DON BOSCO ARTS & SCIENCE COLLEGE ANGADIKADAVU

DEPARTMENT OF MATHEMATICS

2021-2023

Project Report on

HYPERGRAPH

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Project Report on

HYPERGRAPH

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1.

2.

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BONAFIDE CERTIFICATE

Certified that this project report "HYPERGRAPH" is the bonafide work of ANJU MARIYA PAUL who carried out the project work under my supervision.

Signature of Head

Signature of Supervisor

DECLARATION

I, ANJU MARIYA PAUL hereby declare that the Project work entitled HYPERGRAPH has been prepared by me and submitted to Kannur University in partial fulfillment of requirement for the award of Master of Science is a record of original work done by me under the supervision of Mrs. NAJUMUNNISA K, Assistant Professor, Department of Mathematics, Don Bosco Arts & Science College, Angadikadavu. I, also declare that this Project work has not been submitted by me fully for the award of any Degree, Diploma, Title or recognition before any authority.

Place: Angadikadavu ANJU MARIYA PAUL

Date: (C1PSMM1905)

ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also I must express my deepest gratitude to people along the way. No words can adequately express the sense of gratitude, still I try to express my heartfelt thanks through words. The outset, I am deeply indebted to my project supervisor Mrs. NAJUMUNNISA K Assistant Professor, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu, for the invaluable guidance, loving encouragement and meticulous care towards me throughout my career. I express my deep sense of gratitude to all the faculty members of the Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu. I can never forget the support and encouragement rendered by the Principal and the Staff of Don Bosco Arts & Science College, Angadikadavu. I could not name many who sincerely supported and helped for the successful completion of this Project. It is my pleasure and duty to thank each and everyone of them who walked with me.

ANJU MARIYA PAUL

CONTENTS

INTRODUCTION

Hypergraphs are systems of finite sets and form, probably, the most general concept in discrete mathematics. This branch of mathematics has developed very rapidly during the latter part of the twentieth century, influenced by the advent of computer science. Originally, developed in France by Claude Berge in 1960, it is a generalization of graph theory in which an edge can join any number of vertices. The basic idea consists in considering sets as generalized edges and then in calling hypergraph the family of these edges (hyperedges).

Hypergraphs model more general types of relations than graphs do. In the past decades, the theory of hypergraphs has proved to be of a major interest in applications to real-world problems. These mathematical tools can be used to model networks, biology networks, data structures, process scheduling, computations and a variety of other systems where complex relationships between the objects in the system play a dominant role.

My work includes four chapters. First chapter contains basic definitions in hypergraphs like in graph theory. Second chapter provides the first properties such as the Helly property, the König property, and so on. In third chapter, the classical notions of colorings are addressed. Fourth chapter deals with some applications of hypergraph theory.

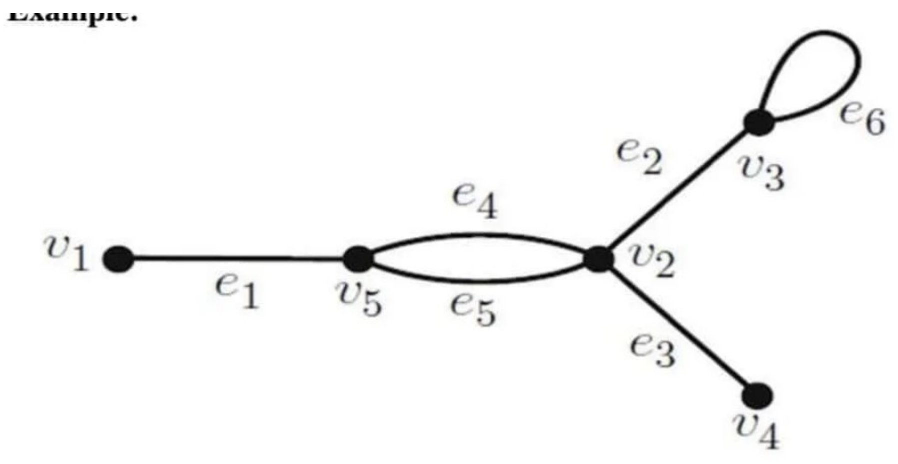
PRELIMINARIES

Definitions

Graph

A graph is an ordered triple $G = (V(G), E(G), I_G)$ where $V(G)$ is a nonempty set, $E(G)$ is a set disjoint from $V(G)$ and I_G is an "incidence" relation that associates with each element of E(G) an unordered pair of elements (same or distinct) of $V(G)$. Elements of $V(G)$ are called the vertices(or nodes or points) of G; and elements of $E(G)$ are called the edges (or lines) of $G.V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively. If, for the edge e of $G, I_G(e) = \{u, v\}$ we write $I_G(e) = uv$.

Example:



Here $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and I_G is given by

$I_G(e_1) = \{v_1, v_5\}, I_G(e_2) = \{v_2, v_3\}$,

$I_G(e_3) = \{v_2, v_4\}, I_G(e_4) = \{v_2, v_5\}$,

$I_G(e_5) = \{v_2, v_5\}, I_G(e_6) = \{v_3, v_3\}$

Vertices u and v are adjacent to each other in G if and only if there is an edge of G with u and v as its ends. Two distinct edges e and f are said to be adjacent if and only if they have a common end vertex.

The number of edges incident at v in G is called the degree of the vertex v in G denoted by $\mathrm{d}(v)$.

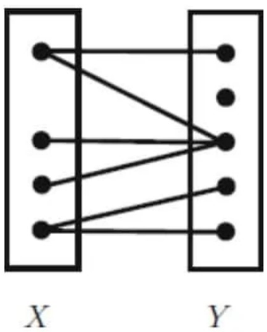
A walk in a graph G is an alternating sequence $W : v_0 e_1 v_l e_2 v_2 \ldots \mathrm{e_p v_p}$ of vertices and edges beginning and ending with vertices in which $v_{\mathrm{i}-1}$ and v_{i} are the ends of $e_{i.}$. A walk is called a trail if all the edges appearing in the walk are distinct. It is called a path if all the vertices are distinct. A cycle is a closed trail in which the vertices are all distinct. The length of a walk is the number of edges in it.

A Graph G is said to be connected if every pair of vertices u, v are connected by a path. Otherwise G is said to be disconnected. The components of G are the maximal connected subgraphs of G

Bipartite graph

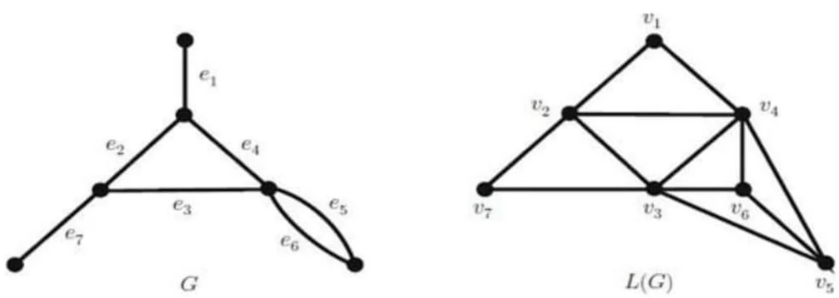
A graph is bipartite if its vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a bipartition of the bipartite graph.

Example:



Let G be a loopless graph. We construct a graph LG) in the following way: The vertex set of L(G) is in 1-1 correspondence with the edge set ofG and twO vertices of L(G) are joined by an edge if and only if the corresponding edges of G are adjacent in G: The graph L(G) (which is always a simple graph) is called the line graph or the edge graph of G.

