicativity of Euler's ϕ Function

Theorem 2.3

$\phi(mn) = \phi(m)\phi(n)$ if m and n are coprime natural numbers.

Proof:

Consider the array of natural numbers not greater than mn arranged in m columns and n rows in the following manner

$$\begin{array}{ccccccc} 1 & 2 & \cdots & r & \cdots & m \\ m+1 & m+2 & & m+r & & 2m \\ 2m+1 & 2m+2 & & 2m+r & & 3m \\ \vdots & \vdots & & \vdots & & \vdots \\ (n-1)m+1 & (n-1)m+2 & & (n-1)m+r & & nm \end{array}$$

Clearly each row of the above array has m distinct residues modulo m. Each column has n distinct residues modulo n: for $1 \leq i, i \leq n-1$

im +j $\equiv$ im + j (mod n)

$\Rightarrow im \equiv im$ (mod n)

$\Rightarrow i \equiv i$ (mod n)   (as gcd(m,n) = 1)

$\Rightarrow i \equiv i$

Each row has ϕ(m) residues coprime to m, and each column has ϕ(n) residues coprime to n. Hence in total ϕ(m)ϕ(n) elements in the above array which are coprime to both m and n, it follows that

$$\phi(\mathrm{mn}) = \phi(\mathrm{m})\phi(\mathrm{n})$$

Theorem 2.4

Let n be any natural numbers, then

$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$

Proof:

By fundamental theorem of arithmetic, we can write

$$n = P_1^{k_1}P_2^{k_2} \ldots \ldots . \ldots P_r^{k_r}$$

Where p_i are the distinct prime factor of n, and k_i are the non negative integers. By previous theorem and proposition,

$$\phi(n) = \phi\left(p_1^{k_1}\right) \cdot \ldots, \phi\left(p_r^{k_r}\right)$$

$$= P_1^{k_1-1}(P_1-1)\cdots P_r^{k_{r-1}}(P_r-1)$$

$$= p_1^{k_1}\left(1-\frac{1}{P_1}\right)\cdots P_r^{k_r}\left(1-\frac{1}{P_r}\right)$$

$$= n\left(1-\frac{1}{p_1}\right)\cdots\cdot\left(1-\frac{1}{p_r}\right)$$

Theorem 2.5

For $n > 2$, $\phi(n)$ is an even integer.

Proof:

First, assume that n is a power of 2, let us say that $n = 2^k$, with $k \geq 2$. By

theorem 2.2,

$$\phi(n) = \phi(2^k) = 2^k\left(1-\frac{1}{2}\right) = 2^{k-1}$$

an even integer. If n does not happen to be a power of 2, then it is divisible by an odd prime p; we therefore may write n as $n = p^k m$, where $k \geq 1$ and gcd $(p^k, m) = 1$. Exploiting the multiplicative nature of the phi-function, we obtain

$$\phi(n) = \phi(p^k)\phi(m) = p^{k-1}(p-1)\phi(m)$$

which again is even because $2 \mid p - 1$.

Theorem 2.6

For each positive integer n,

$$n = \sum_{d|n} \phi(d)$$

Proof:

Let us partition the set {1,2,…….,n} into mutually disjoint subsets S_d for each d/n, where

$$S_d = \{1 \leq m \leq n \mid \text{gcd}\,(m, n) = d\}$$

$$= \{1 \leq \frac{m}{d} \leq \frac{n}{d} \mid \text{gcd}\,(\frac{m}{d}, \frac{n}{d}) = 1\}$$

Then

$$\{1,2,\ldots\ldots,n\} = \sum_{d|n} S_d$$

$$\Rightarrow \qquad n = \sum_{d|n} \phi\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \phi(d)$$

As for each divisor of n, n/d is also a divisor of n

MOBIUS FUNCTION

Definition 2.2

The Mobius function $\mu: N \longrightarrow \{0, \pm 1\}$ is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2/n \text{ for some prime } p \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r, \text{ where } p_i \text{ are distinct primes} \end{cases}$$

For example,

$\mu(1) = 1$ $\mu(2) = -1$ $\mu(3) = -1$

$\mu(4) = 0$ $\mu(5) = -1$ $\mu(6) = 1$

If p is a prime number, it is clear that $\mu(p) = -1$; in addition, $\mu(p^e) = 0$ for $e \geq 2$.

Theorem 2.7

The Mobius function is a multiplicative function i.e.

$\mu(mn) = \mu(m)\mu(n)$, if m and n are relatively prime

Proof:

Let m and n be coprime integers, we can consider the following to cases

Case 1: let $\mu(mn) = 0$ then there is a prime p such that $p^2/_{mn}$. As m and n are coprime p cannot divide both m and n hence either $p^2/_m$ or $p^2/_n$. Therefore either $\mu(m) = 0$ or $\mu(n) = 0$ and we have $\mu(mn) = \mu(m)\mu(n)$

Case 2: suppose that $\mu(mn) \neq 0$ then mn is square free, hence so are m and n. let

$m = p_1 \dots\dots\dots p_r$ and $n = q_1 \dots\dots\dots q_s$ where p_i *and* q_j are all distinct primes then mn = $p_1 \dots\dots\dots p_r q_1 \dots\dots\dots q_s$ where all the primes occurring in the factorization of mn are distinct. Hence

$$\mu(mn) = (-1)^{r+s}$$

$$= (-1)^r(-1)^s$$

$$= \mu(m)\mu(n)$$

Theorem 2.8

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Where d runs through all the positive divisors of n.

Proof:

$$Let \quad F(n) = \sum_{d|n} \mu(d)$$

As μ is multiplicative, so is F(n) by the theorem (F be a multiplicative arithmetic function $F(n) = \sum_{d|n} f(d)$ then F is also a multiplicative arthmetic function)

Clearly

$$F(1) = \sum_{d|n} \mu(d)$$

= μ(1)

= 1

For integers which are prime power, i.e. of the form p^k for some $k \geq 1$

$$F(p^2) = \mu(1) + \mu(p) + \mu(p^2) + \cdots \dots\dots\dots . . + \mu(p^k)$$

= 1 + (-1) + 0..............+ 0

= 0

Now consider any integer n, and consider its prime factorization. Then

$$n = p_1^{k_1} \dots\dots\dots\dots . . p_r^{k_r}, \qquad k_i \geq 1$$

$$\Rightarrow \quad F(n) = \prod F(p_i^{k_i})$$

$$= 0$$

Mobius inversion formula

The following theorem is known as Mobius inversion formula

Theorem 2.9

Let F and f be two function from the set N of natural number to the field complex number C such that

$$F(n) = \sum_{d|n} f(d)$$

Then we can express f(n) as

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d)$$

Proof:

First observe that if d is divisor of n so is n/d. Hence both the summation in the last line of the theorem are same. Now

$$\sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \left(\mu(d) \sum_{c|(n/d)} f(c)\right)$$

The crucial step in the proof is to observe that the set of S of pairs of integers (c,d) with d|n and c|n/d is the same as the set T of pairs (c,d) with c/n and d|n/c.

$$\sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \left(\mu(d) \sum_{c|(n/d)} f(c)\right)$$

$$= \sum_{d|n} \left(\sum_{c|(n/d)} \mu(d) f(c)\right)$$

$$= \sum_{(c,d)\in S} f(c)\mu(d)$$

$$= \sum_{(c,d)\in T} f(c)\mu(d)$$

$$= \sum_{c|n} \left(f(c) \sum_{d|(n/c)} \mu(d) \right)$$

= F(n)

$As \sum_{d|n} \mu(d) = 0 \quad unless\ {}^{n}/_{c} = 1,$ which happens when c = n

Let us demonstrate this with n = 15

$$\sum_{d|15} \mu(d)F\left(\frac{15}{d}\right) = \mu(1)[\, f\ (1) + \ f\ (3) + \ f\ (5) + \ f\ (15)] + \mu(3)[\, f\ (1) + \ f\ (5)]$$

$$+\mu(5)[\, f\ (1) + \ f\ (3)] + \mu(15)[\, f\ (1)]$$

= f (1)[μ(1) + μ(3) + μ(5) + μ(15)] + f (3)[μ(1) + μ(5)] + f (5)[μ(1) + μ (5)] + f(15) μ(1)

= f(1).0 + f(3).0 + f(5).0 + f(15)

= f(15)

The above theorem leads to the following interesting identities

1. we know that for any positive integer n,

$$\sum_{d|n} \phi(d) = n$$

Where $\phi(n)$ is Euler's ϕ function. Hence

$$\phi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d$$

For example,

$$\phi(10) = \mu(1)10 + \mu(2)5 + \mu(5)2 + \mu(10)1$$

= 10 - 5 – 2 + 1

= 4

2. similarly

$$\sigma(n) = \sum_{d|n} d$$
$$n = \sum_{d|n} \mu\left(\frac{n}{d}\right)\sigma(d)$$

For example,

With n = 10

$$\mu(10).1 + \mu(2)(1+5) + \mu(5)(1+3) + \mu(1)(1+3+5+10)$$

= 1 – 1 – 5 – 1 – 3 + 1 + 3 +5 + 10

= 10

We have seen before that if multiplicative so is $F(n) = \sum_{d|n} f(d)$. But we can now

Prove that converse applying the Mobius inversion formula

Theorem 2.10

If F is a multiplicative function and

$$F(n) = \sum_{d|n} f(d)$$

then f is also multiplicative.

Proof:

By the Mobius inversion formula we know that

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d)$$

Let m and n be relatively prime positive integers. We recall that any divisor

d of mn can be uniquely written as $d = d_1, d_2$, where $d_1 | m$, $d_2 | n$, and

gcd(d_1, d_2) = 1 = gcd($\frac{m}{d_1}, \frac{n}{d_2}$).

Conversely if d_1/m and d_2/n then d_1d_2/mn thus,

$$\begin{aligned} f(mn) &= \sum_{d|mn} \mu(d)F\left(\frac{mn}{d}\right) \\ &= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1d_2)F\left(\frac{mn}{d_1d_2}\right) \\ &= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1)\mu(d_2)F\left(\frac{m}{d_1}\right)F\left(\frac{n}{d_2}\right) \\ &= \sum_{d_1|m} \mu(d_1)F\left(\frac{m}{d_1}\right)\sum_{d_2|n} \mu(d_2)F\left(\frac{n}{d_2}\right) \\ &= f(m)f(n) \end{aligned}$$

In view of the above theorem we can say that as N(n) = n is a multiplicative function so is $\phi(n)$ because

$$\sum_{d|n} \phi(d) = n = \mathrm{N(n)}$$

CONCLUSION

The purpose of this project gives a simple account of Arithmetic function, Euler's phi function and Mobius Function. The study of these topics given excellent introduction to the subject called 'NUMBER THEORETIC FUNCTION'

Number Theoretic Function demands a high standard of rigor. Thus, our presentation necessarily has its formal aspect with care taken to present clear and detailed argument. An understanding of the statement of the theorem, number theory proof is the important issue. In the first chapter we discuss about function τ and σ are both multiplicative function. If f is a multiplicative function and F is defined by

$F(n) = \sum_{d|n} f(d)$, then F is also multiplicative. In the second chapter 2 we discuss about that if p is prime the $\phi(p) = p - 1,\ \phi(mn) = \phi(m)\phi(n)$. The Mobius function is multiplicative function if f is multiplicative function and $F(n)$=$\sum_{d|n} f(d)$,

then F is also multiplicative.

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DON BOSCO ARTS & SCIENCE COLLEGE

ANGADIKADAVU

DEPARTMENT OF MATHEMATICS

2018-2021

Project Report on

INNER PRODUCT SPACES

DEPARTMENT OF MATHEMATICS

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ANGADIKADAVU

MARCH 2021

Project Report on

INNER PRODUCT SPACES

Dissertation submitted in the partial fulfilment

of the requirement for the award of

Bachelor of Science in Mathematics of

Kannur University

Name : SHARON JOSEPH

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Examiners: 1.

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KANNUR UNIVERSITY

BONAFIDE CERTIFICATE

Certified that this project report on " INNER PRODUCT SPACES" is the bonafide work of SHARON JOSEPH who carried out the project work under my supervision.

Mrs. Riya Baby

Head of Department

Mr. Anil M V

Supervisor

DECLARATION

I, SHARON JOSEPH hereby declare that the project work entitled ‘INNER PRODUCT SPACES′ has been prepared by me and submitted to Kannur University in partial fulfilment of requirement for the award of Bachelor of Science is a record of original work done by me under the supervision of Mr. ANIL M V, Assistant Professor, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu.

I, also declare that this Project work has been submitted by me fully or partially for the award of any Degree, Diploma, Title or recognition before any authority.

Place : Angadikadavu

SHARON JOSEPH

Date :

DB18CMSR29

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Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express mydeepest gratitude to people along the way.

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SHARON JOSEPH

Contents

INTODUCTION

In linear algebra, an inner product space is a vector space with an additional structure called an inner product. This additional structure associates each pair of vectors in the space with a scalar quantity known as the inner product of the vectors. Inner products allow the rigorous introduction of intuitive geometrical notions such as the length of a vector or the angle between two vectors. They also provide the means of defining orthogonality between vectors (zero inner product). Inner product spaces generalize Euclidean spaces (in which the inner product is the dot product, also known as the scalar product) to vector spaces of any (possibly infinite) dimension and are studied in functional analysis. The first usage of the concept of a vector space with an inner product is due to Peano, in 1898.

An inner product naturally induces an associated norm, thus an inner product space is also a normed vector space. A complete space with an inner product is called a Hilbert space. An (incomplete) space with an inner product is called a pre-Hilbert space.

PRELIMINARIES

LINEAR SPACES

Definition 1: A linear (vector) space X over a field **F** is a set of elements together with a function, called addition, from $X \times X$ into X and a function called scalar multiplication, from $\mathbf{F} \times X$ into X which satisfy the following conditions for all $x, y, z \in X$ and $\alpha, \beta \in \mathbf{F}$;

i. $(x + y) + z = x + (y + z)$

ii. $x + y = y + x$

iii. There is an element 0 in X such that $x + 0 = x$ for all $x \in X$.

iv. For each $x \in X$ there is an element $-x \in X$ such that $x + (-x) = 0$.

v. $(x + y) = \alpha x + \alpha y$

vi. $(\alpha + \beta)x = \alpha x + \beta x$

vii. $\alpha(\beta x) = (\alpha\beta)x$

viii. $1 \cdot x = x$.

Properties i to iv imply that X is an abelian group under addition and v to vi relate the operation of scalar multiplication to addition X and to addition and multiplication in **F**.

Examples:

(a) $V_n(\mathbf{R})$. The vectors are n-tuples of real numbers and the scalars are real

numbers with addition and scalar multiplication defined by

$$(\alpha_1,\cdots,\alpha_n)+(\beta_1,\cdots,\beta_n)=(\alpha_1+\beta_1,\cdots,\alpha_n+\beta_n) \tag{1}$$

$$\beta(\alpha_1,\cdots,\alpha_n)=(\beta\alpha_1,\cdots,\beta\alpha_n) \tag{2}$$

$V_n(\mathbf{R})$ is a linear space over **R**. Similarly, the set of all n-tuples of complex numbers with the above definition of addition and multiplication is a linear space over **C** and is denoted as $V_n(\mathbf{C})$.

(b) The set of all functions from a nonempty set X into a field **F** with addition and scalar multiplication defined by

$$[f+g](t)=f(t)+g(t) \quad \text{and} \quad [\alpha f](t)=\alpha f(t);\ f,g\in X,\ t\in T \tag{3}$$

is a linear space.

Let T = **N** the set of all positive integers and X is the set of all sequences of elements **F** with addition and scalar multiplication defined by

$$(\alpha_n+\beta_n)=\ (\alpha_n+\beta_n) \tag{4}$$

$$\beta\left(\alpha_n\right)=\left(\beta\alpha_n\right) \tag{5}$$

denoted as $V_\infty(\mathbf{F})$, form a linear space.

METRIC SPACES

Remember the distance function in the Euclidean space $\mathbf{R}^n$.

Let $x,y,z \in \mathbf{R}^n$, then

(1) $|x - y| \geq 0$; $|x - y| = 0$ if and only if $x = y$;

(2) $|x - y| = |y - x|$;

(3) $|x - y| \leq |x - z| + z - y|$.

Definition 2: A **metric** or **distance function** on a set X is a real valued function d defined on $X \times X$ which has the following properties: for all $x, y, z \in X$.

(1) $d(x,y) \geq 0$; $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq d(x, z) + d(z, y)$

A **metric space** (X, d) is a nonempty set X and a metric d defined on X.

Examples: In addition to the Euclidean spaces let us have the following examples. *Here all functions are assumed to be continuous*. Let L^p denotes a set of complex valued functions in $\mathbf{R}^n$ such that $|f|^p$ is integrable. Let us recall some results concerning such functions.

Höder's Inequality: If $p > 1$, $1/q = 1 - 1/p$

$$\int |fg| \leq [\int |f|^p]^{1/p}[\int |g|^q]^{1/q} .$$

Minkowski's Inequality: If $p \geq 1$,

$$[\int |f+g|^p]^{1/p} \leq [\int |f|^p]^{1/p} + [\int |g|^p]^{1/p}$$

If x_k and y_k for $k = 1, \ldots, m$ are complex numbers, let $f(t) = |x_k|$ and $g(t) = |y_k|$ for $t \in [k, k+1]$ and $f(t) = 0 = g(t)$ for $t \in [1, m+1]$. Then we obtain the summation form of the above inequalities from the integral form

Hölder's Inequality

$$\sum_{k=1}^{m} |x_k y_k| \le \left[\sum_{k=1}^{m} |x_k|^p\right]^{1/p} \left[\sum_{k=1}^{m} |y_k|^q\right]^{1/q}$$

Minkowski's Inequality:

$$\left[\sum_{k=1}^{m} |x_k + y_k|^p\right]^{1/p} \le \left[\sum_{k=1}^{m} |x_k|^p\right]^{1/p} + \left[\sum_{k=1}^{m} |y_k|^p\right]^{1/p}$$

NORMED LINEAR SPACES

Definition 3. A norm on X is a real valued function, whose value at x is denoted by $||x||$, satisfying the following conditions for all $x, y \in X$ and $\alpha \in \mathbf{F}$;

(1) $||x|| > 0$ if $x \neq 0$

(2) $||\alpha x|| = |\alpha| ||x||$

(3) $||x + y|| \le ||x|| + ||y||$.

A linear space X with a norm defined on it is called a **normed linear space**.

Example: l^p **space**. On the linear space $V_n(\mathbf{F})$, define

$$|| x || = [\sum_{k=1}^{n} | \alpha_i |^p]^{1/p}$$

where $p \ge 1$ is any real number and $x = (\alpha_1, \cdots, \alpha_n)$. This defines a norm (called p-norm) on $V_n(\mathbf{F})$. This space is called l^p space.

CHAPTER 1

INNER PRODUCT SPACES

INNER PRODUCTS

Let F be the field of real numbers or the field of complex numbers, and V a vector space over F an inner product on V is a function which assigns to each ordered' pair of vectors α, β in V a scalar $(\alpha|\beta)$ in F in such a way that for all α, β, γ in V and all scalars c.

(a) $(\alpha + \beta|\gamma) = (\alpha|\gamma) + (\beta|\gamma)$;

(b) $(c\alpha|\beta) = c(\alpha|\beta)$;

(c) $(\beta|\alpha) = (\overline{\alpha|\beta})$, the bar denoting complex conjugation

(d) $(\alpha|\alpha) > 0$ if $\alpha \neq 0$

It should be observed that conditions (a), (b) and (c) implies that

$(e) = (\alpha \mid c\beta + \gamma) = (\bar{c}(\alpha|\beta) + (\alpha|\gamma)$

One other point should be made. When F is the field R of real nunbers. The complex conjugates appearing in (c) and (e) are superflom. However, in the complex case they are necessary for the consistency of the conditions. Without these complex conjugates we would have the contradiction

$$(\alpha|\alpha) > 0 \text{ and } (i\alpha \mid i\alpha) = -1(\alpha|\alpha)$$

Example 1:

On F^n there is an inner product which we call the standard inner product. It is defined on $\alpha = (x_1, \cdots x_n)$ and $\beta = (y_1, \ldots, y_n)$, by

$$(\alpha|\beta) = \sum_i x_i \overline{y_i}$$

When F is R this may be also written as

$$(\alpha|\beta) = \sum_i x_i y_i$$

In the real case, the standard inner product is often called the dot or scalar product and denoted by $\alpha \cdot \beta$.