Example:

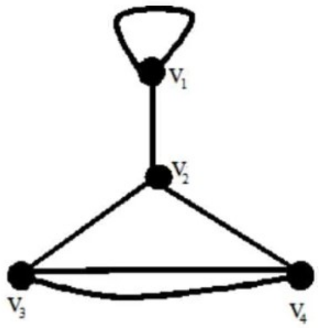


MATRIX REPRESENTATION OF THE GRAPH

Adjacency Matrix

Let G be a graph with " n " vertices listed as $v_1, v_2, \ldots v_n$. The adjacency matrix of G is the n $\times$ n matrix $A(G) = [a_{ij}]$ where a_{ij} is the number of edges joining the vertex v_i to the vertex v_j

Example:



$$A(G) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Incidence Matrix

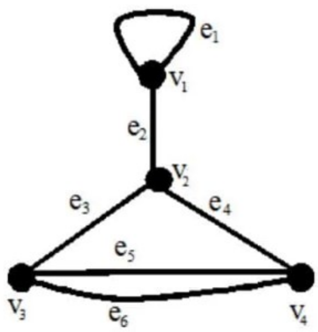
Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges $e_1, e_2, \ldots, e_m$ then the incidence matrix of G is the n $\times$ m matrix $M(G) = [m_{ij}]$ where m_{ij} is the number of times the vertex v_i is incident with the edge e_j. i.e,

0 if v_i is not an end vertex of e_j

$m_{ij} = \{1$ if v_i is an end vertex of non loop edge e_j

2 if e_j is a loop with v_i as end vertex

Example:



$$M(G) = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

CHAPTER 1

HYPERGRAPHS: BASIC CONCEPTS

Hypergraph

A hypergraph H is a pair $H = \big(V; E = (e_i)_{i \in I}\big)$ where V is a set of elements called vertices and E is a set of nonempty subsets of V called hyperedges. Therefore E is a subset of $P(V) \backslash \emptyset$ where $P(V)$ is the power set of V.

Example:

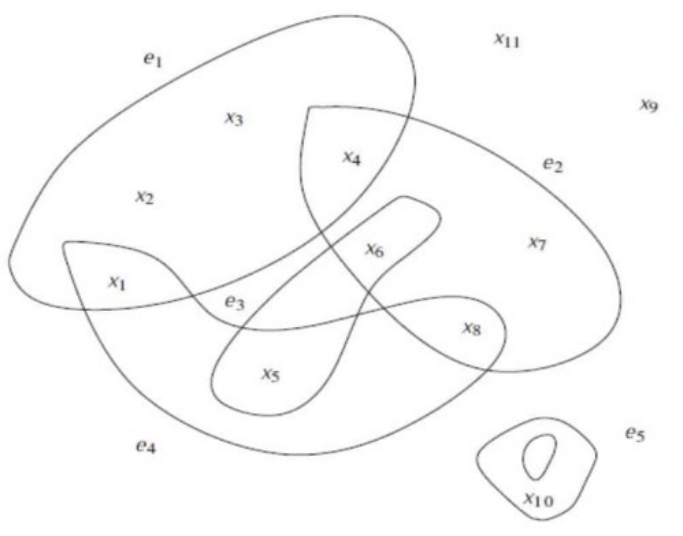


Fig. 1.1 Hypergraph

In this hypergraph H, $V = \{x_1, x_2 \ldots x_{11}\}$ and $E = \{e_{1,} e_2, e_3, e_4, e_5\} = \{\{x_1, x_2, x_3 ,\ x_4\}, \{x_4, x_6, x_7, x_8\}, \{x_5, x_6\}, \{x_1, x_5, x_8\}, \{x_{10}\}\}$

Order of a hypergraph

The order of the hypergraph $H = (V;E)$ is the cardinality of V, i.e.

If $V = \{x_1, x_2 \ldots x_\mathrm{n}\}$ then $/V\ |=$ n; In fig 1.1 , | V| = 11.

Size of a hypergraph

The size of a hypergraph $H = (V;E)$ is the cardinality of E. i.e.

if $E = \{e_{1,} e_2, e_3, e_{4,\ldots,} e_\mathrm{m}\}$ then $|E| = m$. In fig 1.1, $|E| = 5$.

Empty hypergraph

The empty hypergraph is the hypergraph such that:

- $V = \emptyset$
- $E = \emptyset$

Trivial hypergraph

A trivial hypergraph is a hypergraph such that:

- $V \neq \emptyset$
- $E = \emptyset$

Isolated vertex

If $\bigcup_{\mathrm{i}\in I} e_{\mathrm{i}} = V$ the hypergraph is without isolated vertex, where a vertex x is isolated if $x \in \mathrm{V} \backslash \bigcup_{\mathrm{i}\in I} e_{\mathrm{i}}$. In fig 1.1$\mathrm{x}_{11}, \mathrm{x}_9$ are isolated vertices.

Loop

A hyperedge $e \in E$ such that $|e| = 1$ is a loop.

Adjacent vertices

Two vertices in a hypergraph are adjacent if there is a hyperedge which contains both vertices. In particular, if $\{x\}$ is a hyperedge then x is adjacent to itself.

Incident hyperedges

Two hyperedges in a hypergraph are incident if their intersection is nonempty.

Star

The star $H(x)$ centered in x is the family of hyperedges $(e_j)_{j \in J}$ containing x.

Degree of a hypergraph

The number of hyperedges containing the vertex x is called degree of x denoted by $d(x)$ and for a loop $\{x\}$ the degree $d(x) = 2$

If the family of hyperedges is a set, i.e. if $i \neq j \Leftrightarrow \mathrm{e}_i \neq e_j$, we say that H is *without repeated hyperedge.*

If the hypergraph is without repeated hyperedge the degree is denoted by $d(x) = |H(x)|$, excepted for a loop $\{x\}$ where the degree $d(x) = 2$.

The maximal degree of a hypergraph is denoted by $\Delta(H)$.

Regular hypergraph

If each vertex has the same degree, we say that the hypergraph is regular, or k-regular if for every

$$x \in V, d(x) = k$$

Rank $r(H)$

The rank $r(H)$ of H is the maximum cardinality of a hyperedge in the hypergrah:

$$r(H) = \max_{\mathrm{i} \in l} |e_{\mathrm{i}}|$$

Co-rank $\mathrm{cr}(H)$

The minimum cardinality of a hyperedge is the co-rank $\mathrm{cr}(H)$ of H : $\mathrm{cr}(H) = \min_{\mathrm{i} \in I} |e_{\mathrm{i}}|$

Uniform hypergraph

If $r(H) = cr(H) = k$ the hypergraph is k-uniform or uniform

Simple hypergraph

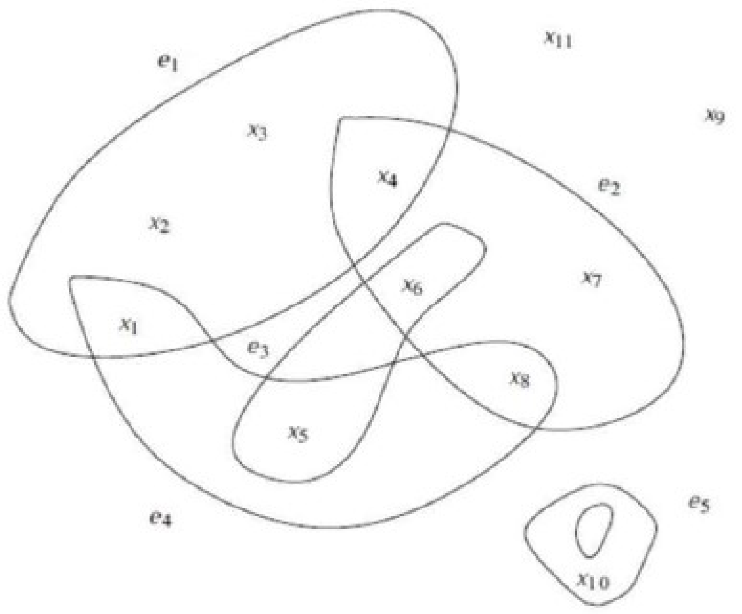
A simple hypergraph is a hypergraph $H = (V;E)$ such that: $e_{\mathrm{i}} \subseteq e_{\mathrm{j}} \Leftrightarrow i = j$.

A simple hypergraph has no repeated hyperedge.

Linear hypergraph

A hypergraph is linear if it is simple and $\left|e_i \cap e_j\right| \leq 1$ for all i,j $\in$ I where $i \neq j$.

Example:



The hypergraph H above has 11 vertices; 5 hyperedges; 1 loop: e_5; 2 isolated vertices: x_{11}, x_9. The rank $r(H) = 4$, the co-rank $cr(H) = 1$. The degree of x_1 is 2. Here x_1 and x_2 are adjacent vertices and e_1 and e_4 are incident hyperedges. H is simple and linear.

Examples of hypergraph

Example 1:

Let M be a computer science meeting with $\mathrm{k} \geq 1$ sessions; $\mathrm{S}_1,\ \mathrm{S}_2,\ \mathrm{S}_3, \ldots \mathrm{S}_\mathrm{k}$. Let V be the set of people at this meeting. Assume that each session is attended by one person at least. We can build a hypergraph in the following way:

- The set of vertices is the set of people who attend the meeting;
- The family of hyperedges $(e_i)_{i \in \{1,2,\ldots k\}}$ is built in the following way: $e_i, i \in \{1, 2, \ldots \mathrm{k}\}$ is the subset of people who attend the meeting S_i.

Example 2:

Fano Plane

The Fano plane is the finite projective plane of order 2, which have the smallest possible number of points and lines, 7 points with 3 points on every line and 3 lines through every point. To a Fano plane we can associate a hypergraph called *Fano hypergraph*:

- The set of vertices is $V = \{0, 1, 2, 3, 4, 5, 6\}$;

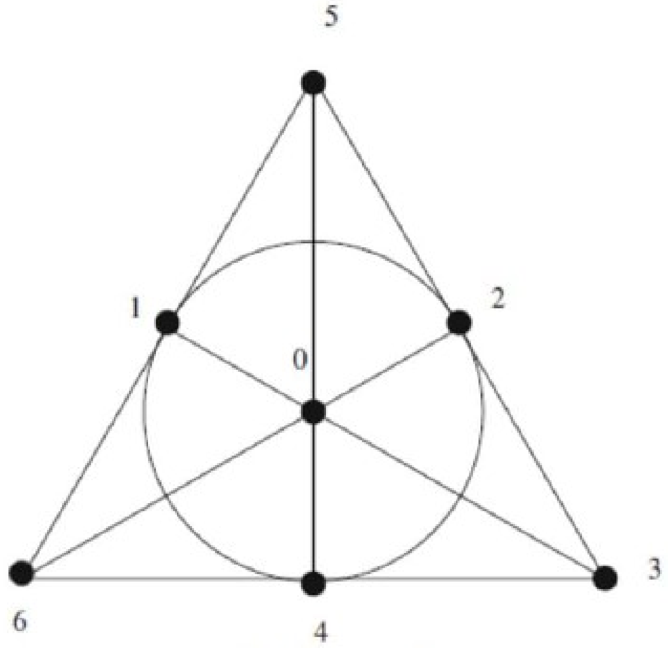


Fig. 1.2 Fanno Hypergraph

- The set of hyperedges is $E = \{013, 045, 026, 124, 346, 235, 156\}$
- The rank is equal to the co-rank which is equal to 3, hence, the Fano hypergraph is 3-uniform.

Path in a hypergraph

Let $H = (V;E)$ be a hypergraph without isolated vertex; a path P in H from x to y, is a vertex- hyperedge alternative sequence:

$x = x_1, e_1, x_2, e_2, \ldots, x_s, e_s, x_{s+1} = y$ such that

- $x_1, x_2, \ldots, x_s, x_{s+1}$ are distinct vertices with the possibility that $x_1 = x_{s+1}$;
- $e_1, e_2, \ldots, e_s$ are distinct hyperedges;
- $x_{\mathrm{i}}, x_{\mathrm{i+1}} \in e_{\mathrm{i},(\mathrm{i}} = 1, 2, \ldots, \mathrm{s})$.

– If $\mathrm{x} = x_1 = x_{s+1} = \mathrm{y}$ the path is called a *cycle*.

– The integer s is the length of path P

– if there is a path from x to y there is also a path from y to x. In this case we say that P *connects* x and y.

Connected hypergraph

A hypergraph is connected if for any pair of vertices, there is a path which connects each pair of vertices; it not connected otherwise. In this case we may also say that it is *disconnected*.

Distance

The distance $d(x,y)$ between two vertices x and y is the minimum length of a path which connects x and y. If there is a pair of vertices x,y with no path from x to y (or from y to x), we define $d(x,y) = \infty(H$ is not connected).

Connected component

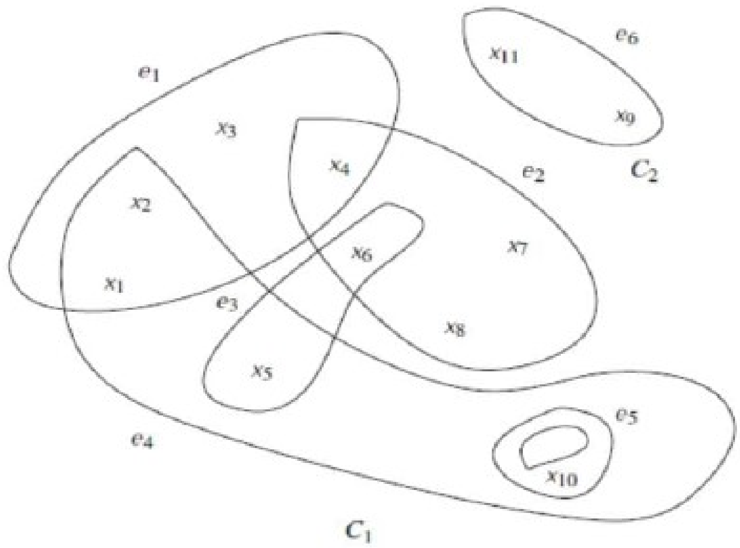
Let $H = (V,E)$ be a hypergraph, a connected component is a maximal set of vertices $X \subseteq V$ such that, for all $x,y \in X, d(x,y) \neq \infty$

Diameter

The diameter $d(H)$ of H is defined by $d(H) = \max\{d(x,y)/x,y \in V\}$.

Example:

The hypergraph above has 2 connected components, $C_1, C_2. P = x_{10}e_4x_5e_3x_6e_2x_4e_1x_3$ is a path from x_{10} to $x_3, P' = x_{10}e_4x_1e_1x_3$ is also a path from x_{10} to x_3 and the distance $d(x_{10}, x_3) = 2$ is the length of P'. Notice that the distance $d(x_{10}, x_3)$ is also the length of the path $P'' = x_{10}e_4x_2e_1x_3$.



Complete hypergraph

A hypergraph H is complete if $H = (V; E = P(V) \setminus \{\emptyset\})$.

For $n = |V|$, a complete k-uniform hypergraph on $n \geq k \geq 2$ vertices is a hypergraph which has all k-subsets of V as hyperedges, i. e. $E = P_k(V)$, where $P_k(V)$ is the set of all k-subset of V; it is denoted by K_n^k.