INNER PRODUCTS SPACES

An inner product space is a real or complex vector space together with a specified inner product on that space.

- A finite-dimensional real inner product space is often called a Euclidean spare. A complex inner product spare often referred to as a unitary spare.
- Every inner product space is a normed linear space and every normed space is a metric space. Hence , every inner product space is a metric space.

Theorem

If V is an inner product space, then for any vector's α, β in V and any scalar c

(1) $||c\alpha|| = |c|||\alpha||$;

(ii) $\left||\alpha|\right| > 0$ for $\alpha \neq 0$

(iii) $|(\alpha \mid \beta)| \leqslant ||\alpha|| \, ||\beta||$

(iv) $\| \alpha + \beta|| \leqslant \| \alpha \| + \| \beta||$

Proof:

Statements (i) and (ii) follow almost immediately form the various definitions involved. The inequality in (iii) is clearly valid when $\alpha = 0.$ if $\alpha \neq 0$, put

$$\gamma = \beta - \frac{(\beta|\alpha)}{\| \alpha \|^2} \alpha$$

Then, $(\gamma \mid \alpha) = 0$ and

$$0 \leqslant \| \gamma \|^2 = \left(\beta - \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha \middle/ \beta - \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha\right)$$

$$= (\beta|\beta) - \frac{(\beta|\alpha)(\alpha|\beta)}{\| \alpha \|^2}$$

$$= \| \beta \|^2 - \frac{|(\alpha|\beta)|^2}{\| \alpha \|^2}$$

Hence,

$$|(\alpha \mid \beta)|^2 \leqslant \| \alpha \|^2 \| \beta \|^2$$

Now using (c) we find that

$$\| \alpha + \beta \|^2 = \| \alpha \|^2 + (\alpha \mid \beta) + (\beta \mid \alpha) + \| \beta \|^2$$

$$\begin{aligned} &= \| \alpha \|^2 + 2\mathrm{Re}\,(\alpha \mid \beta) + \| \beta \|^2 \\ &\leqslant \| \alpha \|^2 + 2 \| \alpha \| \| \beta \| + \| \beta \|^2 \\ &= (\| \alpha \| + \| \beta \|)^2 \end{aligned}$$

Thus,

$$\| \alpha + \beta \| \leqslant \| \alpha \| + \| \beta \|$$

the inequality (iii) is called the Cauchy -Schwarz inequality. It has a wide variety of application

the proof shows that if $\alpha$is non-zero then

$$\big((\alpha \mid \beta)\big) < \| \alpha \| \| \beta \|, \text{unless}$$

$$\beta = \frac{(\beta|\alpha)}{\| \alpha \|^2}\alpha$$

Then equality occurs in (iii) if and only if α and β are linearly independent.

CHAPTER 2

ORTHOGONAL SETS

Definition

Let α and β be the vectors in an inner product space V. Then α is orthogonal to β if (α | β) = 0. We simply say that and are orthogonal.

Definition

If S is a set of vectors in V, S is called an orthogonal set provided all set pairs of distinct vectors in S are orthogonal.

Definition

An orthogonal set is an orthogonal set S with the additional property that $\| \alpha \| = 1$ for every α in S.

- The zero vectors are orthogonal to every vector in V and is the only vector with this property.
- It is an appropriate to think of an orthonormal set as a set of mutually perpendicular vectors each having length l.

Example: the vector (x, y) is R^2 is orthogonal to $(-\mathrm{y}, \mathrm{x})$ with respect to the standard inner product, for,

$$((\mathrm{x}, \mathrm{y})|(-\mathrm{y}, \mathrm{x})) = -\mathrm{xy} + \mathrm{yx} = 0$$

- The standard basis of either R^n or C^n is an orthonormal set with respect to the standard inner product.

Theorem : An orthogonal set of nonzero vectors is linearly independent.

Proof:

Let S be a finite or infinite orthogonal set of nonzero vectors in a given inner product space suppose $\alpha_{1,}\alpha_2, \ldots \alpha_n$ are distinct vectors in S and that β=$c_1\alpha_{1+} + \cdots c_n\alpha_n$

Then (β|α_k)=($c_1\alpha_{1+} + \cdots c_n\alpha_n|\alpha_k$)

$$= c_1(\alpha_1|\alpha_k) + c_2(\alpha_2|\alpha_k) + \ldots + c_n(\alpha_n|\alpha_k)$$

$$= c_k(\alpha_n|\alpha_k) \text{ , since } (\alpha_i\,|\alpha_j) = 0 \text{,if } i \neq j \text{ and } (\alpha_i\,|\alpha_j) = 1 \text{,if } i=j$$

Hence, c_k= (β $|\alpha_k) \,/(\alpha_k\,, \alpha_k)$)

$$c_k = (\beta\,|\alpha_k)/||\alpha_k||^2, 1 \leq k \leq m$$

Thus, when β=0 each c_k=0; so S is a linearly independent set.

Corollary:

If $\{\alpha_{1,}\alpha_2, \ldots \alpha_m\}$ is an orthogonal set of nonzero vectors in a finite dimensional inner product space V, then m ≤ dimV.

That is number of mutually orthogonal vectors in V cannot exceed the dimensional V.

Corollary:

If a vector β is linear combination of an orthogonal of nonzero vectors $\alpha_{1,}\alpha_2, \ldots \alpha_n$, then β is the particular linear combination

$$\beta = \sum_{k=1}^{m} \frac{(\beta\,|\alpha_k)}{||\alpha_k||^2}\alpha_k$$

Proof :

Since β is the linear combination of an orthogonal sequence of nonzero vectors $\alpha_{1,}\alpha_2, \ldots \alpha_n$, we can write β $=c_1\alpha_1 + \cdots c_n\alpha_n$.

Where $c_k = \frac{(\beta|\alpha_k)}{||\alpha_k||^2}$, 1 ≤ k ≤ m (ref. by previous theorem)

Hence, β$= \sum_{k=1}^{m} \frac{(\beta|\alpha_k)}{||\alpha_k||^2}$

Theorem (Gram Schmidt Orthogonalization Process)

Let V be an inner product space and $\{\beta_{1,} \dots , \beta_n\}$ be any linearly independent vectors in V. Then one may construct orthogonal vectors $\{\alpha_{1,}\alpha_2, \dots \alpha_n\}$ in V, such that for each k = 1, 2, …n, the set $\{\alpha_{1,}\alpha_2, \dots \alpha_k\}$is an orthogonal basis for the subspace of V spanned by $\beta_{1,} \dots , \beta_n$.

Proof :

The vectors are obtained by means of a construction known as the Gram Schmidt orthogonalization process.

First let $\alpha_1 = \beta_1$ The other vectors are then given inductively as follows:

Suppose $\alpha_{1,}\alpha_2, \dots \alpha_m$ ($1 \le m \le n$) have been chosen so that for every k

$$\{\alpha_{1,}\alpha_2, \dots \alpha_k\} \ (1 \le k \le m)$$

is an orthogonal basis for the space of v that is spanned by $\beta_{1,} \dots , \beta_n$

To construct the next vector $\alpha_{m+1,}$ let

$$\alpha_{m+1,} = \beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}\alpha_k$$

Then $\alpha_{m+1} \neq 0$. For otherwise, $\beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}\alpha_k = 0$, implies,

$\beta_{m+1} = \beta_{m+1} - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}\alpha_k$,implies, β_{m+1} is a linear combination of $\alpha_{1,}\alpha_2, \dots \alpha_m$ and hence a linear combination of $\beta_1, \beta_2, \dots, \beta_m$, a contradiction.

Furthermore, if $1 \le j \le m$, then,

$$(\alpha_{m+1}|\ \alpha_j) = (\beta_{m+1}|\ \alpha_j) - \sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}(\alpha_k\ |\alpha_j)$$

$$= (\beta_{m+1}|\ \alpha_m) - (\beta_{m+1}|\ \alpha_j)$$, using the orthonormality of $\{\alpha_{1,}\alpha_2, \dots \alpha_m\}$.

Therefore $\{\alpha_{1,}\alpha_2, \dots,\alpha_{m+1}\}$ is an orthogonal set consisting of m+1 nonzero vectors in the subspace spanned by $\beta_{1,} \dots , \beta_{m+1}$. Hence by an earlier Theorem , it is a basis for this subspace .Thus the vectors , $\alpha_{1,}\alpha_2, \dots \alpha_n$ may be constructed using the formula

$$\alpha_{m+1}= \beta_{m+1}-\sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}\alpha_k$$

In particular, when n=3 ,we have

$$\alpha_1=\beta_1$$

$$\alpha_2=\beta_2- \frac{(\alpha_2|\beta_2)}{||\alpha_k||^2}\alpha 1$$

$$\alpha_3=\beta_3 - \frac{(\beta_3|\alpha_1)}{||\alpha_1||^2}\alpha 1 - \frac{(\alpha_2|\beta_3)}{||\alpha_k||^2}\alpha_2$$

Corollary :

Every finite dimensional inner product space has an orthonormal basis.

Proof :

Let V be a finite dimensional inner product space and $\{\beta_{1,} \dots, \beta_n\}$ a basis for V. Apply the gram Schmidt orthogonalization process to construct an orthogonal basis , simply replace each vector α_n by $\frac{\alpha_k}{||\alpha_k||}$.

Gram-Schmidt process can be used to test for linear dependence . For suppose $\beta_{1,} \dots, \beta_n$ are linearly independent vectors in an inner product space; to exclude a trivial case , assume that β≠0. Let m be largest integers for which $\beta_{1,} \dots, \beta_m$are independent. Then 1≤m<n.

Let $\alpha_1, \alpha_2, \dots \alpha_m$ be the vectors obtained by applying the orthogonalization process to $\beta_{1,} \dots, \beta_m$. Then the vector α_{m+1} given by $\alpha_{m+1}= \beta_{m+1}-\sum_{k=1}^{m} \frac{(\beta_{m+1}|\alpha_k)}{||\alpha_k||^2}\alpha_k$ is necessarily 0.

For α_{m+1} is in the subspace spanned by $\alpha_1, \alpha_2, \dots \alpha_m$ and orthogonal to each of the vectors , hence it is 0 as $\beta=\sum_{k=1}^{m} \frac{(\beta|\alpha_k)}{||\alpha_k||^2}\alpha_k$. Conversely, if $\alpha_1, \alpha_2, \dots \alpha_m$ are different from 0 and α_{m+1} =0, then $\beta_{1,} \dots, \beta_{m+1}$ are linearly independent .

Definition:

A best approximation to β ∈ V by vectors in a subspace W of V is a vector $\alpha \in W$ such that

$$\|\beta - \alpha\| \le \|\beta - \gamma\| \text{ for every vector } \gamma \in W .$$

Theorem

Let W be a subspace of an inner product space V and let $\beta \in V$.

1. The vector $\alpha \in W$ is a best approximation to $\beta \in V$ by vectors in W if and only if $\beta - \alpha$ is orthogonal to every vector in W .
2. If a best approximation to $\beta \in V$ by vectors in W exists, it is unique.
3. If W is finite-dimensional and $\{\alpha_{1,}\alpha_2, \dots \alpha_n\}$ is any orthonormal basis for W ,

   then the vector

$$\alpha = \sum_{k=1}^{n} \frac{(\beta \mid \alpha_k)}{\|\alpha_k\|^2} \alpha_k$$

   is the (unique) best approximation to β by vectors in W.

Definition:

Let V be an inner product space and S be any set of vectors in V. The orthogonal complement of S is the set $S^{\perp}$ of all vectors in V which are orthogonal to every vector in S.

That is, $S^{\perp} = \{\alpha \in V : (\alpha \mid \beta) = 0, \forall \beta \in S\}$

Definition:

Whenever the vector α in the above theorem exists it is called the orthogonal projection of β on W. If every vector in V has an orthogonal projection on W, the mapping that assigns to each vector in V its orthogonal projection on W is called the orthogonal projection of V on W.

Corollary :

Let V be an inner product space and W a finite dimensional subspace and E be the orthogonal projection of V on W. Then the mapping

$$\beta \to \beta - E\beta$$

is the orthogonal projection of V on $W^{\perp}$.

Proof :

Let $\beta \in V$. Then $\beta - E\beta \in W^{\perp}$, and for any $\gamma \in W^{\perp}$, $\beta - \gamma = E\beta + (\beta - E\beta - \gamma)$

Since $E\beta \in W$ and $\beta - E\beta - \gamma \in W^{\perp}$,

It follows that

$$||\beta - \gamma||^2 = (E\beta + (\beta - E\beta - \gamma), E\beta + (\beta - E\beta - \gamma))$$
$$= ||E\beta||^2 + ||\beta - E\beta - \gamma||^2$$
$$\geq ||\beta - (\beta - E\beta)||^2$$

with strict inequality when $\gamma \neq \beta - E\beta$. Therefore, $\beta - E\beta$ is the best approximation to β by vectors in $W^{\perp}$.

Theorem

Let W be a finite dimensional subspace of an inner product space V and let E be the orthogonal projection of V on W. Then E is an idempotent linear transformation of V onto W , $W^{\perp}$ is the null space of E , and $V = W \oplus W^{\perp}$.

Proof

Let β be an arbitrary vector in V. Then $E\beta$ is the best approximation to β that lies in W. In particular, $E\beta = \beta$ when β is in W. Therefore, $E(E\beta) = E\beta$ for every β in V; that is, E is idempotent : $E^2 = E$. To prove that E is linear transformation, let α and β be any vectors in V and c an arbitrary scalar ,Then by theorem,

$\alpha - E\alpha$ and $\beta - E\beta$ are each orthogonal to every vector in W. Hence the vector

$c(\alpha - E\alpha) + (\beta - E\beta) = (c\alpha + \beta) - (cE\alpha + E\beta)$

Also belongs to $W^{\perp}$. Since $cE\alpha + E\beta$ is a vector in W, it follows from theorem that

$E(c\alpha + \beta) = cE\alpha + E\beta$.

Again let β be any vector in V. Then $E\beta$ is the unique vector in W such that $\beta - E\beta$ is in $W^{\perp}$.

Thus $E\beta = 0$ when β is in $W^{\perp}$.

Conversely, β is in $W^{\perp}$ when $E\beta = 0$. Thus $W^{\perp}$ is the null space of E.

The equation ,

$$\beta = E\,\beta + \beta - E\beta$$

shows that V = $W + W^{\perp}$; moreover $W \cap W^{\perp} = \{0\}$; for if α is a vector in $W \cap W^{\perp}$,then

$(\alpha \mid \alpha)$=0. Therefore, α=0 and V is the direct sum of W and $W^{\perp}$.

Corollary :

Under the conditions of theorem, $I - E$ is the orthogonal projection of V on $W^{\perp}$.

It is an independent linear transformation of V onto $W^{\perp}$ with null space W.

Proof :

We have seen that the mapping $\beta \rightarrow \beta - E\beta$ is the orthogonal projection of V on $W^{\perp}$.

Since E is a linear transformation , this projection $W^{\perp}$ is the linear transformation $I - E$ from its geometric properties one sees that $I - E$ is an idempotent .Transformation of V onto W.

This also follows from the computation $(I - E)(I - E) = I - E - E + E^2$

$$= I - E$$

Moreover , $(I - E)\beta = 0$ If and only if $\beta = E\beta$, and this is the case if and only if β is in W .

Therefore W is the null space of $I - E$.

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GRAPH COLORING

Project report submitted to

The Kannur University

for the award of the degree

of

Bachelor of Science

by

SHARUNJITH M S

DB18CMSR30

Under the guidance of

MRS. Riya Baby

Department of Mathematics
Don Bosco Arts and Science College
Angadikadavu

March 2021

Examiners 1: Examiner 2:

CERTIFICATE

It is to certify that this project report '**GRAPH COLORING**' is the bona fide project of SHARUNJITH M S who carried out the project under my supervision.

Mrs. Riya Baby

Supervisor, HOD

Department of Mathematics

Don Bosco Arts and Science College

Angadikadavu

DECLARATION

I, SHARUNJITH M S, hereby declare that this project report entitled "**GRAPH COLORING**" is an original record of studies and bona fide project carried out by me during the period from November 2019 to March 2020, under the guidance of **Mrs. Riya Baby**, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not been submitted by me elsewhere for the award of any degree, diploma, title or recognition, before.

SHARUNJITH M S

Department of Mathematics

Don Bosco Arts and Science College

Angadikadavu

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SHARUNJITH M S

CONTENTS

INTRODUCTION

A proper coloring of a graph is an assignment of colors to the vertices of the graph so that no two adjacent vertices have the same color.

Usually we drop the word “proper” unless other types of coloring are also under discussion. Of course, the “colors” don’t have to be actual colors ; may can be any distinct labels - integers ,for examples , if a graph is not connected , each connected component can be colored independently; except where otherwise noted , we assume graphs are connected. We also assume graphs are simple in this section. Graph coloring has many applications in addition to its intrinsic interest.

In the same way the most important concept of graph coloring is utilized in resource allocation, scheduling. Also, paths, walks and circuits in graph theory are used in tremendous applications say travelling salesman problem, database design concepts, resource networking.

This project deals with coloring which is one of the most important topics in graph theory. In this project there are three chapters. First chapter is coloring . The second chapter is chromatic number. The last chapter deals with application of graph coloring.

BASIC CONCEPTS

1. GRAPH

A graph is an ordered triplet. G=(V(G), E(G), I(G)); V(G) is a non empty set, E(G) is a set disjoint from V(G) and I(G) is an incidence map that associates each element of E(G) and unrecorded pair of element of V(G). The elements of V(G) are called vertices (or nodes or points) of G and the elements of E(G) are Called edges or lines of G.

2. MULTIPLE EDGE / PARALLEL EDGE

A set of 2 or more edges of a graph G is called a multiple edge or parallel edge if they have the same end vertices.

3. LOOP

An edge for which the 2 end vertices are same is called a loop.

4. SIMPLE GRAPH

A graph is simple if it has no loop and no multiple edges.

5. DEGREE

Let G be a graph and v € V the number of edge incident at V in G is called the degree or vacancy of the vertex v in G.

CHAPTER - 1

COLORING

Graph coloring is nothing but a simple way of labeling graph components such as vertices , edges and regions under some constraints. In a graph, no two adjacent vertices, adjacent edges , or adjacent regions are colored with minimum number of colors .This number is called the chromatic number and the graph is called properly colored graph.

In graph theory coloring is a special case of graph labeling; it is an assignment of labels traditionally called “colors” to elements of a graph subject to certain constraints. In it simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color, it is called vertex coloring. Similarly, edge coloring assigns a color to each edge so that no two adjacent edges share the common color.

While graph coloring , the constraints that are set on the graph are colors , order of coloring , the way of assigning color , etc. A coloring is given to a vertex or a particular region . Thus, the vertices or regions having same colors form independent sets.

VERTEX COLORING

Vertex coloring is an assignment of colors to the vertices of a graph ‘G’ such that no two adjacent vertices have the same color .Simply put , no two vertices of an edge should be of the same color.

The most common type of vertex coloring seeks to minimize the number of colors for a given graph . Such a coloring is known as a minimum vertex coloring , and the minimum number of colors which with the vertices of a graph may be colored is called the chromatic number .