Matrix representation of hypergraphs

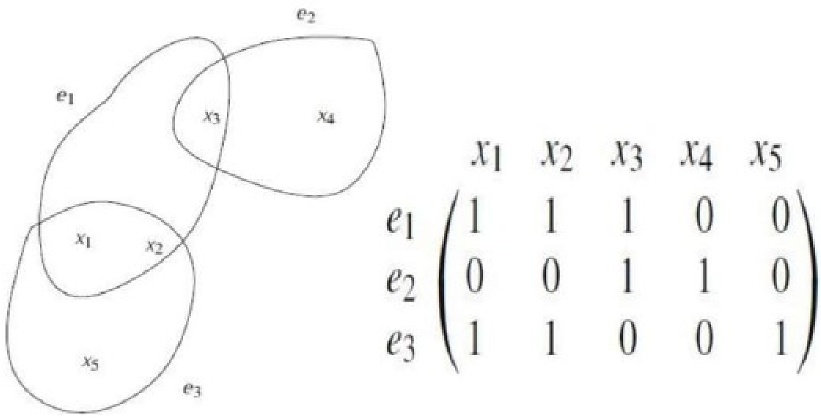
Incidence matrix

Let $H = (V; E)$ be a hypergraph, without isolated vertex, $V = \{v_1, v_2, \ldots, v_n\}$ and $E = (e_1, e_2, \ldots, e_m)$. Then H has an $\mathrm{n} \times \mathrm{m}$ incidence matrix $A = \left(a_{\mathrm{i,j}}\right)$

where

$$(a_{i,j}) = \begin{cases} 0 \text{ if } v_j \in e_j \\ 1 \text{ otherwise} \end{cases}$$

Example:

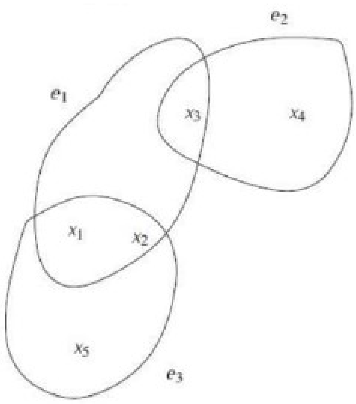


$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Adjacency matrix

It is a square matrix which rows and columns are indexed by the vertices of H and for all $x, y \in V, x \neq y$ the entry $a_{x,y} = |\{e \in E : x, y \in e\}|$ and $a_{x,x} = 0$.

Example:



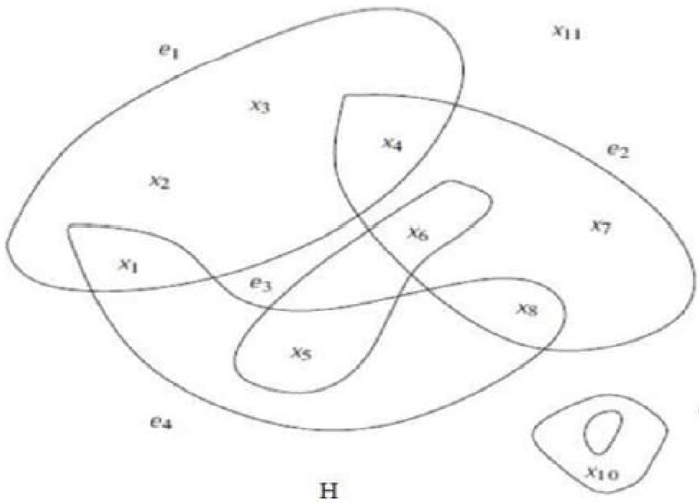
$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & 0 & 2 & 1 & 0 & 1 \\ x_2 & 2 & 0 & 1 & 0 & 1 \\ x_3 & 1 & 1 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 1 & 0 & 0 \\ x_5 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Induced Subhypergraph

Let $H = \left(V; E = (e_i)_{i \in I}\right)$ be a hypergraph. The induced subhypergraph $H(V')$ of the hypergraph H where $V' \subseteq V$ is the hypergraph $H(V') = (V', E')$ defined as $E' = \{V(e_i) \cap V' \neq \emptyset; e_i \in E$ and either e_i is a loop or $|V(e_i) \cap V'| \geq 2\}$.

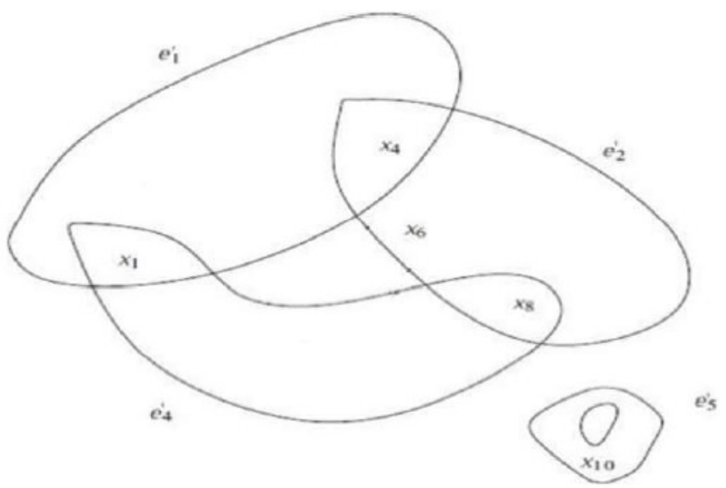
Example:



Consider the above hypergraph H.

$$H\left(V'\right) = \left(V' = \{x_1, x_4, x_6, x_8, x_{10}\};\quad e'_1 = e_1 \cap V' = \{x_1, x_4\};\right.$$
$$e'_2 = e_2 \cap V' = \{x_4, x_6, x_8\};\quad e'_4 = e_4 \cap V' = \{x_1, x_8\};$$
$$\left. e'_5 = e_5 \cap V' = \{x_{10}\}\right) \text{ is an induced subhypergraph.}$$

Notice that $e'_3 = e_3 \cap V' = \{x_6\}$ is not an hyperedge for this induced hypergraph.

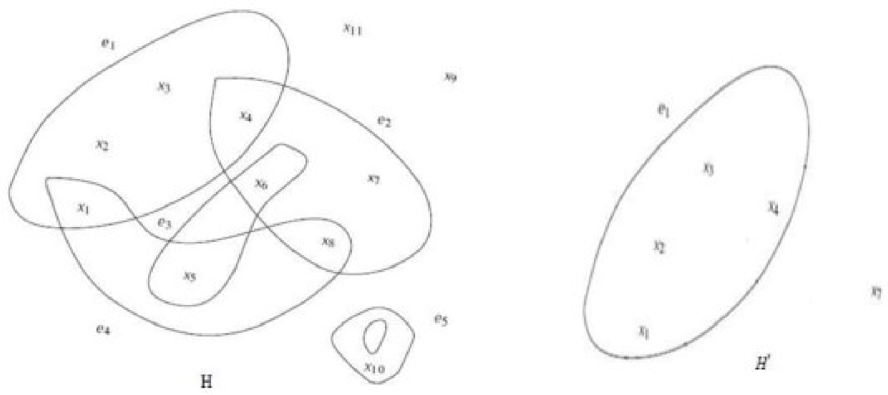


Subhypergraph

Let $H = \left(V; E = (e_j)_{j \in J}\right)$ be a hypergraph. Given a subset $V' \subseteq V$, the subhypergraph H' is the hypergraph $H' = \left(V', E' = (e_j)_{j \in J}\right)$ such that for all $e_j \in E' : e_j \subseteq V'$.

Example:

For the hypergraph $H, H' = (V' = \{x_1, x_2, x_3, x_4, x_7\}, E = \{e_1\})$ is a subhypergraph with 1isolated vertex: x_7.

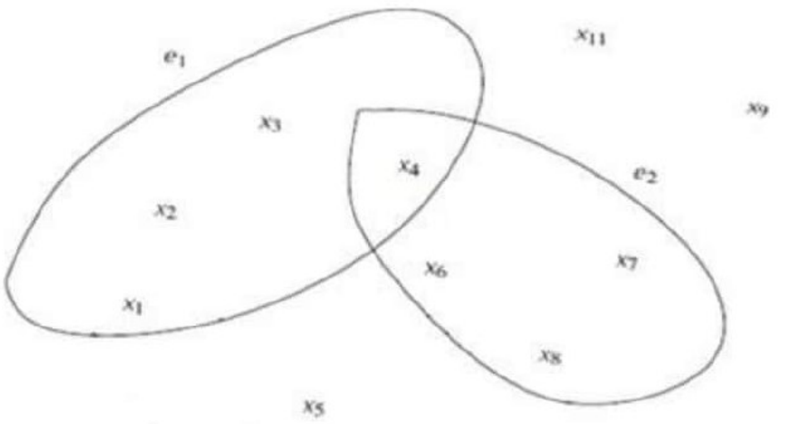


Partial hypergraph

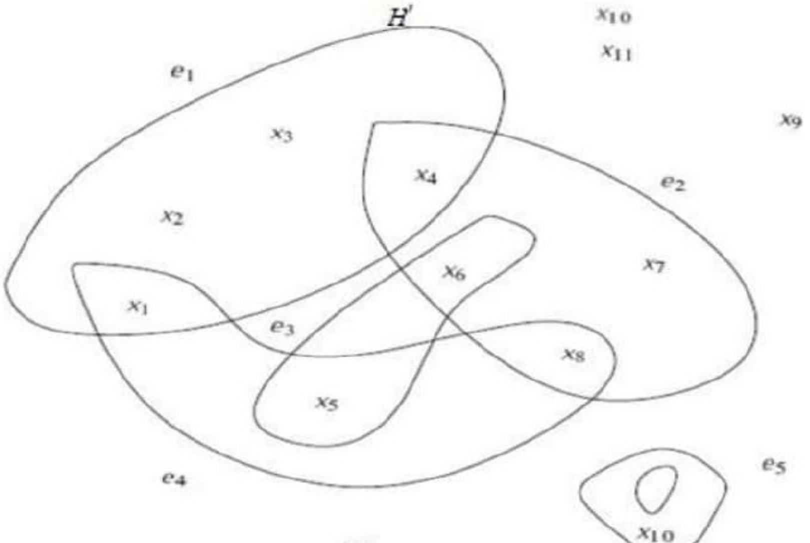
Let $H = \left(V; E = (e_j)_{j \in J}\right)$ be a hypergraph. A partial hypergraph generated by $J \subseteq I$, H' of H is a hypergraph $H' = \left(V', (e_j)_{j \in J}\right)$, where $U_{j \in J} e_j \subseteq V'$. Notice that we may have $V' = V$.

Example

For the hypergraph $H, H' = (V; \{e_l, e_2\})$ is a partial hypergraph generated by $J = \{1, 2\}$



H'



H

CHAPTER 2

HYPERGRAPHS: FIRST PROPERTIES

Line graph

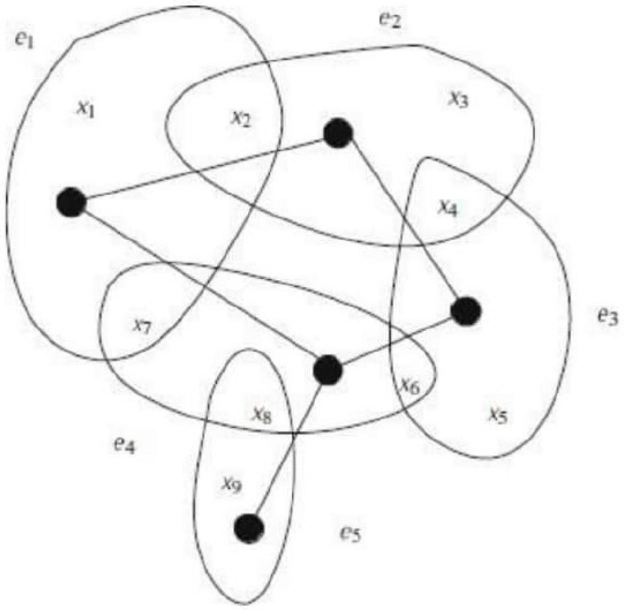
Let $H = \big(V; E = (e_i)_{i \in I}\big)$ be a hypergraph such that $E \neq \emptyset$. The line-graph (or representative graph) of H is the graph $L(H) = (V'; E')$ such that: 1. $V' := I$ or $V' := E$ when H is without repeated hyperedge; 2. $\{i, j\} \in E'(i \neq j)$ if and only if $e_{\mathrm{i}} \cap e_{\mathrm{j}} \neq \emptyset$.

Example

Figure above shows a hypergraph $H = (V; E)$, where $V = \{x_1, x_2, x_3, \ldots, x_9, \}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$, and its representative. The vertices of $L(H)$ are the *black dots* and its edges are the *curves* between these dots.

Lemma

The hypergraph H is connected if and only if $L(H)$ is connected.



Proposition

Any nontrivial graph Γ *is the line-graph of a linear hypergraph*

Proof

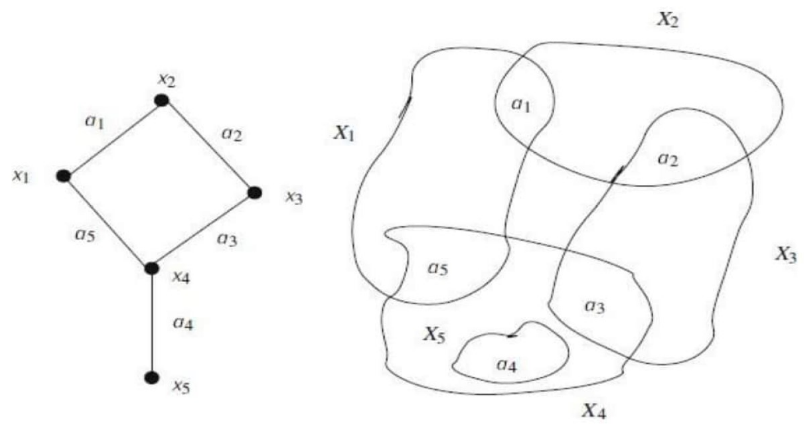
Let $\Gamma = (V, E)$ be a graph with $V = \{x_I, x_2, x_3, \ldots x_n\}$. Without losing of generality we suppose Γ is connected. (Otherwise, we treat connected components one by one). We can construct a hypergraph $H = (W; X)$ in the following way:

- The set of vertices is the set edges of Γ i.e. $W = E$. It is possible since Γ is simple;

- The collection of hyperedges X is the family of X_i where Xi is the set of edges of Γ having x_i as incident vertex.

So we can write; $H = (E;(X_1,X_2,\ldots\ldots..X_n)$ with $X_i = \{e \in E; x_i \in e \mid$ where $i \in \{1,2,3,\ldots \text{n}\}$.

Notice that if Γ has only one edge then $V = \{x_1,x_2\}$ and $X_1 = X_2$.It is the only case where H has a repeated hyperedge.
If $|E| > 1$, if i $\neq$ j and $X_i \cap X_j \neq \phi$; there is exactly one, (since Γ is a simple graph) $e \in E$ suchthat $e \in X_i \cap X_j$ with $e = (x_i,x_j)$. It is clear that Γ is a linear graph of H.