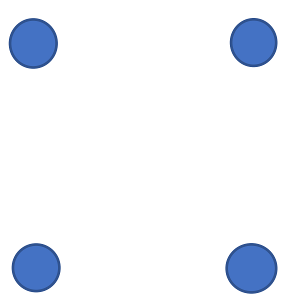
CHROMATIC NUMBER:

The minimum number of colors required for vertex coloring of graph ‘G’ is called as the chromatic number of G , denoted by X (G) .

X(G) = 1 iff ‘G’ is a null graph. If ‘G’ is not a null graph , then X(G) ≥ 2.

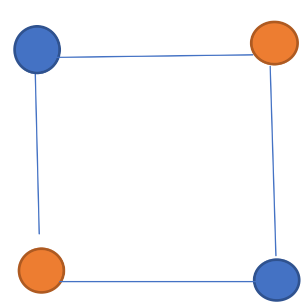
EXAMPLES;

1.



Null Graph (X (G) = 1)

2.

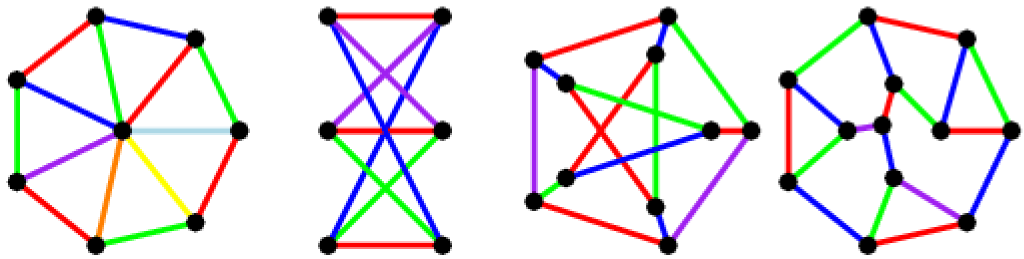


Not Null Graph (X (G) = 2)

EDGE COLORING

An edge coloring of a graph G is a coloring of the edges of G such that adjacent edges (or the edges bounding different regions) receive different colors. An edge coloring containing the smallest possible number of colors for a given graph is known as a minimum edge coloring.

The edge chromatic number gives the minimum number of colours with which graph's edges can be colored.



CHROMATIC INDEX

The minimum number of colors required for proper edge coloring of graph is called chromatic index.

A complete graph is the one in which each vertex is directly connected with all

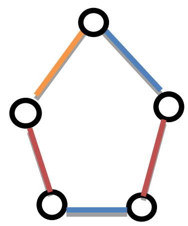
other vertices with an edge. If the number of vertices of a complete graph is n, then the chromatic

index for an odd number of vertices will be n and the chromatic index for even number of

vertices will be n-1.

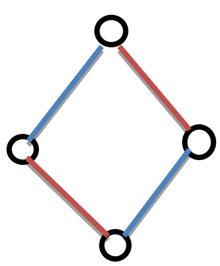
EXAMPLES;

1.



The given graph will require 3 unique colors so that no two incident edges have the Same color. So its chromatic index will be 3.

2.



The given graph will require 2 unique colors so that no two incident edges have the same color. So its chromatic index will be 2.

CHAPTER 2

Chromatic Number

The chromatic number of a graph is the smallest number of colors needed to color the vertices of so that no two adjacent vertices share the same color. That is the smallest value of possible to obtain a k-coloring.

- Graph Coloring is a process of assigning colors to the vertices of a graph.
- It ensures that no two adjacent vertices of the graph are colored with the same color.
- Chromatic Number is the minimum number of colors required to properly color any graph.

Graph Coloring Algorithm

- There exists no efficient algorithm for coloring a graph with minimum number of colors.

However, a following greedy algorithm is known for finding the chromatic number of any given graph.

Greedy Algorithm

<u>Step-01:</u>

Color first vertex with the first color.

Step-02:

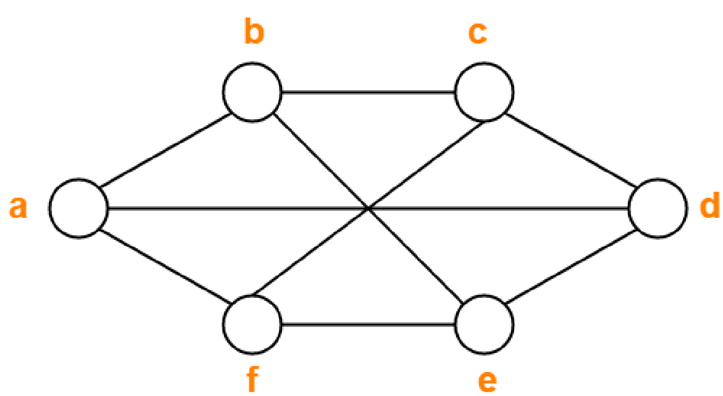
Now, consider the remaining (V-1) vertices one by one and do the following-

- Color the currently picked vertex with the lowest numbered color if it has not been used to color any of its adjacent vertices.
- If it has been used, then choose the next least numbered color.
- If all the previously used colors have been used, then assign a new color to the currently picked vertex.

Problems Based On Finding Chromatic Number of a Graph

Problem-01:

Find chromatic number of the following graph-



Solution-

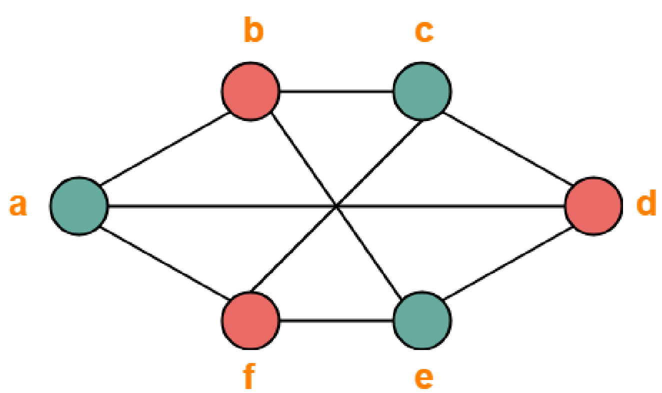
Applying Greedy Algorithm, we have

Vertex	a	b	C	d	e	f
Color	C1	C2	C1	C2	C1	C2

From here,

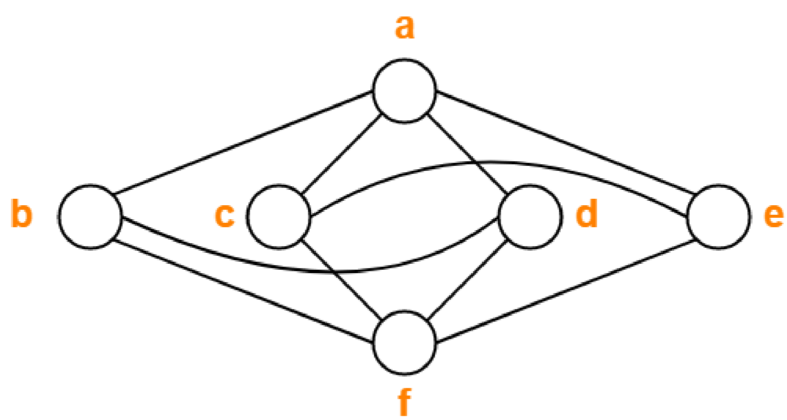
- Minimum numbers of colors used to color the given graph are 2.
- Therefore, Chromatic Number of the given graph = 2.

The given graph may be properly colored using 2 colors as shown below-



Problem-02:

Find chromatic number of the following graph-



Solution-

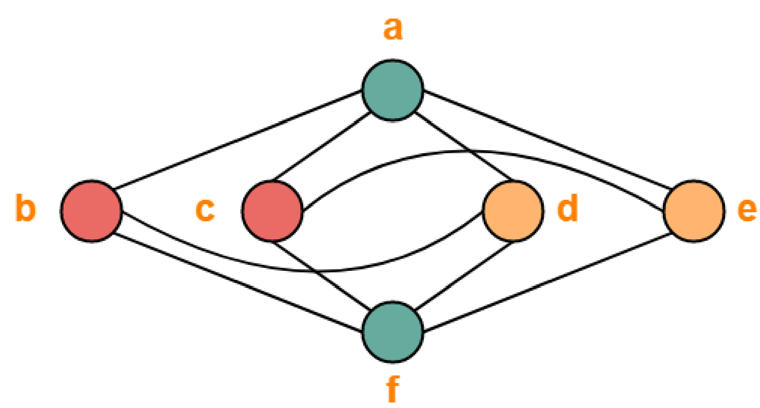
Applying Greedy Algorithm, we have-

Vertex	a	b	C	d	e	f
Color	C1	C2	C2	C3	C3	C1

From here,

- Minimum numbers of colors used to color the given graph are 3.
- Therefore, Chromatic Number of the given graph = 3.

The given graph may be properly colored using 3 colors as shown below-



Chromatic Number of Graphs

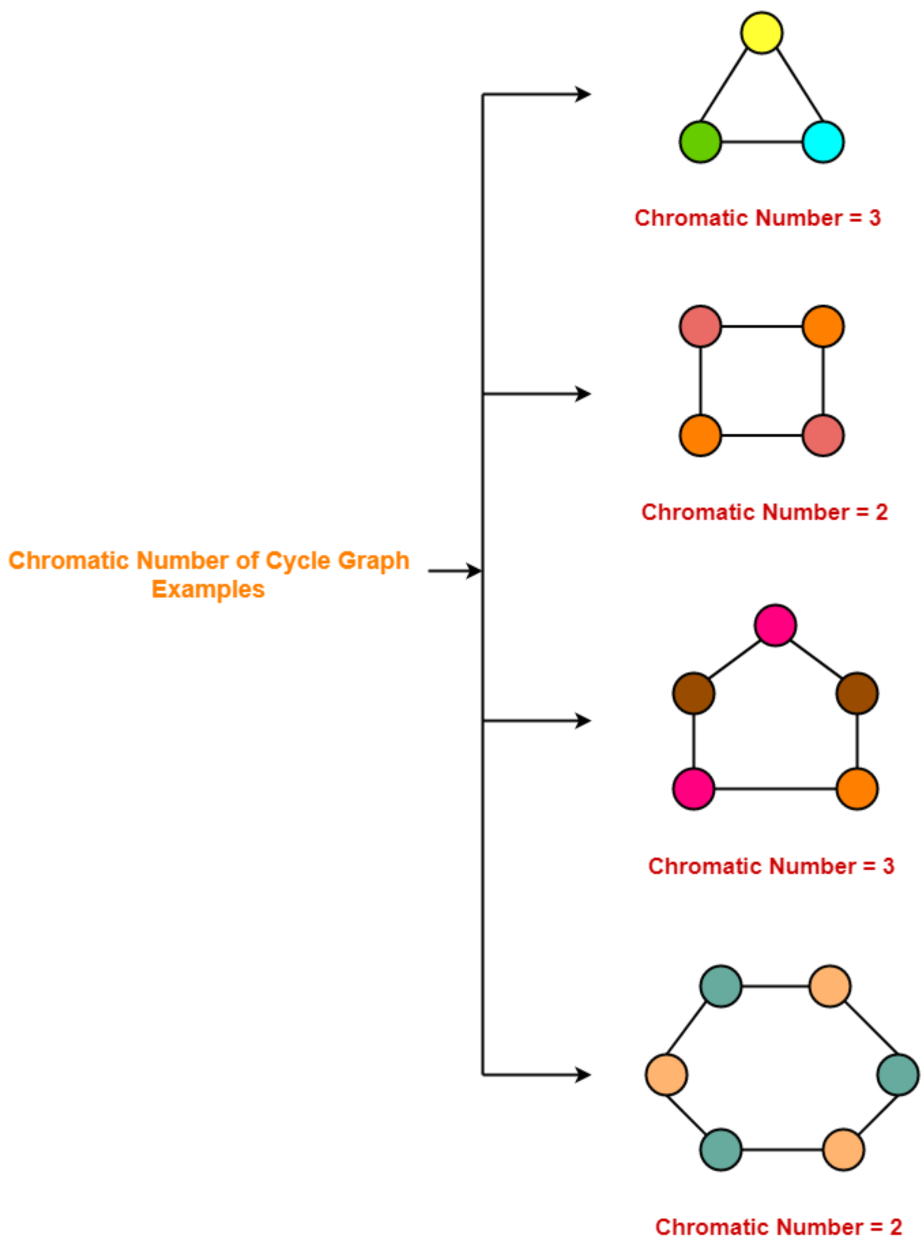
Chromatic Number of some common types of graphs are as follows-

1. Cycle Graph-

- A simple graph of ‘n’ vertices (n>=3) and ‘n’ edges forming a cycle of length ‘n’ is called as a cycle graph.
- In a cycle graph, all the vertices are of degree 2.

Chromatic Number
• If number of vertices in cycle graph is even, then its chromatic number = 2. • If number of vertices in cycle graph is odd, then its chromatic number = 3.

Examples-



2. Planar Graphs-

A planar graph is a graph that can be embedded in the plane, that is it can be drawn on the plane in such a way that its edges intersect only at their endpoint. In other words, it can be drawn in such a way that no edges cross each other.

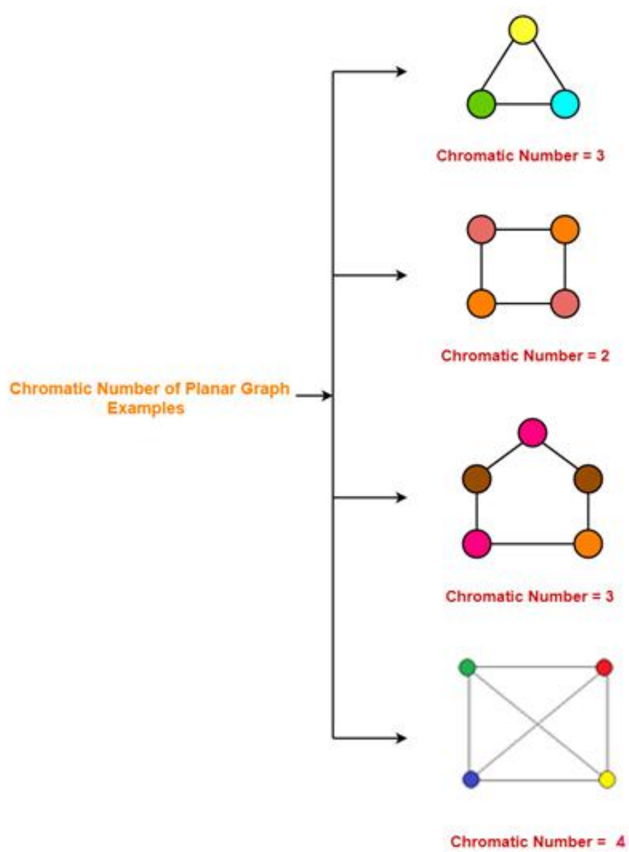
A **Planar Graph** is a graph that can be drawn in a plane such that none of its edges cross each other.

Chromatic Number Chromatic Number of any Planar Graph is less than or equal to 4

Examples-

+

- All the above cycle graphs are also planar graphs.
- Chromatic number of each graph is less than or equal to 4.

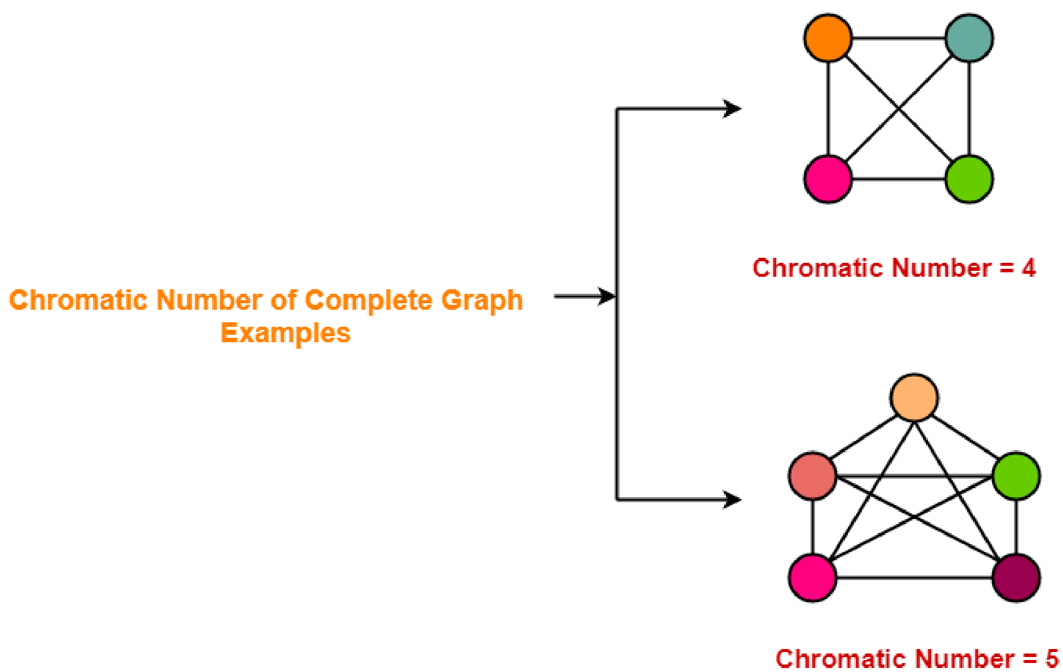


3. Complete Graphs-

- A complete graph is a graph in which every two distinct vertices are joined by exactly one edge.
- In a complete graph, each vertex is connected with every other vertex.
- So to properly it, as many different colors are needed as there are number of vertices in the given graph.

Chromatic Number Chromatic Number of any Complete Graph = Number of vertices in that Complete Graph

Examples-



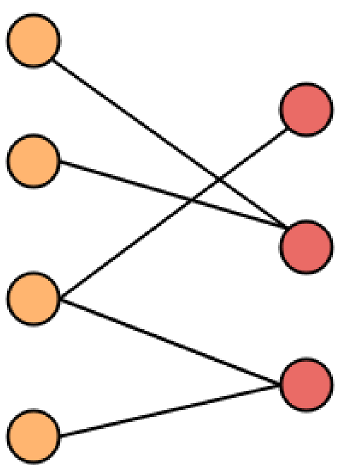
4. Bipartite Graphs-

A bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to one in V. Vertex sets U and V are usually called the parts of the graph.

- A **Bipartite Graph** consists of two sets of vertices X and Y.
- The edges only join vertices in X to vertices in Y, not vertices within a set.

Chromatic Number Chromatic Number of any Bipartite Graph = 2

Example-



Chromatic Number = 2

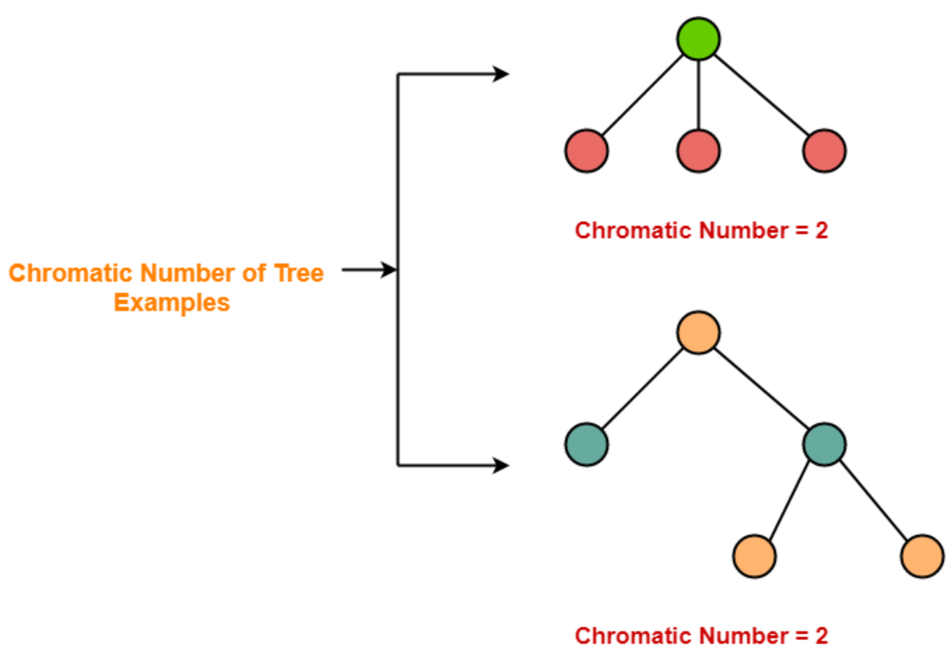
5. Trees-

A tree is an undirected graph in which any two vertices are connected by exactly one path, or equivalently a connected acyclic undirected graph.

- A **Tree** is a special type of connected graph in which there are no circuits.
- Every tree is a bipartite graph.
- So, chromatic number of a tree with any number of vertices = 2.

Chromatic Number
Chromatic Number of any tree = 2

Examples-



CHAPTER-3

APPLICATIONS OF GRAPH COLORING

1) Making Schedule or Time Table:

Suppose we want to make an exam schedule for a university. We have list different subjects and students enrolled in every subject. Many subjects would have common students (of same batch, some backlog students, etc). How do we schedule the exam so that no two exams with a common student are scheduled at same time? How many minimum time slots are needed to schedule all exams? This problem can be represented as a graph where every vertex is a subject and an edge between two vertices mean there is a common student. So this is a graph coloring problem where minimum number of time slots is equal to the chromatic number of the graph.

2) Mobile Radio Frequency Assignment:

When frequencies are assigned to towers, frequencies assigned to all towers at the same location must be different. How to assign frequencies with this constraint? What is the minimum number of frequencies needed? This problem is also an instance of graph coloring problem where every tower represents a vertex and an edge between two towers represents that they are in range of each other.

3) Register Allocation:

In compiler optimization, register allocation is the process of assigning a large number of target program variables onto a small number of CPU registers. This problem is also a graph coloring problem.

4) Sudoku:

Sudoku is also a variation of Graph coloring problem where every cell represents a vertex. There is an edge between two vertices if they are in same row or same column or same block.

5) Map Coloring:

Geographical maps of countries or states where no two adjacent cities cannot be assigned same color. Four colors are sufficient to color any map.

6) Bipartite Graphs:

We can check if a graph is bipartite or not by coloring the graph using two colors. If a given graph is 2-colorable, then it is Bipartite, otherwise not. See this for more details.

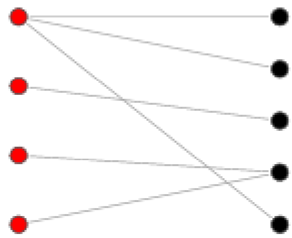
Explanation;

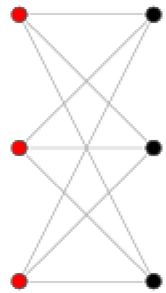
Algorithm:

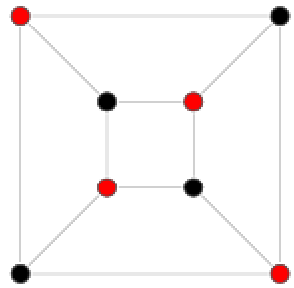
A bipartite graph is possible if it is possible to assign a color to each vertex such that no two neighbour vertices are assigned the same color. Only two colors can be used in this process.

Steps:

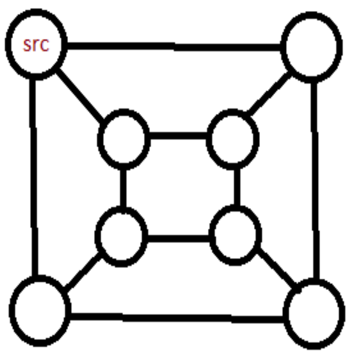
1. Assign a color (say red) to the source vertex.
2. Assign all the neighbours of the above vertex another color (say blue).
3. Taking one neighbour at a time, assign all the neighbour's neighbours the color red.
4. Continue in this manner till all the vertices have been assigned a color.
5. If at any stage, we find a neighbour which has been assigned the same color as that of the current vertex, stop the process. The graph cannot be colored using two colors. Thus the graph is not bipartite.



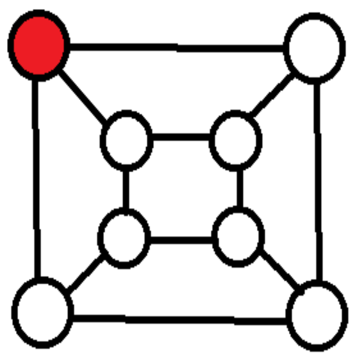




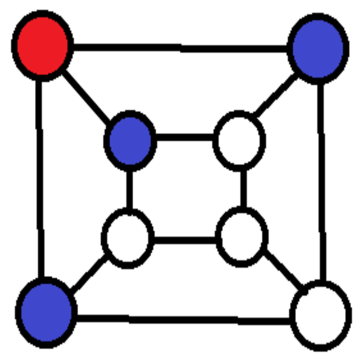
Example:



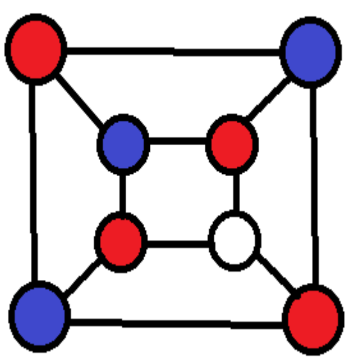
given a graph with source vertex



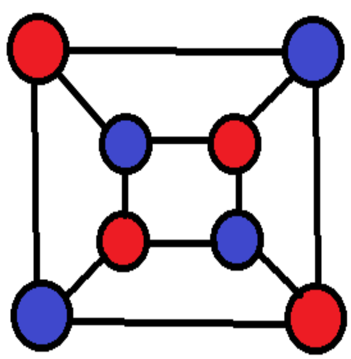
colour src vertex, say red



assign another colour to the neighbours, say blue



assign the neighbours of the vertices of the previous step the colour red



repeat till all vertices are coloured, or a conflicting colour assignment occurs.

set U: red colour
set V: blue colour

CONCLUSION

This project aims to provide a solid background in the basic topics of graph coloring. Graph coloring problem is to assign colors to certain elements of a graph subject to certain constraints. The nature of coloring problem depends on the number of colors but not on what they are.

The study of this topic gives excellent introduction to the subject called "Graph Coloring".

This project includes two important topics such as vertex coloring and edge coloring and came to know about different ways and importance of coloring.

Graph coloring enjoys many practical applications as well as theoretical challenges. Besides the applications, different limitations can also be set on the graph or on the away a color is assigned or even on the color itself. It has been reached popularity with the general public in the form of the popular number puzzle Sudoku and it is also use in the making of time management which is an important application of coloring. So graph coloring is still a very active field of research.

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NUMBER THEORETIC FUNCTION

Project report submitted to
The Kannur University
for the award of the degree
of

Bachelor of Science

by

SHILJI KURIAN

DB18CMSR08

Under the guidance of

Ms. Ajeena joseph

Department of Mathematics
Don Bosco Arts and Science College
Angadikadavu

CERTIFICATE

Certified that this project **'Number Theoretic Function'** is a bona fide project of **SHILJI KURIAN** carried out the project work under my supervision.

Mrs. Riya Baby **Ms. Ajeena joseph**

Head of Department **Supervisor**

Department of Mathematics

Don Bosco Arts and Science College

Angadikadavu

DECLARATION

I **SHILJI KURIAN** hereby declare that the project **'Number Theoretic Function'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Ajeena joseph, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

Name

SHILJI KURIAN

Department of Mathematics

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Angadikkadavu

ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express my deepest gratitude to people along the way.

No words can adequately express the sense of gratitude; still, I try to express my heartfelt thanks through words. At the outset, I am deeply indebted to my project supervisor Ms. Ajeena joseph, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu, for the invaluable guidance, loving encouragement and meticulous care towards me throughout my career. I express my deep sense of gratitude to all the faculty members of the Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu.

I can never forget the support and encouragement rendered by the Principal and the staff of Don Bosco Arts and Science College, Angadikkadavu.

I could not name many who sincerely supported and helped for the successful completion of this project. It is my pleasure and duty to thank each and every one of them who walked with me.

My greatest debt is always, to God Almighty.

CONTENTS

INTRODUCTION

A Number Theoretic Function is a complex valued function defined for all positive integers. In Number Theory, there exist many number theoretic functions. This includes Divisor Function, Sigma Function, Euler's-Phi Function and Mobius Function. All these functions play a very important role in the field of Number Theory.

In the first chapter we will discuss about Arithmetic Function. In the second chapter we will introduce Euler's-Phi Function and Mobius Function.

PRELIMINARY

Let n be a fixed positive integer. Two integers a and b are said to be *congruent modulo n,* symbolized by

$$a \equiv b \pmod{n}$$

if n divides the difference $a - b$; that is, provided that $a - b = kn$ for some integer k.

Example:

To fix the idea, consider $n = 7$. It is routine to check that

$3 \equiv 24 \pmod{7}$ $\quad -31 \equiv 11 \pmod{7}$ $\quad -15 \equiv -64 \pmod{7}$

Because $3 - 24 = (-3)7$, $-31 - 11 = (-6)7$ and $-15 - (-64) = 77$. When n does not divide $(a - b)$, we say that a is *incongruent to b modulo n*, and in this case we write

$a \not\equiv b \pmod{n}$. For a simple example: $25 \not\equiv 12 \pmod{7}$, because 7 fails to divide

$25 - 12 = 13$.

It is to be noted that any two integers are congruent modulo 1, whereas two integers are congruent modulo 2 when they are both even or both odd. In as much as congruence modulo 1 is not particularly interesting, the usual practice is to assume that $n > 1$.

Remark:

Given an integer a, let q and r be its quotient and remainder upon division by n, so that

$a = qn + r \quad 0 \leq r < n$

Then, by definition of congruence, $a \equiv r \pmod{n}$. Because there are n choices for r , we see that every integer is congruent modulo n to exactly one of the values $0, 1, 2, \ldots, n - 1$; in particular, $a \equiv 0 \pmod{n}$ if and only if $n \mid a$.

Fundamental Theorem of Arithmetic

is Every integer $n > 1$ can be represented as Product of prime factor in only one way, apart from the order of the factors.

Residue

If a is an integer and m is a positive integer then the residue class of a modulo m is denoted by $\hat{a}$ and is given by

$$\begin{aligned}\hat{a} \quad &= \{x : x \equiv a (mod m)\} \\ &= \{x : x = a + mk,\ k = 0, \pm 1, \pm 2, \cdots\}\end{aligned}$$

CHAPTER 1

ARITHMETIC FUNCTION

An arithmetic Function is a function defined on the positive integers which take values in the real or complex numbers. i.e., A function $f: N \rightarrow C$ is called an arithmetic function.

An arithmetic function is called multiplicative if $f(mn) = f(m)f(n)$ for all coprime natural numbers m and n.

Examples

a) Sum of divisors $\sigma(n)$

b) Number of divisors $\tau(n)$

c) Euler's function $\phi(n)$

d) Mobius function $\mu(n)$

Definition 1.1

Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of n and $\sigma(n)$ denote the sum of positive divisors of n.

Example

Consider $n = 12$. Since 12 has the positive divisors 1, 2, 3, 4, 6, 12, we find that

$\tau(12) = 6$ and $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$

For the first few integers,

$\tau(1) = 1 \quad \tau(2) = 2 \quad \tau(3) = 2 \quad \tau(4) = 3 \quad \tau(5) = 2 \quad \tau(6) = 4, \ldots$

$\sigma(1) = 1, \quad \sigma(2) = 3, \quad \sigma(3) = 4, \quad \sigma(4) = 7, \quad \sigma(5) = 6, \quad \sigma(6) = 12, \ldots$

It is not difficult to see that $\tau(n) = 2$ if and only if n is a prime number; also, $\sigma(n) = n + 1$ if and only if n is a prime.

Theorem 1.1

If n = $p_1^{k_1} \ldots\ldots\ldots\ldots p_r^{k_r}$ is the prime factorization of $n > 1$, then

(a) $\tau(n) = (k_1 + 1)(k_2 + 1) \cdot \cdot \cdot (k_r + 1)$, and

(b) $\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{p_r^{k_r+1}-1}{p_r-1}$

Proof

The positive divisors of n are precisely those integers

$$d = p_1^{a_1}\, p_2^{a_2} \ldots\ldots p_r^{a_r}$$

where $0 \le a_i \le k_i$. There are $k_1 + 1$ choices for the exponent a_1; $k_2 + 1$ choices for a_2, . . . ; and $k_r + 1$ choices for a_r . Hence, there are

$(k_1 + 1)(k_2 + 1) \cdot \cdot \cdot (k_r + 1)$

possible divisors of n.

To evaluate σ(n), consider the product

$$\left(1 + p_1 + P_1^2 + \cdots \ldots\ldots\ldots P_1^{K_1}\right)\left(1 + p_2 + P_2^2 + \cdots \ldots\ldots\ldots P_2^{K_2}\right)\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\left(1 + p_r + P_r^2 + \cdots \ldots\ldots\ldots P_r^{K_r}\right)$$

Each positive divisor of n appears once and only once as a term in the expansion of this product, so that

$$\sigma(n) = \left(1 + p_1 + P_1^2 + \cdots \ldots\ldots\ldots P_1^{K_1}\right)\left(1 + p_2 + P_2^2 + \cdots \ldots\ldots\ldots P_2^{K_2}\right)\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\left(1 + p_r + P_r^2 + \cdots \ldots\ldots\ldots P_r^{K_r}\right)$$

Applying the formula for the sum of a finite geometric series to the ith factor on the right-hand side, we get

$$\left(1 + p_i + P_i^2 + \cdots \ldots\ldots\ldots P_i^{K_i}\right) = \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

It follows that

$$\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1} \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \frac{p_r^{k_r+1}-1}{p_r-1} \ .$$

Corresponding to the $\sum$ notation for sums, the notation for products may be defined using $\prod$, the Greek capital letter pi. The restriction delimiting the numbers over which the product is to be made is usually put under the $\prod$ sign.

Examples

$$\prod_{1 \le d \le 5} f(d) = f(1)f(2)f(3)f(4)f(5)$$

$$\prod_{d \mid 9} f(d) = f(1)f(3)f(9)$$

$$\prod_{\substack{p \mid 30 \\ p \text{ prime}}} f(p) = f(2)f(3)f(5)$$

With this convention, the conclusion to Theorem 1.1 takes the compact form: if $n = p_1^{k_1}\, p_2^{k_2} \ldots \ldots . . p_r^{k_r}$ is the prime factorization of n > 1, then

$$\tau(n) = \prod_{1 \le i \le r} (k_i + 1)$$

and

$$\sigma(n) = \prod_{1 \le i \le r} \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

Theorem 1.2

The functions τ and σ are both multiplicative functions

Proof

Let m and n be relatively prime integers. Because the result is trivially true if either m or n is equal to 1, we may assume that m > 1 and n > 1. If

$$m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \qquad \text{and} \qquad n = q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$$

are the prime factorizations of m and n . It follows that the prime factorization of the product mn is given by

$$mn = p_1^{k_1} \cdots p_r^{k_r} q_1^{j_1} \cdots q_s^{j_s}$$

Applying to theorem 1.1, we obtain

$$\begin{aligned}\tau(mn) &= [(k_1+1)\cdots(k_r+1)][(j_1+1)\cdots(j_s+1)] \\ &= \tau(m)\tau(n)\end{aligned}$$

In a similar fashion, theorem 1.1 gives

$$\begin{aligned}\sigma(mn) &= \left[\frac{p_1^{k_1+1}-1}{p_1-1} \cdots \frac{p_r^{k_r+1}-1}{p_r-1}\right]\left[\frac{q_1^{j_1+1}-1}{q_1-1} \cdots \frac{q_s^{j_s+1}-1}{q_s-1}\right] \\ &= \sigma(m)\sigma(n)\end{aligned}$$

Thus, τ and σ are multiplicative functions.

Theorem 1.3

If f is a multiplicative function and F is defined by

$$F(n) = \sum_{d \mid n} f(d)$$

then F is also multiplicative.

Proof

Let m and n be relatively prime positive integers. Then

$$\begin{aligned} F(mn) &= \sum_{d \mid mn} f(d) \\ &= \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1 d_2) \end{aligned}$$

because every divisor d of mn can be uniquely written as a product of a divisor d_1 of m and a divisor d_2 of n, where $\gcd(d_1, d_2) = 1$. By the definition of a multiplicative function,

$$f(d_1 d_2) = f(d_1)\, f(d_2)$$

It follows that

$$\begin{aligned} F(mn) &= \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1) f(d_2) \\ &= \left(\sum_{d_1 \mid m} f(d_1) \right) \left(\sum_{d_2 \mid n} f(d_2) \right) \\ &= F(m)F(n) \end{aligned}$$

It might be helpful to take time out and run through the proof of Theorem 1.3 in a concrete case. Letting m = 8 and n = 3, we have

$$F(8 \cdot 3) = \sum_{d \mid 24} f(d)$$

$= f(1) + f(2) + f(3) + f(4) + f(6) + f(8) + f(12) + f(24)$

$= f(1 \cdot 1) + f(2 \cdot 1) + f(1 \cdot 3) + f(4 \cdot 1) + f(2 \cdot 3) + f(8 \cdot 1) + f(4 \cdot 3) + f(8 \cdot 3)$

$= f(1)\, f(1) + f(2)\, f(1) + f(1)\, f(3) + f(4)\, f(1) + f(2)\, f(3) + f(8)\, f(1)$

$+ f(4) f(3) + f(8)\, f(3)$

$= [f(1) + f(2) + f(4) + f(8)][f(1) + f(3)]$

$$= \sum_{d \mid 8} f(d) \cdot \sum_{d \mid 3} f(d)$$

$= F(8)F(3)$

Theorem 1.3 provides a deceptively short way of drawing the conclusion that τ and σ are multiplicative

The Mangoldt function $\Lambda(n)$

Definition 1.2

For every integer $n \geq 1$ we define

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here is a short table of values of $\Lambda(n)$:

n:	1	2	3	4	5	6	7	8	9	10
$\Lambda(n)$:	0	log 2	log 3	log 2	log 5	0	log 7	log 2	log 3	0

The proof of the next theorem shows how this function arises naturally from the fundamental theorem of arithmetic.

Theorem 1.4

If $n \geq 1$ we have

$$\log n = \sum_{d \mid n} \Lambda(d) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Proof

The theorem is true if $n = 1$ since both members are 0 . Therefore, assume that $n > 1$ and write

$$n = \prod_{k=1}^{r} p_k{}^{a_k}$$

Taking logarithms we have

$$\log\ n = \sum_{k=1}^{r} a_k \log\ p_k$$

Now consider the sum on the right of (1). The only nonzero terms in the sum come from those divisors d of the form $p_k{}^m$ for $m = 1,2, \dots, a_k$ and $k = 1,2, \dots, r$. Hence

$$\sum_{d|n} \Lambda(d) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \Lambda(p_k^m) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \log\ p_k = \sum_{k=1}^{r} a_k \log\ p_k = \log\ n$$

which proves (1).