Dual of a hypergraph

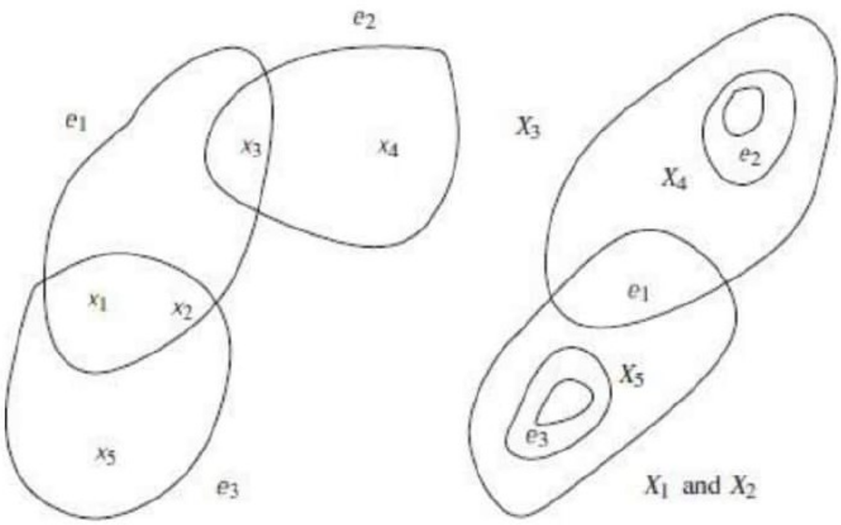
Let $H = (V;E)$ be a hypergraph which is without isolated vertex. A dual $H^* = (V^*;\ E^*)$ of H isa hypergraph such that:

- the set of vertices, $V^* = \{x_1*, x_2^*, \ldots, x_m^*\}$ is in bijection f with the set of hyperedges E;

- the set of hyperedges is given by: $e_1* = X_1, e_2* = X_2, \ldots, e_n* = X_n$, where $e_j^* = X_j = \left\{f(e_i) = x_i^* : x_j \in e_j\right\}$. Without loss of generality, we identify V^* with E. Hence $e_j^* = X_j = \left\{e_i : x_j \in e_j\right\}$, for $j \in \{1, 2, \ldots n\}$.

- So there is a bijection g from the hyperedges E of H to the vertices V^* of H^*.

Example



Left side of the above figure represents a hypergraph $H = (V;E)$ with $V = \{x_1, x_2, x_3, x_4, x_5\}$ and set of hyperedges $E = \{e_1, e_2, e_3\}$. Right side represents the dual $H^* = (V^*;E^*)$ with $V^* = \{e_1, e_2, e_3\}$ and $E^* = (X_i)_{i\in\{1,2,3,4,5\}}$.

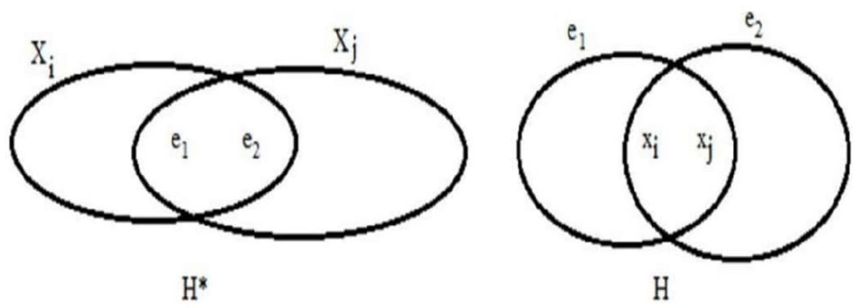
We notice that H is without repeated hyperedge but its dual has one.

Proposition

The dual H^* of a linear hypergraph without isolated vertex is also linear.

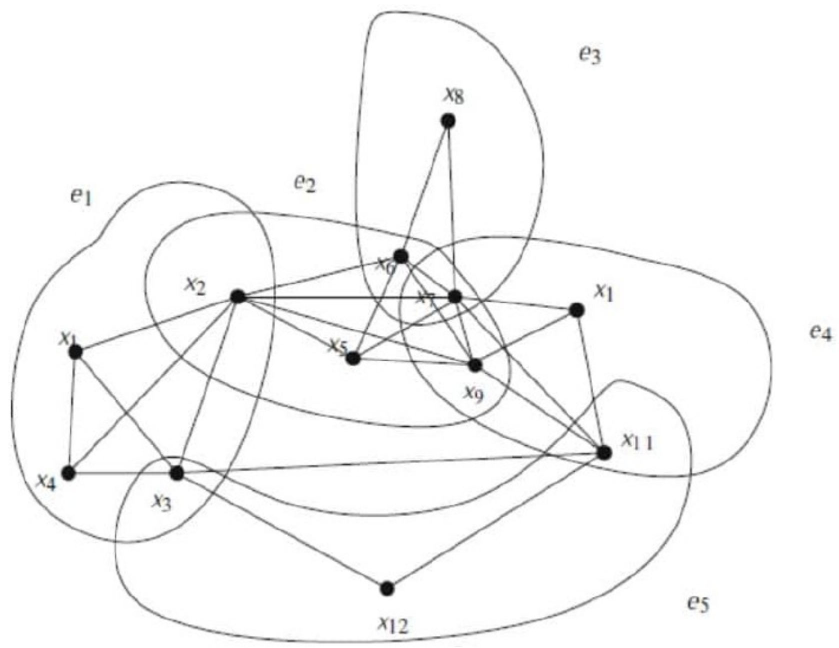
Proof

Let H be a linear hypergraph. Assume that H^* is not linear. There is two distinct hyperedges X_i and X_j of H^* which intersect with at least two vertices e_1 and e_2. The definition of duality implies that x_i and x_j belong to e_1 and e_2 (the hyperedges of H standing for the vertices e_1, e_2 of H^* respectively) so H is *not* linear. Contradiction.



2-Section of a hypergraph

Let $H = (V;E)$ be a hypergraph, the 2-section of H is the graph, denoted by [H$_2$, which vertices are the vertices of H and where two distinct vertices form an edge if and only if they are in the same hyperedge of H. Example :

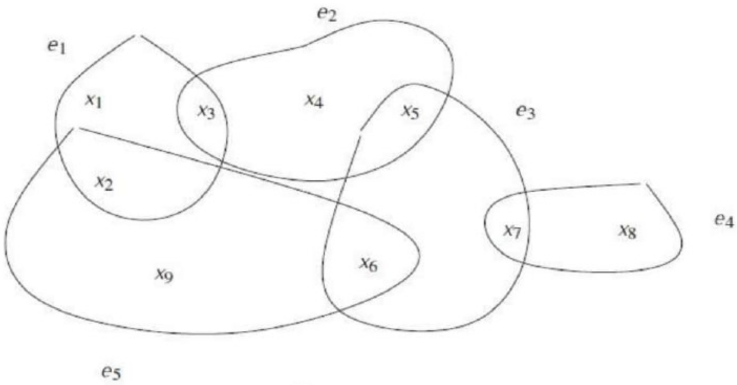


Degree of a hyperedge

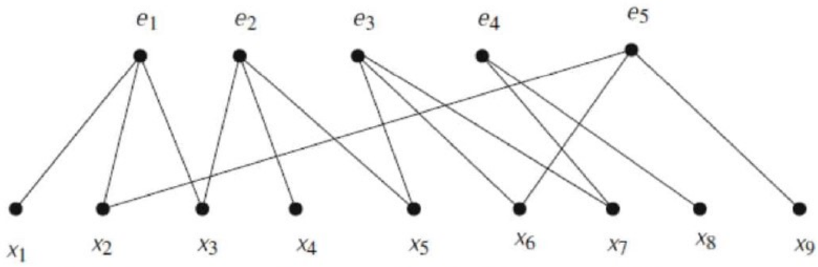
Let $H = (V;E)$ be a hypergraph, the degree of a hyperedge, $e \in E$ is its cardinality, that is $d(e) = |e|$.

Incidence graph

The incidence graph of a hypergraph $H = (V;E)$ is a bipartite graph $IG(H)$ with a vertex set $S = VUE$, and where $x \in V$ and $e \in E$ are adjacent if and only if $x \in e$.



H



The incidence graph associated with the hypergraph H

Proposition

Let $H = (V;E)$ be a hypergraph, we have: $\sum_{x \in V} d(x) = \sum_{e \in E} d(e)$.

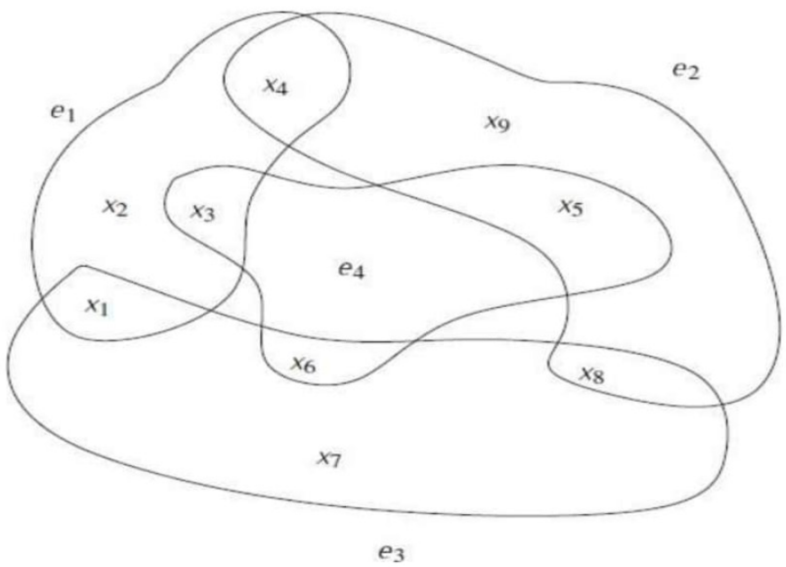
Proof :

Let $IG(H)$ be the incidence graph of H. We sum the degrees in the part E and in the part V in $IG(H)$. Since the sum of the degrees in these two parts are equal we obtain the result.

Intersecting Families

Let $H = \left(V; E = (e_i)_{i \in I}\right)$ be a hypergraph. A subfamily of hyperedges $(e_j)_{j \in J}$, where J $\subseteq$ I is an intersecting family if every pair of hyperedges has a nonempty intersection.

Example:



The maximum cardinality of $|J|$ (of an intersecting family of H) is denoted by $\Delta_0(H)$. An intersecting family with 3 hyperedges e_1, e_2, e_3 and $e_1 \cap e_2 \cap e_3 = \emptyset$ is called a *triangle*.

Helly Property and Strong Helly Property

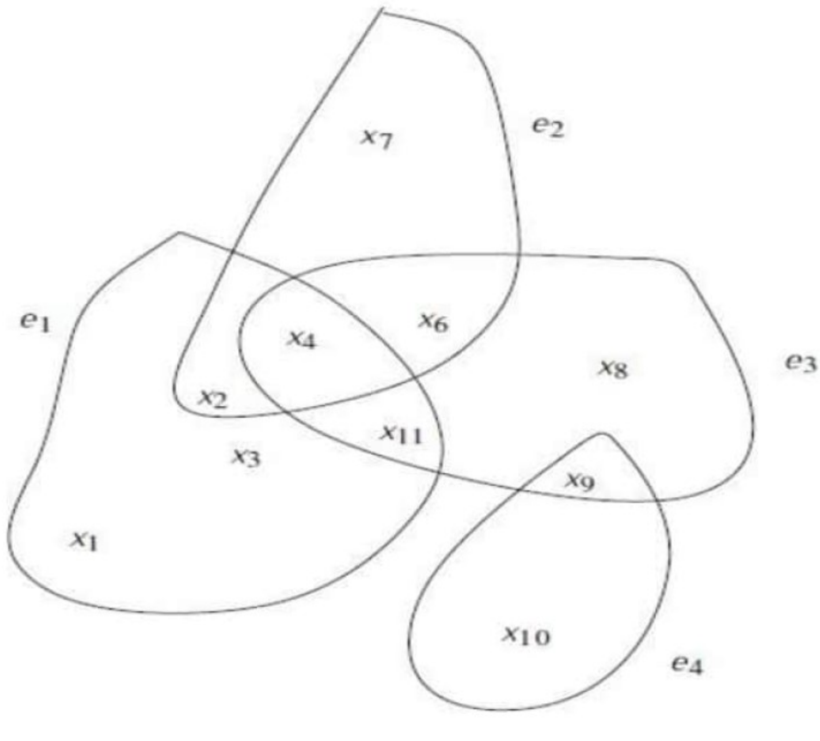
A hypergraph has the Helly property if each intersecting family has a non-empty intersection. Itis obvious that if a hypergraph contains a triangle it has not the Helly property. A hypergraph having the Helly property will be called Helly hypergraph.

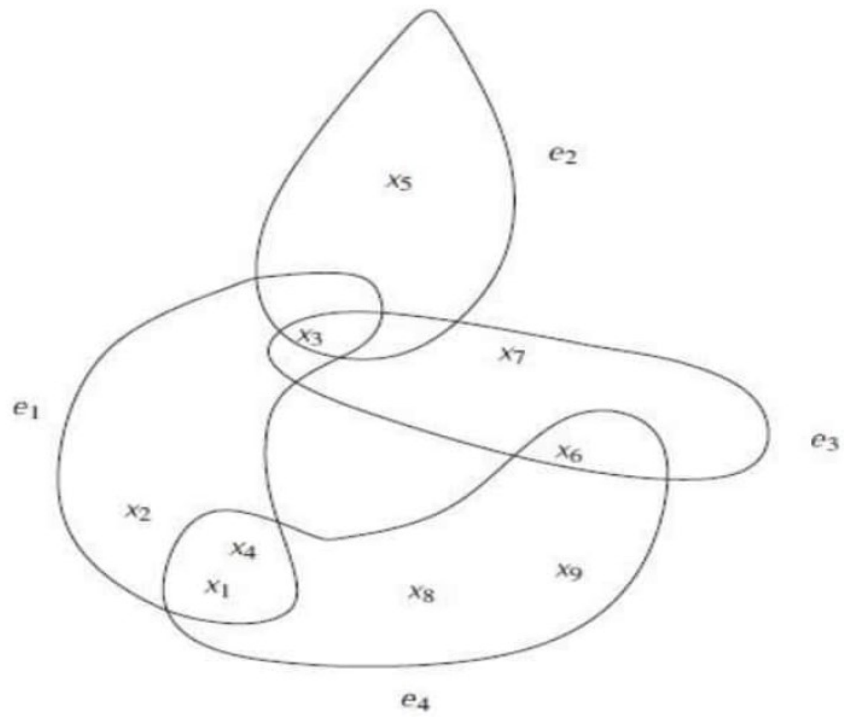
A hypergraph has the strong Helly property if each partial induced subhypergraph has the Helly property.

Example :

The hypergraph above has the Helly property but not the strong Helly property because the induced subhypergraph on $Y = V \setminus \{x_4\}$ contains the triangle:

$$e'_1 = e_1 \cap Y, e'_2 = e_2 \cap Y, e'_3 = e_3 \cap Y$$


x_7
e_2
e_1
x_4
x_6
e_3
x_8
x_2
x_{11}
x_3
x_9
x_1
x_{10}
e_4



This hypergraph has no helly property since, intersecting family e_1, e_3, e_4 has an empty intersection, that is, $e_1 \cap e_3 \cap e_4 = \emptyset$.

CHAPTER 3

HYPERGRAPH COLORING

Coloring

Let $H = \big(V; E = (e_{\mathrm{i}})_{i \in I}\big)$ be a hypergraph and $k \geq 2$ be an integer. A k-coloring of the vertices of H is an allocation of colors to the vertices such that:

(i) A vertex has just one color.