CHAPTER 2

EULER'S ϕ FUNCTION

Let n be positive integer. Let U_n denote the set of all positive integers less than n and coprime to it

For example,

$$U_6 = \{1,5\}$$

$$U_{10} = \{1,3,7,9\}$$

$$U_{18} = \{1,5,7,11,13,17\}$$

Definition 2.1

Euler's ϕ function is a function ϕ: N→N such that for any n ∈ N, ϕ (n) is the number of integers less than n and coprime to it

In other words

'Euler's ϕ function counts the number of elements in U_n'

For example,

$$\phi(1) = 1, \phi(2) = 1, \phi(3) = 2, \phi(4) = 2, \phi(5) = 4$$
$$\phi(6) = 2 \ldots.$$

Theorem 2.1

Let p be a prime. Then ϕ (p) = p-1

Proof:

By definition, any natural number strictly less than p is coprime to p, hence

$$\phi\ (p) = p-1$$

Theorem 2.2

If p is a prime and $k > 0$, then

$$\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1)$$

Proof:

Consider the successive p^k natural numbers not greater than p^k arranged in the following rectangular array of p columns and p^{k-1} rows

1	2	.	.	p
p+1	p+2	.	.	2p
.	.	.	.	.
.	.	.	.	.
p^k-p+1	p^k-p+2	.	.	p^k

among these numbers only the ones at the rightmost sides are not coprime to p^k and there are p^{k-1} members in that column. So

$$\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1).$$

For example, $\phi(8) = 2^3 - 2^2 = 4$ which counts the number of elements in the set $\mathrm{U}_8 = \{1,3,5,7\}$

By the fundamental theorem of arithmetic, we can write any natural number n as

$$n = p_1^{k_1} \dots \dots \dots .. p_r^{k_r}$$

where P_i 's are distinct prime and $k_i \geq 1$ are integers. We already know how to find $\phi\left(p_i^{k_i}\right)$ we would lie to see how $\phi(n)$ is related to $\phi\left(p_i^{k_i}\right)$. This follows from a very important property of Euler's ϕ Function

Multiplicativity of Euler's ϕ Function

Theorem 2.3

$\phi(\mathrm{mn}) = \phi(\mathrm{m})\phi(\mathrm{n})$ if m and n are coprime natural numbers.

Proof:

Consider the array of natural numbers not greater than mn arranged in m columns and n rows in the following manner

$$\begin{array}{cccccc} 1 & 2 & \cdots & r & \cdots & m \\ m+1 & m+2 & & m+r & & 2m \\ 2m+1 & 2m+2 & & 2m+r & & 3m \\ \vdots & \vdots & & \vdots & & \vdots \\ (n-1)m+1 & (n-1)m+2 & & (n-1)m+r & & nm \end{array}$$

Clearly each row of the above array has m distinct residues modulo m. Each column has n distinct residues modulo n: for $1 \leq i, i \leq n-1$

im +j $\equiv$ $im + j$ (mod n)

$\Rightarrow im \equiv im$ (mod n)

$\Rightarrow i \equiv i$ (mod n)   (as gcd(m,n) = 1)

$\Rightarrow i \equiv i$

Each row has ϕ(m) residues coprime to m, and each column has ϕ(n) residues coprime to n. Hence in total ϕ(m)ϕ(n) elements in the above array which are coprime to both m and n, it follows that

$$\phi(\mathrm{mn}) = \phi(\mathrm{m})\phi(\mathrm{n})$$

Theorem 2.4

Let n be any natural numbers, then

$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$

Proof:

By fundamental theorem of arithmetic, we can write

$$n = P_1^{k_1} P_2^{k_2} \ldots \ldots . \ldots P_r^{k_r}$$

Where p_i are the distinct prime factor of n, and k_i are the non negative integers. By previous theorem and proposition,

$$\phi(n) = \phi\left(p_1^{k_1}\right) \cdot \ldots, \phi\left(p_r^{k_r}\right)$$

$$= P_1^{k_1-1}(P_1 - 1) \cdots P_r^{k_{r-1}}(P_r - 1)$$

$$= p_1^{k_1}\left(1-\frac{1}{P_1}\right)\cdots P_r^{k_r}\left(1-\frac{1}{P_r}\right)$$

$$= n\left(1-\frac{1}{p_1}\right)\cdots\cdot\left(1-\frac{1}{p_r}\right)$$

Theorem 2.5

For $n > 2$, $\phi(n)$ is an even integer.

Proof:

First, assume that n is a power of 2, let us say that $n = 2^k$, with $k \geq 2$. By

theorem 2.2,

$$\phi(n) = \phi(2^k) = 2^k\left(1-\frac{1}{2}\right) = 2^{k-1}$$

an even integer. If n does not happen to be a power of 2, then it is divisible by an odd prime p; we therefore may write n as $n = p^k m$, where $k \geq 1$ and gcd $(p^k, m) = 1$. Exploiting the multiplicative nature of the phi-function, we obtain

$$\phi(n) = \phi(p^k)\phi(m) = p^{k-1}(p-1)\phi(m)$$

which again is even because $2 \mid p-1$.

Theorem 2.6

For each positive integer n,

$$n = \sum_{d|n} \phi(d)$$

Proof:

Let us partition the set {1,2,…….,n} into mutually disjoint subsets S_d for each d/n, where

$$S_d = \{1 \leq m \leq n \mid \text{gcd}\,(m,n) = d\}$$

$$= \{1 \leq \frac{m}{d} \leq \frac{n}{d} \mid \text{gcd}\,(\frac{m}{d},\frac{n}{d}) = 1\}$$

Then

$$\{1,2,\ldots\ldots,\mathrm{n}\} = \sum_{d|n} S_d$$

$$\Rightarrow \qquad n = \sum_{d|n} \phi\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \phi(d)$$

As for each divisor of n, n/d is also a divisor of n

MOBIUS FUNCTION

Definition 2.2

The Mobius function $\mu: N \longrightarrow \{0, \pm 1\}$ is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2/n \text{ for some prime } p \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r, \text{ where } p_i \text{ are distinct primes} \end{cases}$$

For example,

$\mu(1) = 1$ $\quad \mu(2) = -1$ $\quad \mu(3) = -1$

$\mu(4) = 0$ $\quad \mu(5) = -1$ $\quad \mu(6) = 1$

If p is a prime number, it is clear that $\mu(p) = -1$; in addition, $\mu(p^e) = 0$ for $e \geq 2$.

Theorem 2.7

The Mobius function is a multiplicative function i.e.

$\mu(mn) = \mu(m)\mu(n)$, if m and n are relatively prime

Proof:

Let m and n be coprime integers, we can consider the following to cases

Case 1: let $\mu(mn) = 0$ then there is a prime p such that $p^2/_{mn}$. As m and n are coprime p cannot divide both m and n hence either $p^2/_m$ or $p^2/_n$. Therefore either $\mu(m) = 0$ or $\mu(\mathrm{n}) = 0$ and we have $\mu(mn) = \mu(m)\mu(n)$

Case 2: suppose that $\mu(mn) \neq 0$ then mn is square free, hence so are m and n. let

$m = p_1 \dots\dots\dots p_r$ and $n = q_1 \dots\dots\dots q_s$ where p_i *and* q_j are all distinct primes then mn = $p_1 \dots\dots\dots p_r q_1 \dots\dots\dots q_s$ where all the primes occurring in the factorization of mn are distinct. Hence

$$\mu(mn) = (-1)^{r+s}$$

$$= (-1)^r(-1)^s$$

$$= \mu(m)\mu(n)$$

Theorem 2.8

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Where d runs through all the positive divisors of n.

Proof:

$$Let \quad F(n) = \sum_{d|n} \mu(d)$$

As μ is multiplicative, so is F(n) by the theorem (F be a multiplicative arithmetic function $F(n) = \sum_{d|n} f(d)$ then F is also a multiplicative arthmetic function)

Clearly

$$F(1) = \sum_{d|n} \mu(d)$$

$$= \mu(1)$$

$$= 1$$

For integers which are prime power, i.e. of the form p^k for some $k \geq 1$

$$F(p^2) = \mu(1) + \mu(p) + \mu(p^2) + \cdots \dots\dots\dots.. + \mu(p^k)$$

$$= 1 + (-1) + 0 \dots\dots\dots\dots + 0$$

$$= 0$$

Now consider any integer n, and consider its prime factorization. Then

$$n = p_1^{k_1} \dots\dots\dots\dots\dots. p_r^{k_r}, \qquad k_i \geq 1$$

$$\Rightarrow \quad F(n) = \prod F(p_i^{k_i})$$

$$= 0$$

Mobius inversion formula

The following theorem is known as Mobius inversion formula

Theorem 2.9

Let F and f be two function from the set N of natural number to the field complex number C such that

$$F(n) = \sum_{d|n} f(d)$$

Then we can express f(n) as

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d)$$

Proof:

First observe that if d is divisor of n so is n/d. Hence both the summation in the last line of the theorem are same. Now

$$\sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \left(\mu(d) \sum_{c|(n/d)} f(c) \right)$$

The crucial step in the proof is to observe that the set of S of pairs of integers (c,d) with d|n and c|n/d is the same as the set T of pairs (c,d) with c/n and d|n/c.

$$\sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \left(\mu(d) \sum_{c|(n/d)} f(c) \right)$$

$$= \sum_{d|n} \left(\sum_{c|(n/d)} \mu(d) f(c) \right)$$

$$= \sum_{(c,d)\in S} f(c)\mu(d)$$

$$= \sum_{(c,d)\in T} f(c)\mu(d)$$

$$= \sum_{c|n}\left(f(c)\sum_{d|(n/c)} \mu(d)\right)$$

= F(n)

$As \sum_{d|n} \mu(d) = 0 \quad unless\ {}^{n}/_{c} = 1,$ which happens when c = n

Let us demonstrate this with n = 15

$$\sum_{d|15} \mu(d)F\left(\frac{15}{d}\right) = \mu(1)[\,f\ (1) + \ f\ (3) + \ f\ (5) + \ f\ (15)] + \mu(3)[\,f\ (1) + \ f\ (5)]$$

$$+\mu(5)[\,f\ (1) + \ f\ (3)] + \mu(15)[\,f\ (1)]$$

= f (1)[μ(1) + μ(3) + μ(5) + μ(15)] + f (3)[μ(1) + μ(5)] + f (5)[μ(1) + μ (5)] + f(15) μ(1)

= f(1).0 + f(3).0 + f(5).0 + f(15)

= f(15)

The above theorem leads to the following interesting identities

1. we know that for any positive integer n,

$$\sum_{d|n} \phi(d) = n$$

Where $\phi(n)$ is Euler's ϕ function. Hence

$$\phi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d$$

For example,

$$\phi(10) = \mu(1)10 + \mu(2)5 + \mu(5)2 + \mu(10)1$$

$= 10 - 5 - 2 + 1$

$= 4$

2. similarly

$$\sigma(n) = \sum_{d|n} d$$
$$n = \sum_{d|n} \mu\left(\frac{n}{d}\right) \sigma(d)$$

For example,

With n = 10

$$\mu(10).1 + \mu(2)(1+5) + \mu(5)(1+3) + \mu(1)(1+3+5+10)$$

$= 1 - 1 - 5 - 1 - 3 + 1 + 3 + 5 + 10$

$= 10$

We have seen before that if multiplicative so is $F(n) = \sum_{d|n} f(d)$. But we can now

Prove that converse applying the Mobius inversion formula

Theorem 2.10

If F is a multiplicative function and

$$F(n) = \sum_{d|n} f(d)$$

then f is also multiplicative.

Proof:

By the Mobius inversion formula we know that

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d)$$

Let m and n be relatively prime positive integers. We recall that any divisor

d of mn can be uniquely written as $d = d_1, d_2$, where $d_1 | m$, $d_2 | n$, and

$\gcd(d_1, d_2) = 1 = \gcd(\frac{m}{d_1}, \frac{n}{d_2})$.

Conversely if d_1/m and d_2/n then d_1d_2/mn thus,

$$\begin{aligned}
f(mn) &= \sum_{d|mn} \mu(d)F\left(\frac{mn}{d}\right) \\
&= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1d_2)F\left(\frac{mn}{d_1d_2}\right) \\
&= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1)\mu(d_2)F\left(\frac{m}{d_1}\right)F\left(\frac{n}{d_2}\right) \\
&= \sum_{d_1|m} \mu(d_1)F\left(\frac{m}{d_1}\right)\sum_{d_2|n} \mu(d_2)F\left(\frac{n}{d_2}\right) \\
&= f(m)f(n)
\end{aligned}$$

In view of the above theorem we can say that as N(n) = n is a multiplicative function so is $\phi(n)$ because

$$\sum_{d|n} \phi(d) = n = \text{N(n)}$$

CONCLUSION

The purpose of this project gives a simple account of Arithmetic function, Euler's phi function and Mobius Function. The study of these topics given excellent introduction to the subject called 'NUMBER THEORETIC FUNCTION'

Number Theoretic Function demands a high standard of rigor. Thus, our presentation necessarily has its formal aspect with care taken to present clear and detailed argument. An understanding of the statement of the theorem, number theory proof is the important issue. In the first chapter we discuss about function τ and σ are both multiplicative function. If f is a multiplicative function and F is defined by

$F(n) = \sum_{d|n} f(d)$, then F is also multiplicative. In the second chapter 2 we discuss about that if p is prime the $\phi(p) = p - 1,\ \phi(mn) = \phi(m)\phi(n)$. The Mobius function is multiplicative function if f is multiplicative function and $F(n)$=$\sum_{d|n} f(d)$,

then F is also multiplicative.

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NORMED LINEAR SPACES

Project report submitted to
The Kannur University
for the award of the degree
of

Bachelor of Science

by

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DB18CMSR09

Under the guidance of

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CERTIFICATE

It is to certify that this project report **'Normed Linear Spaces'** is the bonafide project of **Soorya Devi K S** carried out the project work under my supervision.

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DECLARATION

I **Soorya Devi K S** hereby declare that the project **'Normed Linear Space'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Ms. Athulya P , Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu and has not submitted by me elsewhere for the award of my degree, diploma, title or recognition, before.

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CONTENTS

INTRODUCTION

This chapter gives an introduction to the theory of normed linear spaces. A skeptical reader may wonder why this topic in pure mathematics is useful in applied mathematics. The reason is quite simple: Many problems of applied mathematics can be formulated as a search for a certain function, such as the function that solves a given differential equation. Usually the function sought must belong to a definite family of acceptable functions that share some useful properties. For example, perhaps it must possess two continuous derivatives. The families that arise naturally in formulating problems are often linear spaces. This means that any linear combination of functions in the family will be another member of the family. It is common, in addition, that there is an appropriate means of measuring the “distance” between two functions in the family. This concept comes into play when the exact solution to a problem is inaccessible, while approximate solutions can be computed. We often measure how far apart the exact and approximate solutions are by using a norm. In this process we are led to a normed linear space, presumably one appropriate to the problem at hand. Some normed linear spaces occur over and over again in applied mathematics, and these, at least, should be familiar to the practitioner. Examples are the space of continuous functions on a given domain and the space of functions whose squares have a finite integral on a given domain.

PRELIMINARIES

1) LINEAR SPACES

We introduce an algebraic structure on a set $X$and study functions on X which are well behaved with respect to this structure. From now onwards , $\boldsymbol{K}$ will denote either $\boldsymbol{R}$, the set of all real numbers or $\boldsymbol{C}$**,** the set of all complex numbers. For $k \in \boldsymbol{C}$, $Re\ k$ and $Im\ k$ will denote the real and imaginary part of k.

A linear space(or a vector space) over $\boldsymbol{K}$ is a non-empty set X along with a function $+\ : X \times X \to X$, called addition and a function $\cdot$: $\boldsymbol{K}\times\ X \to X$ called scalar multiplication, such that for all $x\ ,\ y\ ,\ z\ \in\ X$and $k\ ,\ l \in \boldsymbol{K}$, we have

$$x + y = y + x$$

$$x + (y + z) = (x + y) + z$$

$$\exists 0 \in X\ such\ that\ x + 0 = x,$$

$$\exists - x \in X\ such\ that\ x + (-\ x) = 0\ ,$$

$$k \cdot (x + y) = k \cdot x + k \cdot y,$$

$$(k + l)\cdot x = k \cdot x + l \cdot x,$$

$$(kl)\cdot x = k \cdot (l \cdot x),$$

$$1\cdot x = x.$$

We shall write kx in place of $k \cdot x$. We shall also adopt the following notations. For $x, y \in X, k \in \boldsymbol{K}$ and subsets $E, F$of X ,

$$x + F = \{x + y : y \in F\},$$

$$E + F = \{x + y : x \in E, y \in F\},`$$

$$kE = \{kx : x \in E\}.$$

2) BASIS

A nonempty subset E of X is said to be a subspace of X if $kx + ly \in E$ whenever $x, y \in E$ and $k, l \in \boldsymbol{K}$. If $\emptyset \neq E \subset X$, then the smallest subspace of X containing E is

$$spanE = \{k_1x_1 + \cdots + k_nx_n : x_1, ..., x_n \in E,\ k_1, ..., k_n \in K\}$$

It is called the span of E. If span $E = X$, we say that E spans X. A subset E of X is said to be linearly independent if for all $x_1, ..., x_n \in E$ and $k_1, ..., k_n \in \boldsymbol{K}$, the equation $k_1x_1 + \cdots + k_nx_n = 0$implies that $k_1 = \cdots = k_n = 0$. It is called linearly dependent if it is not linearly independent, that is, if there exist $x_1, ..., x_n \in E$ and $k_1, ..., k_n \in \boldsymbol{K}$ such that $k_1x_1 + \cdots + k_nx_n = 0$, where at least one $k_j$is nonzero.

A subset E of X is called a Hamel basis or simply basis for X if *span of* $E = X$ and E is linearly independent .

3) DIMENSION

If a linear space X has a basis consisting of a finite number of elements , then X is called finite dimensional and the number of elements in a basis for X is called the dimension of X, denoted as $dimX$. Every basis for a finite dimensional linear space has the same (finite) number of elements and hence the dimension is well-defined. The space $\{0\}$ is said to have zero dimension. Note that it has no basis !

If a linear space contains an infinite linearly independent subset, then it is said to be infinite dimensional.

4)METRIC SPACE

We introduce a distance structure on a set X and study functions on X which are well-behaved with respect to this structure.

A metric d on a nonempty set X is a function $d: X \times X \to \boldsymbol{R}$
such that for all $x, y, z \in X$

$$d(x, y) \geq 0 \text{ and } d(x, y) = 0 \text{ iff } x=y$$

$$d(y, x) = d(x, y)$$

$$d(x, y) \leq d(x, z) + d(z, y).$$

The last condition is known as the triangle inequality. A metric space is a nonempty set X along with a metric on it.

5)CONTINUOUS FUNCTIONS

Roughly speaking, a function from a metric space to a metric space is continuous if it sends 'nearby' points to 'nearby' points. If X and Y are metric spaces with metrics d and e respectively, then a function $F: X \rightarrow Y$ is said to be continuous at $x_0 \in X$ if for every $\epsilon \rangle\, 0$, there is some $\delta > 0$ (possibly depending on ϵ and x_0) such that $e\left(F(x), F\left(x_0\right)\right) < \epsilon$ for all $x \in X$ satisfying $d\left(x, x_0\right) < \delta$. Further, F is said to be continuous on X if it is continuous at every point of X. It is easy to see that F is continuous on X if and only if the set $F^{-1}(E)$ is open in X whenever the set E is open in$Y$. Also , this happens iff $F(x_n) \rightarrow F(x)$ in Y whenever $x_n \rightarrow x$ in X.

6) UNIFORM CONTINUITY

We note that a continuous function $F: T \rightarrow S$ is, in fact, uniformly continuous, that is, for every $\epsilon > 0$, there exists some $\delta > 0$ such that $e(F(t), F(u)) < \epsilon$ whenever $d\left(t, u\ \right) < \delta$. This can be seen as follows. Let $t \in T$. By the continuity of F at $t \in T$, there is some δ_t, such that $e(F(t), F(u)) < \frac{\varepsilon}{2}$whenever $d(t, u) < \delta_t$.

7) FIELD

A ring is a set R together with two binary operations + and ·(which we call addition and multiplication) such that the following axioms are satisfied .

- ➢ R is an abelian group with respect to addition
- ➢ Multiplication is associative
- ➢ $\forall a\ , b, c \in R$the left distributive law $a(b + c) = (a \cdot b) + (a \cdot c)$ and the right distributive law $(a + b)c = (a \cdot c) + (b \cdot c)$, hold .

A field is a commutative division ring

CHAPTER 1

NORMED LINEAR SPACE

Let X be a linear space over $\boldsymbol{K}$. A norm on X is the function $|| \ ||$ from X to $\boldsymbol{R}$ such that $\forall$ $x,y \in X$ *and* $k \in \boldsymbol{K}$,

$$||x|| \geq 0 \text{ and } ||x|| = 0 \text{ if and only if } x = 0 \ ,$$

$$||x + y|| \leq ||x|| + ||y|| \ ,$$

$$||kx|| = |k| \ ||x|| \ .$$

A norm is the formalization and generalization to real vector spaces of the intuitive notion of " length" in the real world .

A normed space is a linear space with norm on it .

For x and y in X , let

$$d(x,y) = ||x - y||$$

Then d is a metric on X so that (X,d) is a metric space , thus every normed space is a metric space

- *Every normed linear space is a metric space . But converse may not be true .*

Example :

$$d(x,y) = \frac{|x - y|}{1 + |x - y|} \ , \forall x \ , y \in X$$

$$\Rightarrow \ ||x - y|| = \frac{|x - y|}{1 + |x - y|}$$

$$\Rightarrow \ ||z|| = \frac{|z|}{1 + |z|} \ , z = x - y \in X$$

$$||\alpha z|| = \frac{|\alpha z|}{1 + |\alpha z|}$$

$$= \frac{|\alpha|\,|z|}{1 + |\alpha|\,|z|}$$

$$= |\alpha| \left(\frac{|z|}{1 + |\alpha|\,|z|} \right)$$

$$\neq |\alpha|\;||z|| .$$

➢ *Result*

Let X be a normed linear space . Then ,

$$\big|\; ||x|| - ||y|| \;\big| \leq ||x - y|| \;, \forall x , y \in X$$

Proof :

$$||x|| = ||\,(\,x - y\,) + y|| \leq ||x - y|| + ||y||$$

$$\Rightarrow ||x|| - ||y|| \leq ||x - y|| \rightarrow (1)$$

$x \leftrightarrow y$

$$||y|| - ||x|| \leq ||y - x||$$

$$\Rightarrow -\,(\,||x|| - ||y||\,) \leq \;||x - y|| \rightarrow (2)$$

From (1) and (2)

$$\big|||x|| - ||y||\big| \leq \;||x - y||$$

➢ *Norm is a continuous function*

Let $x_n \rightarrow x$, as $n \rightarrow \infty$

$\Rightarrow x_n - x \to 0 \text{ as } n \to \infty$

$\Rightarrow ||x_n - x|| \to 0 \text{ as } n \to \infty$

$\big|\, ||x_n|| - ||x|| \,\big| \leq ||x_n - x|| \to 0 \text{ as } n \to \infty$

$\Rightarrow ||x_n|| - ||x|| \to 0 \text{ , as } n \to \infty$

$\Rightarrow ||x||$ is continuous

➢ *Norm is a uniformly continuous function*

We have , $|| \; || : X \to \boldsymbol{R}$. Let $x,y \in X$ and $\varepsilon > 0$

Then $||x|| = ||x - y + y||$

$\leq ||x - y|| + ||y||$

$\Rightarrow ||x|| - ||y|| \leq ||x - y|| \to (1)$

Interchanging x and y ,

$||y|| - ||x|| \leq ||y - x||$

$\Rightarrow -(\, ||x|| - ||y|| \,) \leq ||x - y||$

$\Rightarrow \quad ||x|| - ||y|| \geq - ||x - y|| \to (2)$

Combining (1) and (2)

$- ||x - y|| \leq ||x|| - ||y|| \leq ||x - y||$

That is ,

$\big|\, ||x|| - ||y|| \,\big| \leq ||x - y||$

Take $\delta = \varepsilon$, then whenever $||x - y|| < \delta$, $\big|\, ||x|| - || y|| \,\big| < \varepsilon$

Therefore $\|\ \|$ is a uniformly continuous function .

- <u>*Continuity of addition and scalar multiplication*</u>

To show that $+ : X\times X \to X$ and $\cdot : \boldsymbol{K} \times X \to X$ are continuous functions.

Let $(x,y) \in X\times X$. To show that + is continuous at (x,y) , that is , to show that for each $(x,y) \in X\times X$ *if* $x_n \to x$ and $y_n \to y$ in X , then

$$+(x_n , y_n) \to +(x , y) ;$$

That is ,

$$x_n + y_n \to x + y .$$

Consider

$$\|(x_n + y_n) - (x + y)\| = \|x_n - x + y_n - y\|$$

$$\leq \|x_n - x\| + \|y_n - y\|$$

Given $x_n \to x$ *and* $y_n \to y$ *, for each* $\epsilon > 0$, $\exists N_1 \ni$

$$\|x_n - x\| < \tfrac{\varepsilon}{2} \forall n \geq N_1 ,$$ *and* $\exists N_2 \ni$

$$\|y_n - y\| < \tfrac{\varepsilon}{2} \quad \forall n \geq N_2$$

\

Take $N = max \{ N_1, N_2\}$

Then $\|x_n - x\| < \frac{\varepsilon}{2}$ and $\|y_n - y\| < \frac{\varepsilon}{2} \forall n \geq N$

Therefore $\|(x_n + y_n) - (x + y)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \forall n \geq N$

That is , $x_n + y_n \to x + y$

Now to show that $\cdot : \boldsymbol{K} \times X \to X$ is continuous

Let $(k , x) \in \boldsymbol{K}\times X$

To show that if $k_n \to k$ and $x_n \to x$, then $k_n x_n \to kx$

Since $k_n \to k$, $\forall\ \varepsilon > 0\ \exists\ N_1 \ni |k_n - k| < \frac{\varepsilon}{2}\ \ \forall\ n \geq N_1$

Since $x_n \to x$, $\forall\ \varepsilon > 0\ \exists\ N_2 \ni ||x_n - x|| < \frac{\varepsilon}{2}\ \ \forall\ n \geq N_2$

Consider $||k_n x_n - kx|| = ||k_n x_n - kx + x_n k - x_n k||$

$$= ||x_n (k_n - k) + k(x_n - x)||$$

$$\leq ||x_n(k_n - k)|| + ||k(x_n - x)||$$

$$= ||x_n||\ |k_n - k| + |k|\ ||x_n - x||$$

$$\leq ||x_n||\frac{\varepsilon}{2} + |k|\frac{\varepsilon}{2}$$

$\therefore\ \ k_n x_n \to kx$

➢ <u>*Examples of normed space*</u>

1) Spaces $\boldsymbol{K}^n$ ($\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$)

For $n = 1$, the absolute value of function | | is a norm on $\boldsymbol{K}$, since $\forall k \in \boldsymbol{K}$

We have ,

$||k|| = ||k \cdot 1|| = |k|\ ||1||$, by definition .

But $||1||$ is a positive scalar .

$\therefore\ ||k||$ is a positive scalar multiple of the absolute value function .

$\therefore$ any norm on $\boldsymbol{K}$ is a positive scalar multiple of the absolute value function

For $n > 1$, let $p \geq 1$ be a real number

$K^n = \{ (x(1) , x(2) , . . . , x(n)) : x(i) \in K , i = 1 , 2 , . . . , n \}$

For $x \in K^n$, that is , $x = (x(1) , x(2) , . . . , x(n))$, define

$$||x||_p = (|x(1)|^p + . . . + |x(n)|^p)^{1/p}$$

Then $|| \, ||_p$ is a norm on K^n

When $p = 1$,

Then , $||x||_1 = |x(1)| + |x(2)| + . . . + |x(n)|$

Since $|x(i)| \geq 0 \;\; \forall \, i = 1 , 2 , . . . , n$, $||x||_1 \geq 0$

And $||x||_1 = 0 \Leftrightarrow |x(1)| + . . . + |x(n)| = 0$

$\Leftrightarrow |x(i)| = 0 \;\; \forall \, i$

$\Leftrightarrow x(i) = 0 \; \forall \, i$

$\Leftrightarrow x = (x(1) , . . . , x(n)) = 0$

Now $||kx||_1 = |kx(1)| + |kx(2)| + . . . + |kx(n)|$

$= |k| \, |x(1)| + . . . + |k| \, |x(n)|$

$= |k| \, (|x(1)| + . . . + |x(n)|)$

$= |k| \, ||x||_1$

$||x + y||_1 = |(x + y)(1)| + . . . + |(x + y)(n)|$

$= |x(1) + y(1)| + . . . + |x(n) + y(n)|$

$\leq |x(1)| + |y(1)| + . . . + |x(n)| + |y(n)|$

$= |x(1)| + . . . + |x(n)| + |y(1)| + . . . + |y(n)|$

$= ||x||_1 + ||y||_1$

Consider $1<p<\infty$

Now , $||x||_p = (|x(1)|^p +. . . + |x(n)|^p)^{1/p}$

Since $|x(i)|^p \geq 0 \ \forall i$, we have $||x||_p \geq 0$

And $||x||_p = 0 \Leftrightarrow (|x(1)|^p +. . . + |x(n)|^p)^{1/p} = 0$

$\Leftrightarrow |x(i)|^p = 0 \ \forall i$

$\Leftrightarrow |x(i)| = 0 \ \forall i$

$\Leftrightarrow x(i) = 0 \ \forall i$

$\Leftrightarrow x = (x(1) , . . . , x(n)) = 0$.

Now

$$||kx||_p = (|kx(1)|^p +. . . + |kx(n)|^p)^{1/p}$$

$$= (|k|^p |x(1)|^p +. . . + |k|^p |x(n)|^p)^{1/p}$$

$$= |k| (|x(1)|^p +. . . + |x(n)|^p)^{1/p}$$

$$= |k| \, ||x||_p .$$

$$||x + y||_p = (|x(1) + y(1)|^p +. . . + |x(n) + y(n)|^p)^{1/p}$$

We have by Minkowski's inequality ,

$$\left(\sum_{i=1}^{n} |x(i) + y(i)|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y(i)|^p\right)^{1/p}$$

Then

$$||x + y||_p \leq \left(|x(1)|^p + \ldots + |x(n)|^p \right)^{1/p} + \left(|y(1)|^p + \ldots + |y(n)|^p \right)^{1/p}$$

$$= ||x||_p + ||y||_p$$

Then , for $1 \leq p < \infty$, $|| \ ||_p$ is a norm on $\boldsymbol{K^n}$

When $p = \infty$, define $||x||_\infty = max\{ |x(1)| , |x(2)| ,\ldots , |x(n)| \}$

Then it is a norm on $\boldsymbol{K^n}$

$||x||_p \geq 0$ since each values $|x(i)| \geq 0$

So that

$$max\{|x(i)| , i=1, \ldots , n\} \geq 0$$

$$||x||_\infty = 0 \Leftrightarrow max\{ |x(i)| : i = 1, \ldots , n \} = 0$$

$$\Leftrightarrow |x(i)| = 0 \ \ \forall i$$

$$\Leftrightarrow x(i) = 0 , \forall i$$

$$\Leftrightarrow x = 0$$

$$||kx||_\infty = max\{ |kx(1)| ,\ldots , |kx(n)| \}$$

$$= max\{ |k| \, |x(1)| , \ldots , |k| \, |x(n)| \}$$

$$= |k| \, max\{|x(1)| , \ldots , |x(n)|\}$$

$$= |k| \, ||x||_\infty$$

$$||x + y||_\infty = max\{ |x(1) + y(1)| , \ldots , |x(n) + y(n)| \}$$

$$\leq max\{ |x(1)| + |y(1)| , \ldots , |x(n)| + |y(n)| \}$$

$\leq max \{ |x(1)| , \ldots , |x(n)| \} + max \{ |y(1)| , \ldots , |y(n)| \}$

$= ||x||_\infty + ||y||_\infty$

2) Sequence space

Let $1 \leq p < \infty$, $l^p = \{ x = (x(1) , x(2) , \ldots) ; \ x(i) \in \boldsymbol{K} \text{ and } \sum_{j=1}^{\infty} |x(j)|^p < \infty \}$, that is , l^p is the space of p-summable scalar sequences in $\boldsymbol{K}$. For $x = (x(1) , x(2) , \ldots) \in l^p$,

let $||x||_p = (|x(1)|^p + |x(2)|^p + \ldots)^{1/p}$. Then it is a norm on l^p.

That is , $|| \, ||_p$ is a function from l^p to $\boldsymbol{R}$.

If $p = 1$, then l^1 is a linear space and $||x||_1 = (|x(1)| + |x(2)| + \ldots)$ is a norm on l^1

Let $p = \infty$. Then l^∞ is the linear space of all bounded scalar sequences . And ,

$$||x||_\infty = sup \{ |x(j)| : j = 1, 2, 3, \ldots \}$$

Then $|| \, ||_\infty$ is a norm on l^∞

CHAPTER 2

THEOREMS ON NORMED SPACES

a) Let Y be a subspace of a normed space X, then Y and its closure $\overline{Y}$ are normed spaces with the induced norm.

b) Let Y be a closed subspace of a normed space X, for $x + Y$ in the quotient space X/Y, let $|||x + Y||| = \inf\{ ||x+y|| : y \in Y\}$. Then $|||\ \ |||$ is a norm on X/Y, called the quotient norm.

A sequence $(x_n + Y)$ converges to $x + Y$ in X/Y iff there is a sequence (y_n) in Y, $(x_n + y_n)$ converges to x in X.

c) Let $||\ \ ||_p$ be a norm on the linear space X_p, $j = 1,2,\ldots$. Fix p such that $1 \le p \le \infty$

For $x = (x(1), x(2), \ldots, x(m))$ that is the product space $X = X_1 \times X_2 \times \ldots \times X_m$,

Let $||x||_p = \left(||x(1)||_1^p + ||x(2)||_2^p + \ldots + ||x(m)||_m^p \right)^{1/p}$, if $1 \le p < \infty$

$||x||_p = \max\{ ||x(1)||_1, \ldots, ||x(m)||_m \}$, if $p = \infty$.

Then $||\ \ ||_p$ is a norm on X.

A sequence (x_n) converges to x in X $\Leftrightarrow$ $(x_n(j))$ converges to $x(j)$ in X_j $\forall\, j=1,2,\ldots,m$.

Proof:

a) Since X is a normed space, there is a norm on X to Y. Since Y is a subspace of X,

$||\ \ ||_y : Y \to \boldsymbol{R}$ is a function. To show that $||\ \ ||_y$ is a norm on Y.

For $y \in Y$, $||y||_Y = ||y||$, then

$$||y||_Y \geq 0 \quad (\because ||y|| \geq 0) \quad and \quad ||y||_Y = 0 \Leftrightarrow y = 0$$

$$||ky||_Y = ||ky|| = |k|\ ||y|| = |k|\ ||y||_y .$$

Let $y_1, y_2 \in Y$. Then ,

$$||y_1 + y_2||_y = ||y_1 + y_2|| \le ||y_1|| + ||y_2|| = ||y_1||_y + ||y_2||_y$$

Now the continuity of addition and scalar multiplication shows that $\overline{Y}$ is a subspace of X, since if $x_n \to x$ and $y_n \to y$, $x_n, y_n \in \overline{Y}$, then

$x_n + y_n \to x + y$ (by continuity of addition) and

$kx_n \to kx$ (by continuity of scalar X^n) .

Since $\overline{Y}$ is closed , $x + y \in \overline{Y}$ and $kx \in \overline{Y}$. Therefore $\overline{Y} \leq X$.

$\therefore$ norm on X induces a norm on Y and $\overline{Y}$

b) X/Y , the quotient space equals $X/Y = \{ x + Y : x \in X \}$.

$$|||x + y||| = inf \{ ||x + y|| : y \in Y \}$$

Claim: $|||\ \ |||$ is a norm on X/Y , called quotient norm

- Let $x \in X$,

$$|||x + Y||| = inf \{ ||x + y|| : y \in Y \} \geq 0.$$

$\therefore$ $|||x + Y||| \geq 0$.

If $|||x + y||| = 0$ (0 in X/Y is Y) , then there is a sequence (y_n) in Y $\ni$

$$||x + y_n || \to 0$$

$$\Rightarrow \quad x + y_n \to 0$$

$$\Rightarrow \quad y_n \to -x$$

Since $y_n \in Y$ and Y is closed

$-x \in Y \quad \Leftrightarrow x \in Y$ ($\because$ Y is a subspace)

$\Leftrightarrow x + Y = Y$, zero in X/Y.

- For $k \in \boldsymbol{K}$,

$$|||k(x + Y)||| = |||kx + Y|||$$
$$= inf \{ ||k(x + y)|| : y \in Y\}$$
$$= inf \{ |k|\ ||x + y|| : y \in Y\}$$
$$= |k|\ inf \{ ||x + y|| : y \in Y\}$$
$$= |k|\ |||x + Y||| .$$

- Let x_1 , $x_2 \in X$. Then

$|||x_1 + Y||| = inf \{ ||x_1 + y|| : y \in Y \}$. Then $\exists\ y_1 \in Y \ni$

$$|||x_1 + Y||| + \frac{\varepsilon}{2} > ||x_1 + y_1|| , \ and$$

$$|||x_2 + Y||| = \inf\{\, ||x_2 + y|| : y \in Y\} \text{ , Then } \exists y_2 \in Y \ni$$

$$|||x_2 + Y||| + \frac{\varepsilon}{2} > ||x_2 + y_2|| \,.$$

$$||x_1 + y_1 + x_2 + y_2|| \le ||x_1 + y_1|| + ||x_2 + y_2||$$

$$\le |||x_1 + Y||| + \frac{\varepsilon}{2} + |||x_2 + Y||| + \frac{\varepsilon}{2}$$

Let $y = y_1 + y_2 \in Y$. Then ,

$$||(x_1+x_2) + y|| \le |||x_1 + Y||| + |||x_2 + Y||| + \varepsilon \quad \text{—(1)}$$

Now , $|||(x_1 + Y) + (x_2 + Y)||| = |||x_1 + x_2 + Y|||$

$$= \inf\{\, ||x_1 + x_2 + y|| : y \in Y\,\}$$

$$< ||x_1 + x_2 + y||$$

$$\le |||x_1 + Y||| + |||x_2 + Y||| + \varepsilon \qquad \text{(by (1))}$$

since ε is arbitrary , we have

$$|||(x_1 + Y) + (x_2 + Y)||| \le |||x_1 + Y||| + |||x_2 + Y|||$$

$\therefore$ $|||\quad|||$ is a norm on X/Y.

Let $(x_n + Y)$ be a sequence in X/Y . Assume that (y_n) is a sequence in $Y \ni (x_n + y_n)$ converges to x in X.

That is , $(x_n - x + y_n)$ converges to 0 . —(1)

Claim: $(x_n + Y)$ converges to $x + Y$.

Consider

$$|||x_n + Y - (x+Y)||| = |||(x_n - x) + Y|||$$

$$= \inf\{\, ||x_n - x + y_n|| : y \in Y\,\}$$

$$\le ||x_n - x + y_n|| \quad \forall y_n \in Y \,.$$

Then by (1) , $x_n + Y$ converges to $x + Y$ in X/Y.

Conversely assume that the sequence $(x_n + Y) \to x + Y$ in X/Y.

Consider $|||x_n + Y - (x + Y)||| = |||x_n - x + Y|||$

$$= \inf\{\, ||x_n - x + y|| : y \in Y\,\}$$

Then we can choose $y_n \in Y \ni$

$$||x_n - x + y_n|| < \; |||(x_n - x) + Y||| + \frac{1}{n} \;\; , \;\; n=1,2,3,\ldots.$$

Since $x_n+Y \;\to x+Y$, we get

$(x_n - x + y_n)$ converges to zero as $n \to\infty$

That is , $(x_n + y_n)$ converges to x in X as $n \to\infty$

c) Consider $1\leq p < \infty$

Given that

$$||x||_p = (||x(1)||_1^p + ||x(2)||_2^p +. \; . \; . \; + ||x(m)||_m^p)^{1/p}$$

Clearly , $||x||_p \geq 0$.

Since each $||x(i)||_i^p \geq 0$.

$$||x||_p = 0 \Leftrightarrow |x(j)|_j^p = 0 \quad \forall \; j = 1, \ldots, m$$

$$\Leftrightarrow x(j) = 0 \qquad \forall \; j.$$

$$\Leftrightarrow x = (x(1), \ldots, x(m)) = 0$$

$$||kx||_p = \left(||kx(1)||_1^p +. \; . \; .+ ||kx(m)||_m^p \right)^{1/p}$$

$$= \left(|k|^p ||x(1)||_1^p +. \; . \; .+ |k|^p ||x(m)||_m^p \right)^{1/p}$$

$$= |k| \left(||x(1)||_1^p +. \; . \; .+ ||x(m)||_m^p \right)^{1/p}$$

$$= |k| \; ||x||_p \qquad , k\in \boldsymbol{K} \text{ and } x\in X$$

Now, $||x + y||_p = \left(||x(1) + y(1)||_1^p +. \; . \; .+ ||x(m) + y(m)||_m^p \right)^{1/p}$

(by Minkowski's inequality)

$$\leq$$

$$\left(\left(||x(1)||_1 + ||y(1)||_1\right)^p +. \; . \; . \; + \left(||x(m)||_m + ||y(m)||_m\right)^p \right)^{1/p}$$

$$\leq \left(\sum_{j=1}^{m} ||x(j)||_j^p\right)^{1/p} + \left(\sum_{j=1}^{m} ||y(j)||_j^p\right)^{1/p} \qquad \text{(Minkowski's inequality)}$$

$$= \left(||x(1)||_1^p +. . . + ||x(m)||_m^p \right)^{1/p}$$

$$= ||x||_p + ||y||_p$$

Now suppose $p = \infty$

$||x||_\infty = max \{ ||x(1)||_1 , . . . , ||x(m)||_m \}$

$||x||_\infty \geq 0$ Since $||x(j)|| \geq 0 , \quad \forall j$

$||x||_\infty = 0 \quad \Leftrightarrow ||x(m)|| = 0 \quad \forall m$

$\Leftrightarrow x(m) = 0 \quad \forall m$

$\Leftrightarrow x = 0$

$||kx||_\infty = max \{ ||kx(1)||_1 , . . . , ||kx(m)||_m \}$

$= |k| \, max \{ ||x(1)||_1 , . . . , ||x(m)||_m \}$

$= |k| \, ||x||_\infty$

$||x + y||_\infty = max \{ ||x(1) + y(1)||_1, . . . , ||x(m) + y(m)||_m \}$

$\leq max \{ ||x(1)||_1 + ||y(1)||_1 , . . . , ||x(m)||_m + ||y(m)||_m \}$

$= max \{ ||x(1)||_1 , . . . , ||x(m)||_m \} + max \{ ||y(1)||_1 , . . . , ||y(m)||_m \}$

$= ||x||_\infty + ||y||_\infty$

We now consider ,

$$||x_n - x(1)||_p = \left(||x_n(1) - x(1)\right)||_1^p +. . .+ ||x_n(m) - x(m)||_m^p)^{1/p}$$

Then

$x_n \to x \text{ in } X \quad \Leftrightarrow ||x_n - x ||_p \to 0$

$\Leftrightarrow ||x_n(j) - x(j)||_j^p \to 0$

$\Leftrightarrow x_n(j) - x(j) \to 0$

$\Leftrightarrow x_n(j) \to x(j)$ in $X \, \forall j$.

RIESZ LEMMA

Let X be a normed space . Y be a closed subspace of X and $X \neq Y$. Let r be a real number such that $0 < r < 1$. Then there exist some $x_r \in X$ such that $||x_r|| = 1$ and

$r < dist\,(x_r\,,\,Y) \leq 1$

Proof :

We have ,

$$dist\,(x\,,\,Y) = inf\{\,d(x\,,\,y) : y \in Y\}$$

$$= inf\{\,||x - y|| : y \in Y\}$$

Since $Y \neq X$, consider $x \in X \ni x \notin Y$.

If $dist(x\,,\,Y) = 0$, then $||x - y|| = 0 \Rightarrow x \in \overline{Y} = Y$ ($\because$ Y is closed)

Therefore ,

$$dist\,(x\,,\,Y) \neq 0$$

That is ,

$$dist\,(x\,,\,Y) > 0$$

Since $0 < r < 1$, $\frac{1}{r} > 1$

$$\Rightarrow \frac{dist\,(x\,,Y)}{r} > dist\,(x\,,\,Y)$$

That is , $\frac{dist\,(x\,,Y)}{r}$ is not a lower bound of $\{\,||x - y|| : y \in Y\,\}$

Then $\exists$ $y_0 \in Y \ni ||x - y_0|| < \frac{dist(x\,,Y)}{r}$ $\rightarrow$(1)

Let $x_r = \frac{x - y_0}{||x - y_0||}$. Then $x_r \in X$

($\because y_0 \in Y$, $x \notin Y \Rightarrow x - y_0 \in X$ and $||x - y_0|| \neq 0$)

Then $||x_r|| = \left|\left|\frac{x - y_0}{||x - y_0||}\right|\right| = \frac{||x - y_0||}{||x - y_0||} = 1$

Now to prove $r < dist(x_r, Y) \leq 1$

We have $\quad dist(x_r , Y) = inf\{ ||x_r - y|| : y \in Y \}$

$$\leq ||x_r - y|| \quad \forall y \in Y$$

In particular, $0 \in Y$, so that $dist(x_r , Y) \leq ||x_r - 0|| = 1$

That is ,

$$dist\ (x_r , Y) \leq 1$$

Now ,

$$dist\ (x_r , Y) = dist\left(\frac{x - y_0}{||x - y_0||}, Y\right)$$

$$= \frac{1}{||x - y_0||} dist\ (x - y_0 , Y)$$

$$= \frac{1}{||x - y_0||} inf\{ ||x - y_0 - y|| : y \in Y\}$$

$$= \frac{1}{||x - y_0||} inf\{||x - (y_0 + y)|| : y_0 + y \in Y\}$$

$$= \frac{1}{||x - y_0||} dist\ (x , Y)$$

$$> \frac{r}{dist\ (x , Y)} dist\ (x , Y) \quad by\ (1)$$

$$\Rightarrow dist\ (x_r , Y) > r$$

That is ,

$$r < dist\ (x_r , Y) \leq 1$$

CONCLUSION

This project discusses the concept of normed linear space that is fundamental to functional analysis . A normed linear space is a vector space over a real or complex numbers ,on which the norm is defined . A norm is a formalization and generalization to real vector spaces of the intuitive notion of “length” in real world

In this project , the concept of a norm on a linear space is introduced and thus illustrated . It mostly includes the properties of normed linear spaces and different proofs related to the topic.

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PLANARITY IN GRAPH THEORY

Project report submitted to
The Kannur University
for the award of the degree
of

Bachelor of Science

by

SREESHMA SHIBU

DB18CMSR10

Under the guidance of

Mrs. PRIJA V

Department of Mathematics
Don Bosco Arts and Science College
Angadikkadavu
June 2021

CERTIFICATE

Certified that this project **'Planar Graph'** is a bona fide project of **Sreeshma Shibu** carried out the project work under my supervision.

Mrs. Riya Baby | **Mrs. Prija V**

Head of Department | **Supervisor**

Department of Mathematics

Don Bosco Arts and Science College

Angadikkadavu

DECLARATION

I **Sreeshma Shibu** hereby declare that the project **'Planar Graph'** is an original record of studies and bona fide project carried out by me during the period of 2018 – 2021 under the guidance of Mrs. Prija V, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu and has not submitted by me elsewhere for the award of my degree, diploma, title, or recognition, before.

Sreeshma Shibu

DBI8CMSR10

Department of Mathematics

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ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also, I must express my deepest gratitude to people along the way.

No words can adequately express the sense of gratitude; still, I try to express my heartfelt thanks through words. At the outset, I am deeply indebted to my project supervisor Mrs. Prija V, Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu, for the invaluable guidance, loving encouragement, and meticulous care towards me throughout my career. I express my deep sense of gratitude to all the faculty members of the Department of Mathematics, Don Bosco Arts and Science College, Angadikkadavu.

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My greatest debt is always, to God Almighty.

CONTENTS

INTRODUCTION

In recent years, Graph Theory has established itself as an important mathematical tool in a wide variety of subjects, ranging from Operational Research and Chemistry to Genetics and Linguistics, and from Electrical Engineering and Geography to Sociology and Architecture. At the same time, it has also emerged as a worthwhile mathematical discipline in its own right.

A great mathematician, Euler become the Father of Graph Theory, when in 1736, he solved a famous unsolved problem of his days, called Konigsberg Bridge Problem. This is today, called as the First Problem of the Graph theory. This problem leads to the concept of the planar graph as well as Eulerian Graphs, while planar graphs were introduced for practical reasons, they pose many remarkable mathematical properties. In 1936, the psychologist Lewin used planar graphs to represent the life space of an individual.

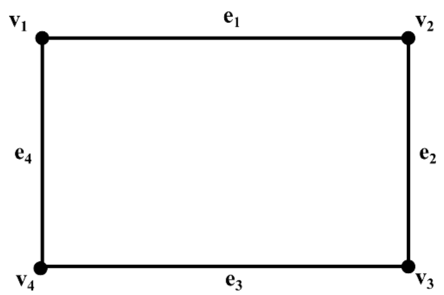
Chapter 1

BASIC CONCEPTS

Graph

A **graph** is an ordered triple $G = \{V(G), E(G), I_G\}$ where $V(G)$ is a non-empty set, $E(G)$ is a set disjoint from $V(G)$ and $I(G)$ is an incidence map that associates each element of $E(G)$ and unordered pair of elements of $V(G)$. The elements of $V(G)$ are called vertices (or nodes or points) of G and the element of $E(G)$ are called edges or lines of G.

Example:



Here $V(G) = \{v_1, v_2, v_3, v_4\}$

$E(G) = \{e_1, e_2, e_3, e_4\}$

$I_G(e_1) = \{v_1, v_2\}$ or $\{v_2, v_1\}$

$\mathrm{I}_G(e_2) = \{v_2, v_3\}$ or $\{v_3, v_2\}$

$I_G(e_3) = \{v_3, v_4\}$ or $\{v_4, v_3\}$

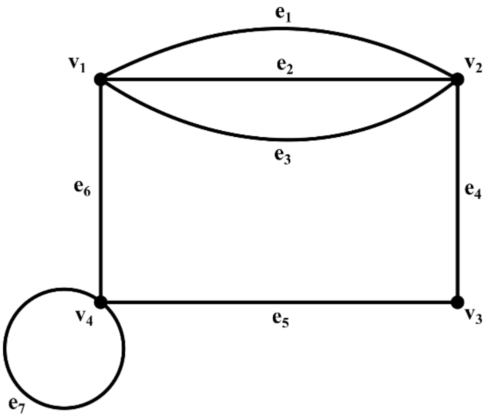
$I_G(e_4) = \{v_4, v_1\}$ or $\{v_1, v_4\}$

Multiple edges

A set of two or more edges of a graph G is called multiple edges or parallel edges if they have the same end vertices.

Loop

An edge for which the two end vertices are same is called a loop.



Here $\{e_1, e_2, e_3, e_4\}$ form the parallel edges.

e_7 is the Loop.

Simple Graph

A graph is simple if it has no loops and no multiple edges.

Finite & Infinite Graphs

A graph is called finite if both $V(G)$ & $E(G)$ are finite. A graph that is not finite is called infinite graph.

Adjacent Vertices

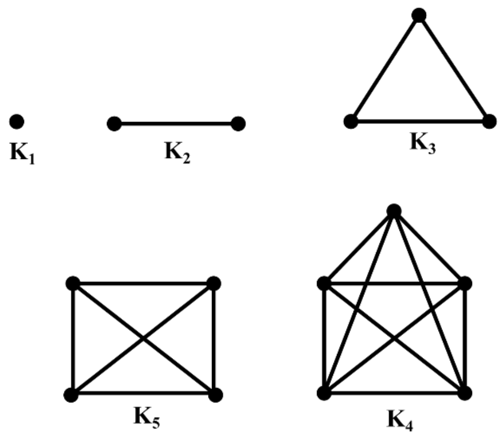
Two vertices u and v are said to be adjacent vertices if and only if there is an edge with u and v as its end vertices.

Adjacent Edges

Two distinct edges are said to be adjacent edges if and only if they have a continuous end vertex.

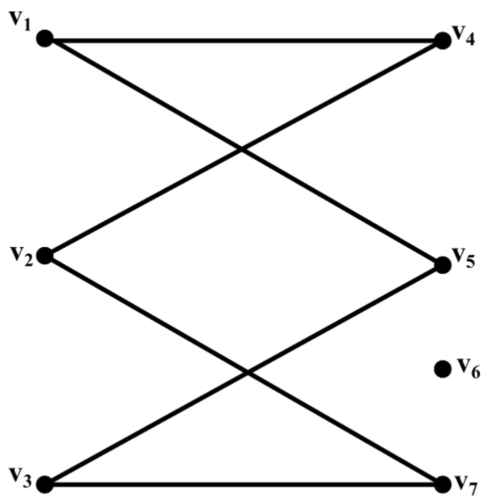
Complete Graph

A simple graph G is said to be complete if every pair of distinct vertices of G are adjacent in G. A complete graph with n vertices is denoted by K_n.



Bipartite Graph

A graph is bipartite if its vertex set can be partitioned into two non-empty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a bipartition of the bipartite graph G. The bipartite graph G with bipartition (X, Y) denoted by $G\ (X, Y)$.



Here $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$

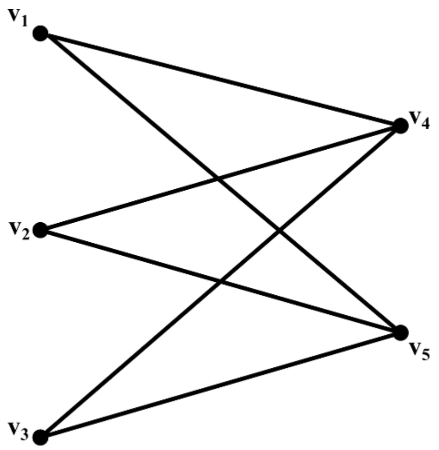
The Bipartition is

$X = \{v_1, v_2, v_3\}$

$Y = \{v_4, v_5, v_6, v_7\}$

Complete Bipartite Graph

A simple bipartite graph $G\ (X, Y)$ is complete if each vertex X is adjacent to all the vertices of Y.

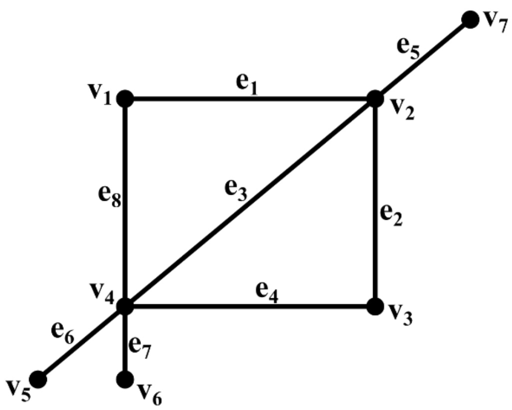


Here $X = \{v_1, v_2, v_3\}$

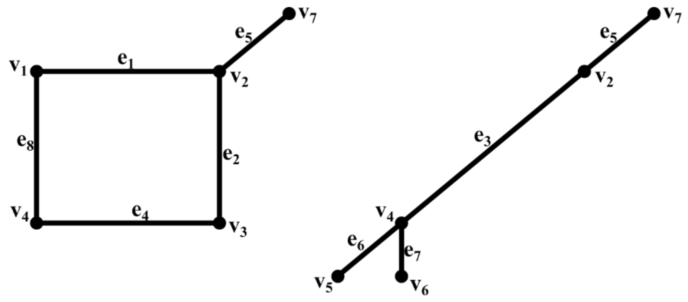
$Y = \{v_4, v_5\}$

Subgraph

A graph H is called subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and I_H is the restriction of I_G to $E(H)$ [ie, $I_H(e) = I_G(e)$ whenever $e \in E(H)$.



G



Subgraphs

Degrees of Vertices

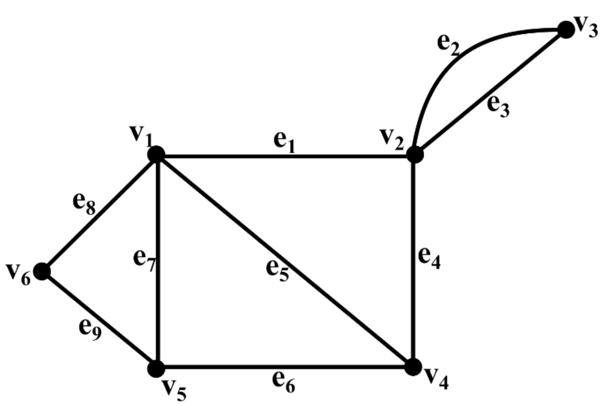
The number of edges incident with vertex V is called degree of a vertex or valency of a vertex and it is denoted by $d(v)$.

Isomorphism of Graph

A graph isomorphism from a graph G to a graph H is a pair (ϕ, θ), where $\phi : V(G) \to V(H)$ and $\theta : E(G) \to E(H)$ are bijection with a property that $I_G(e) = \{u, v\}$ and $I_H(\theta(e)) = \{\phi(u), \phi(v)\}$.

Walk

A walk in a graph G is an alternative sequence $W = v_0 v_1 e_1 v_2 e_2 \ldots v_n e_n$ vertices and edges, beginning and ending with vertices where v_0 is the origin and v_n is the terminus of W.



$W = v_6 e_8 v_1 e_1 v_2 e_2 v_3 e_3 v_2 e_1 v_1$

Closed Walk

A walk to begin and ends at the same vertex is called a closed walk. That is, the walk W is closed if $v_0 = v_n$.

Open Walk

If the origin of the walk and terminus of the walk are different vertices, then it is called an open walk.

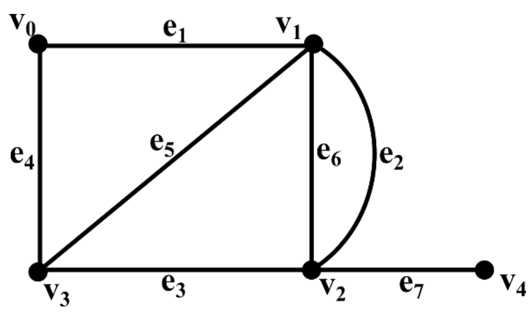
Trail

A walk is called a trail if all the edges in the walk are distinct.

Path

A walk is called a path if all the vertices are distinct.

Example:



$v_0 e_1 v_1 e_2 v_2 e_6 v_1 \rightarrow$ A trail

$v_0 e_1 v_1 e_2 v_2 e_3 v_3 \rightarrow$ A path

$v_0 e_1 v_1 e_2 v_2 e_3 v_3 e_5 v_1 \rightarrow$ A trail, but not a path

Euler's Theorem

The sum of the degrees of the vertices of a graph is equal to the twice the number of edges.

ie: $\sum_{i=1}^{n} d(v_i) = 2m$

Isomorphic Graph

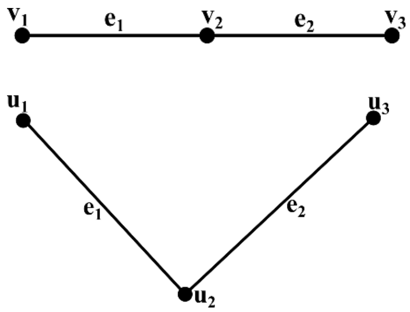
$$I_H(\theta(e)) = \{\phi(u), \phi(v)\}$$

A graph $G_1 = (V_1, E_1)$ is said to be isomorphic to graph $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and v_1 in G_1 then the corresponding edge e_2 in G_2 has its end vertices u_2 and v_2 in G_2. This correspondence is called a graph isomorphism.

Example:

$G =$

$H =$



ie: G and H are isomorphic.

Components

A connected component of a graph is a maximal connected subgraph. The term is also used for maximal subgraph or subset of a graph 's vertices that have some higher order of connectivity, including bi-connected components, tri-connected components and strongly connected components.

Tree

A connected graph without cycles is called a tree.

Vertex Cut

Let G be a connected graph. The set V' subset of $V(G)$ is called a Vertex cut of G, if $G - V'$ is a disconnected graph.

Cut Vertex

If $V' = \{v\}$ is a Vertex cut of the connected Graph G, then the vertex $\boldsymbol{v}$ is called a Cut vertex.

Edge Cut

Let G be a non-trivial connected graph with vertex set V and let S be a non-empty subset of V and $\bar{S} = V - S$. Let $E' = [S, \bar{S}]$ denote the set of all edges of G that have one end vertex is S and the other is $\bar{S}$. Then $G - E'$ is a disconnected graph and $E' = [S, \bar{S}]$ is called an edge cut of G.

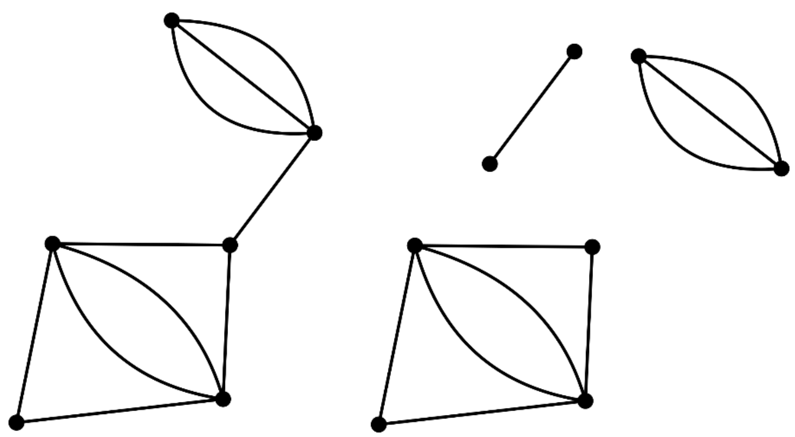
Cut Edge

If $E' = \{e\}$ is an edge cut of G then e is called a cut edge of G.

Block

A block is a Connected graph without any cut vertices.

Eg:



Graph G Blocks of G

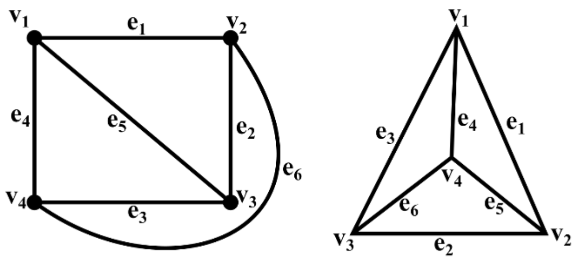
Chapter 2

PLANAR GRAPHS

Plane Graph

A plane graph is a graph drawn in the plane, such a way that any pair of edges meet only at their end vertices.

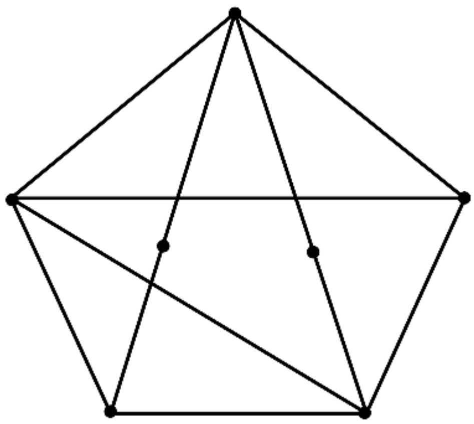
Example:



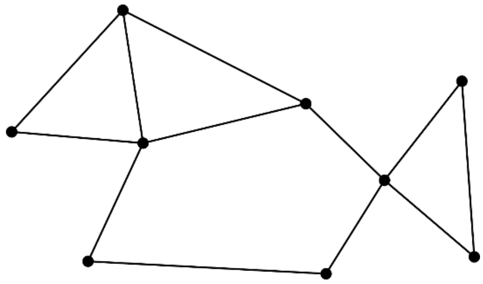
Planar Graph

A planar graph is a graph which is isomorphic to a plane graph, ie: it can be drawn as a plane graph.

A plane graph is a graph that can be drawn in the plane without any edge crossing.



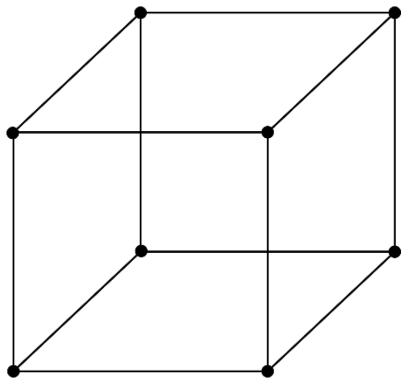
Example of Planar graph:



Planar Representation

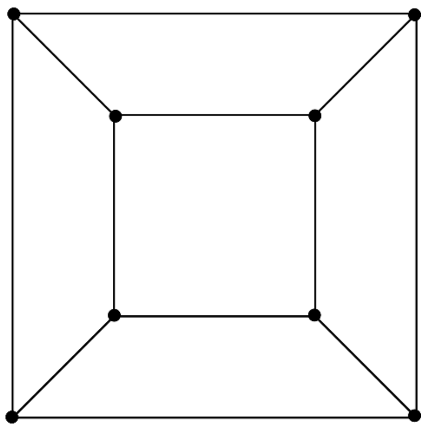
The pictorial representation of a planar graph as a plane graph is called a planar representation.

Eg: Is Q_3 shown below, planar?



The graph Q_3

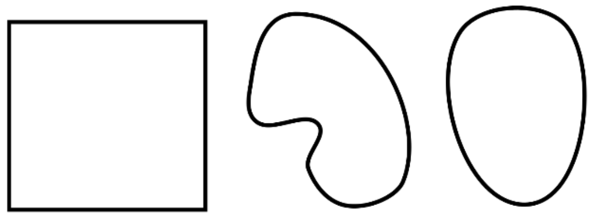
Planar representation of Q_3 is:



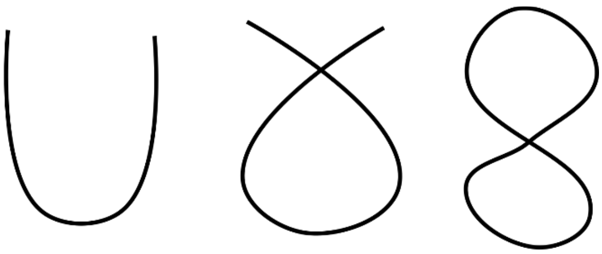
Jordan Curve

A Jordan Curve in the plane is a continuous non-self-intersecting curve where Origin and Terminals coincide.

Example:



Jordan Curves

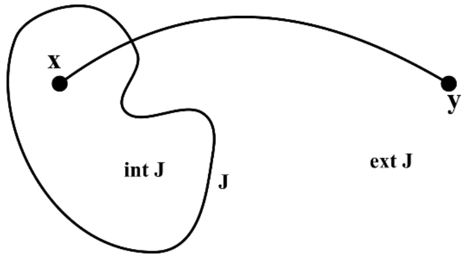


Non-Jordan Curves

Remark

If J is a Jordan Curve in the plane, then the part of the plane enclosed by J is called interior of J and is denoted by ‘int J’. We exclude from ‘int J’ the points actually lying on J. Similarly, the part of the plane lying outside J is called the exterior of J and is denoted by ‘ext J’.

Example:



Arc connecting point x in int J with point y in ext J.

Theorem

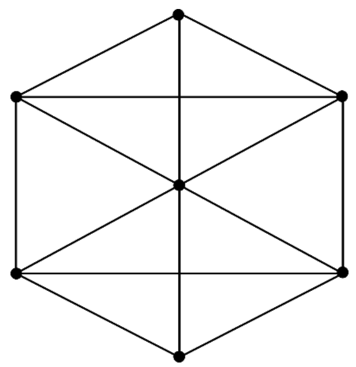
Let J be a Jordan Curve, if x is a point in int J and y is a point in ext J then any line joining x to y must meet J at some point, ie: must cross J. this is called Jordan Curve Theorem.

Boundary

The set of edges that bound a region is called its boundary.

Definition

A graph which is not planar is known as non-planar graph or a graph that cannot be drawn in the plane without any edge crossing is known as non-planar graph.



Theorem

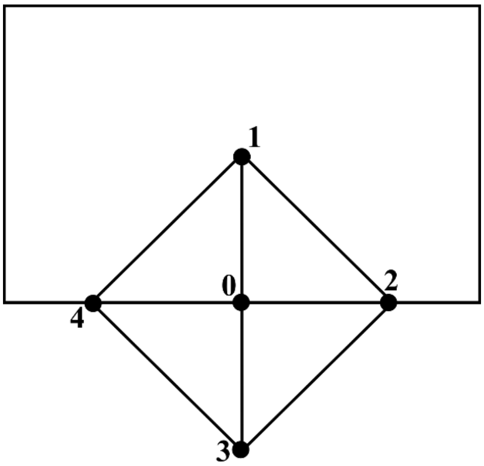
K_5 is nonplanar:

Every drawing of the complex graph K_5 in the plane (or sphere) contains at least one edge crossing.

Proof:

Label the vertices 0, 1, 2, 3, 4. By the Jordan Curve theorem any drawing of the cycle (1, 2, 3, 4, 1) separates the plane into two regions. Consider the region with

vertex 0 in its interior as the 'inside' of the circle. By the Jordan Curve theorem, the edges joining vertex 0 to each of its vertices 1, 2, 3 and 4 must also lie entirely inside the cycle, as illustrated below.



Drawing most of the K_5 in the plane

Moreover, each of the 3-cycles $\{0, 1, 2, 0\}$, $\{0, 2, 3, 0\}$, $\{0, 3, 4, 0\}$ and $\{0, 4, 1, 0\}$ also separates the plane and hence the edges $(2, 4)$ must also lie to the exterior of the cycle $\{1, 2, 3, 4\}$ as shown. It follows that the cycle formed by edges $(2, 4)$, $(4, 0)$ and $(0, 2)$ separates the vertices 1 and 3, again by Jordan Curve theorem. Thus, it is impossible to draw edge $(1, 3)$ without crossing an edge of that cycle. So, it is proven that the drawing of the K_5 in the plane contains at least one edge-crossing.

Theorem

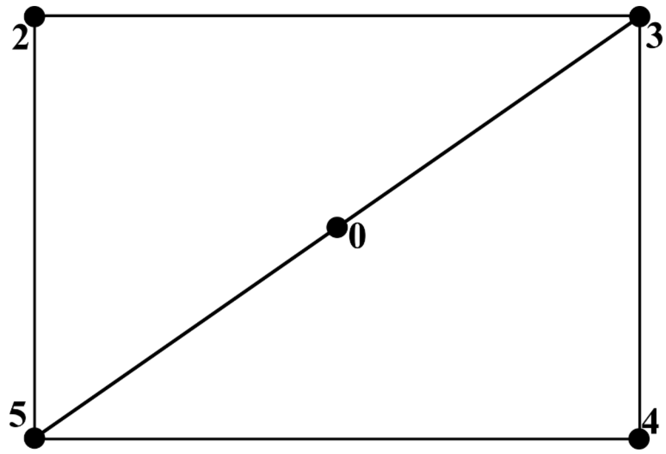
K_{33} is nonplanar:

Every drawing of the complete bipartite graph K_{33} in the plane (or sphere) contains at least one edge crossing.

Proof:

Label the vertices of one partite set 0, 2, 4 and of the order 1, 3, 5. By the Jordan Curve theorem, cycle $\{2, 3, 4, 5, 2\}$ separates the plane into two regions,

and as in the previous proof (K_5), we regard the region containing the vertex 0 as the 'inside' of the cycle. By the Jordan Curve theorem, the edges joining vertex 0 to each of the vertices 3 and 5 lie entirely inside that cycle, and each of the cycle $\{0, 3, 2, 5, 0\}$ and $\{0, 3. 4, 5, 0\}$ separates the plane, as illustrated below.



Drawing most of the K_{33} in the plane

Thus, there are 3 regions: the exterior of cycles $\{2, 3, 4, 5, 2\}$ and the inside of each of the other two cycles. It follows that no matter which region contains vertex 1, there must be some even numbered vertex that is not in that region, and hence the edge from vertex 1 to that even-numbered vertex would have to cross some cycle edge.

Corollary

Subgraph of a planar graph is planar.

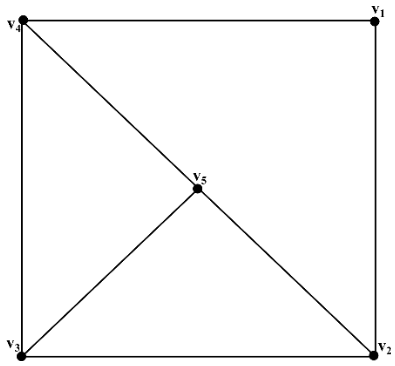
Definition

A plane graph partitions the plane into number of regions called faces.

Let G be plane graph. If x is a point on the plane which is not in G, ie: x is not a vertex of G or a point on any edge of G, then we define the faces of G containing x to be the set of all points on the plane which can be reached from x by a line which does not cross any edge of G or go through any vertex of G.

The number of faces of a plane graph G denoted by $f(a)$ or simply f.

Each plane graph has exactly one unbounded face called the exterior face.



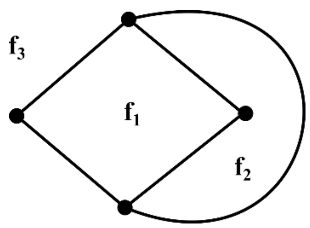
Here $f(G) = 4$

Degree of faces

The degree $d(f)$ of a face f is the number of edges with which it is incident, that is the number of edges in the boundary of a face.

Cut edge being counted twice.

Eg:



$d(f_2) = 3$

$d(f_1) = 4$

$d(f_3) = 3$

Theorem

A graph is planar if and only if each of its blocks is planar.

Proof:

If G is planar, then each of its blocks is planar since a subgraph of planar graph is planar.

Conversely, suppose that each block of G is planar. We now use induction on the number of blocks of G to prove the result. Without loss of generality, we

assume that G is connected. If G has only one block, then G itself is a block, and hence G is planar.

Now suppose G has k planar blocks and that the result has been proved for all connected graph having (k-1) planar blocks. Choose any end block B_0 of G and delete from G all the vertices of B_0 except the unique cut vertex, say v_0 of G in B_0. The resulting connected graph G` of G contains (k-1) planar blocks. Hence, by the induction hypothesis G` is planar. Let $G^{\sim}$` be plane embedded of G` such that v_0 belongs to the boundary of unbounded face, say f `. Let $B_0^{\sim}$ be a plane embedding of $B_0$ in $f$ `, so that v_0 is in the exterior face of $B_0^{\sim}$. Then $G^{\sim}$` and $B_0^{\sim}$ is a plane embedding of G.

Chapter 3

EULER'S FORMULA

Theorems

Euler Formula:

For a connected plain graph G, $n - m + f = 2$ where n, m, and f denote the number of vertices, edges and faces of G respectively.

Proof:

We apply the induction on f.

If $f = 1$ the G is a tree and $m = n - 1$.

Hence $n - m + f = 2$ and suppose that G has f faces.

Since $f \geq 2$, G is not a tree and hence contains a cycle C. Let e be an edge of C. Then e belongs to exactly 2 faces, say $f_1$and f_2 and the deletion of e from G results in the formation of a single face from $f_1$and f_2. Also, since e is not a cut edge of G. $G - e$ is connected.

Further the number of faces of $G - e$ is $f - 1$, number of edges in $G - e$ is $m - 1$ and number of vertices in $G - e$ is n. So, applying induction to $G - e$, we get $n - (m - 1) + (f - 1) = 2$ and this implies that $n - m + f = 2$. This completes the proof of theorem.

Corollary 1

All plane embedding of a planar graph have the same number of faces.

Proof:

Since $f = m - n + 2$ the number of faces depends only on n and m and not on the particular embedding.

Corollary 2

If G is a simple planar graph with at least 3 vertices, then $m \leq 3n - 6$.

Proof:

Without the generality we can assume that G is a simple connected plane graph. Since G is simple and $n \geq 3$, each face of G has degree at least 3. Hence if f denote the set of faces of G $\sum_{f \epsilon F} d(f) \geq 3f$. But $\sum_{f \epsilon F} d(f) = 2m$.

Consequently $2m \geq 3f$ so that $f \leq \frac{2m}{3}$.

By the Euler formula $m = n + f - 2$ now $f \leq \frac{2m}{3}$ implies $\mathrm{m} \leq \mathrm{n} + \left(\frac{2\mathrm{m}}{3}\right) - 2$. This gives. $m \leq 3n - 6$.

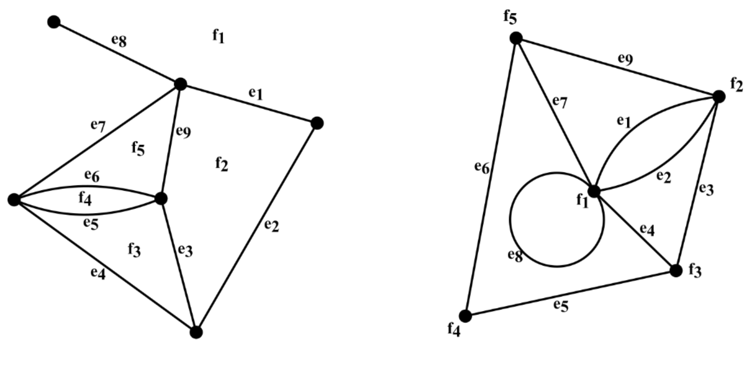
Chapter 4

DUAL OF A PLANE GRAPH

Definition

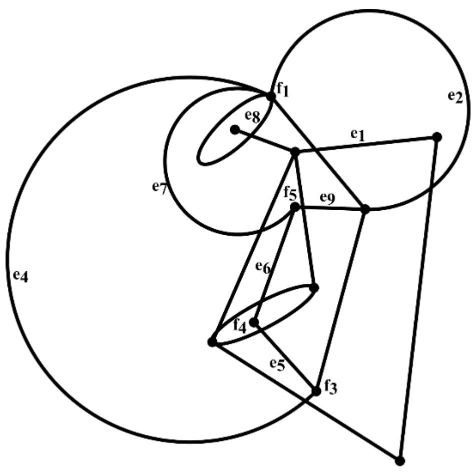
Let G be a plane graph. One can form out of G a new graph H in the following way corresponding to each face f(g), take the vertex f* and corresponding to each edge e(g), take an edge e*. Then edge e* joins vertices f* and g* in H iff edge e is common to the boundaries of faces f and g in G. The graph H is then called dual of G.

Example:



(a) (b)

Plane graph and its Dual



(c)

CONCLUSION

In this project we discussed the topic planar graph in graph theory.

We discussed about Euler formula and verified that some graphs are planar, and some are non-planar. A related important property of planar graphs, maps and triangulations is that they can be enumerated very nicely.

We also discussed about duality of a graph.in mathematical discipline of graph theory, the dual graph of a plane graph G is a graph that has a vertex of each face of G .it has many applications in mathematical and computational study.

In fact, graph theory is being used in our so many routine activities. For eg; using GPS or google maps to determine a route based on used settings.

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1) R. Balakrishnan and K. Ranganathan, A textbook of Graph Theory, University Text

2) J. A. Bondy and U. S. R. Murty, Graph Theory with Great Britain: The Macmillian Press Ltd. 1976