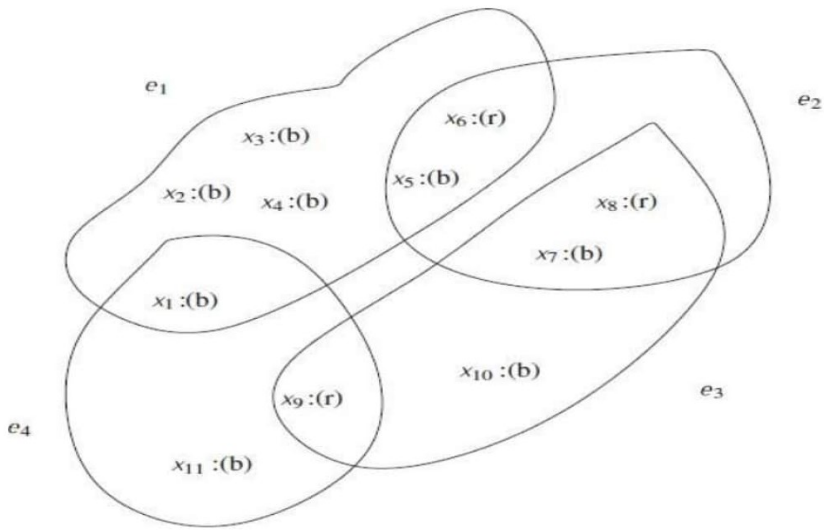
(ii) We use k colors to color the vertices.

(iii) No hyperedge with a cardinality more than 1 is monochromatic.

From this definition it is easy to see that any coloring induces a partition of the set of vertices in k - classes: $(C_l, C_2, C_3, \ldots, C_k)$ such that for $e \in E(H), |e| > 1$ then e not a subset of C_i for all $i \in \{1, 2, \ldots \mathrm{k}\}$.

Example

The figure below shows a colored hypergraph H where (r) is *red* and (b) is *blue*.



Chromatic number $\chi(H)$

The chromatic number $\chi(H)$ of H is the smallest k such that H has a k-coloring. Here we have $\chi(H) = 2$

Example:

If H is the hypergraph which vertices are the different waste products of a chemical production factors, and which hyperedges are the dangerous combination of these waste products. The chromatic number of H is the smallest number of waste disposal sites that the factory needs in order to avoid any dangerous situation.

Proposition

For any hypergraph $H = (V;E)$ with an order equal to n, we have: $\chi(H) \cdot \alpha(H) \geq n$.

Proof

Let $(C_1, C_2, C_3, \ldots, C_k\ )$ be a k - coloring of H with $k = \chi(H)$, we know that for $e \in E$, $|e| > 1$ then $e \leftarrow C_i$, for all $i \in \{1,2,3,\ldots k\}$, consequently C_i is a stable for all $i \in \{1,2,3,\ldots k\}$. Hence $|Ci| \leq \alpha(H)$, for all $i \in \{1,2,3,\ldots k\}$. We have: $n = \sum_{i=1}^{k} |c_i| \leq k \cdot \alpha(H) = \chi(H) \cdot \alpha(H)$.

Proposition

If $H = (V;E)$ is a hypergraph with an order equal to n, we have: $\chi(H) + \alpha(H) \leq n+1$

Proof

Assume that S is a stable with $|S| = \alpha(H)$. We can color any vertex of S with the same color andusing $n - \alpha(H)$ other colors to color the set $V - S$

with different colors. Hence we have :

$$\chi(H) \leq n - \alpha(H) + 1$$

that leads to

$$\chi(H) + \alpha(H) \leq n + 1.$$

Particular Colorings

Strong Coloring

Let $H = (V; E)$ be a hypergraph, a strong k-coloring is a partition $(C_1, C_2, \ldots, C_K)$ of V such that the same color does not appear twice in the same hyperedge. In another word: $|e \cap C_l| \leq 1$ for any hyperedge and any element of the partition.

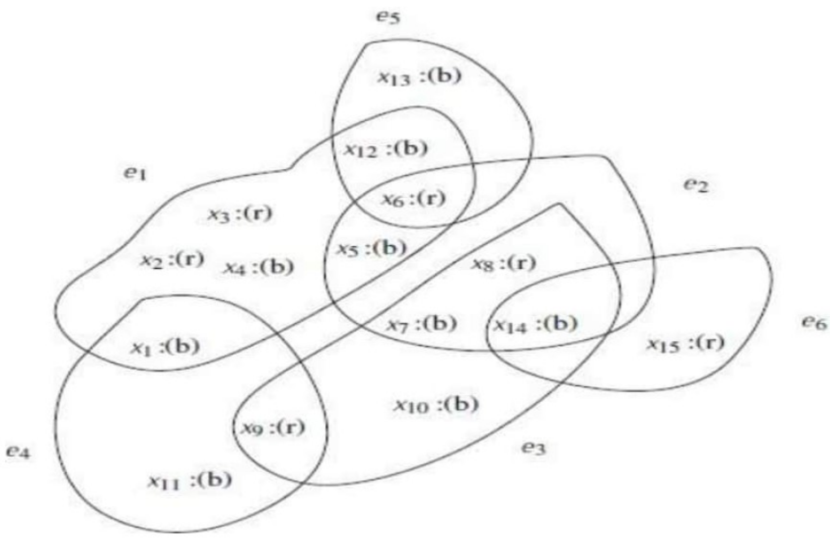
The strong chromatic number denoted by $\chi'(H)$ is the smallest k such that H has a strong k-coloring.

Equitable Coloring

Let $H = (V; E)$ be a hypergraph, an equitable k-coloring is a k-partition $(C_1, C_2, \ldots,\ C_K\ )$ of V such that, in every hyperedge e, all the colors $\{1, 2, \ldots, \mathrm{k}\}$ appear the same number of times, to within one, if k does not divide $|e_i|$. That is for all $e \in E$,

$$\left\lfloor \frac{|e|}{k} \right| \leq |e \cap C_i| \leq \left\lceil \frac{|e|}{k} \right\rceil i \in \{1,2,3\ldots k\}$$

Example:



The figure shows a hypergraph $H = (V;E)$ and an equitable 2- coloring of it, where (r) is red and (b)is blue.

Good Coloring

Let $H = (V;E)$ be a hypergraph, a good k-coloring is a k-partition $(C_1, C_2, \ldots, C_K)$ of V such that every hyperedge e contains the largest possible number of different colors, i.e. for every $e \in E$, the number of colors in e is $\min\{|e|;k\}$.

Lemma

Let $H = (V;E)$ be a hypergraph (with $m = |E|$, and $C = (C_1, C_2, \ldots, C_K)$ be a good k-coloring of H, we have:

(i) *if $k \leq c\ r(H)$,($c\ r(H)$ is the co-rank of H) then C is a partition in k transversal sets;*

(ii) *if $k \geq r(H)$ then the good coloring C is a strong coloring*

Proof:

Assume that $k \leq cr(H)$. By definition of a good coloring, if C_i is a set of vertices with color i, we must have:

$$C_i \cap e_j \neq \emptyset, \forall j \in \{1, 2, \ldots, \mathrm{m}\}.$$

Hence C_i is a transversal of H. Assume now that $k \geq r(H)$. Let $e \in E$, then $k \geq| e$, any two vertices belonging to e have different colors. Consequently, by definition of a strong coloring, the good coloring C is a strong coloring.

Uniform Coloring

Let $H = (V;E)$ be a hypergraph with $|V| = n$. A uniform k-coloring is a k-partition: $(C_1, C_2, \ldots, C_K)$ of V such that the number of vertices of the same color is always the same, to within one, if k does not divide n, i.e

$$\left\lceil \frac{|n|}{k} \right\rceil \leq |\mathrm{C_i}| \leq \left\lceil \frac{|n|}{k} \right\rceil, \mathrm{i} \in \{1, 2, 3 \ldots \mathrm{k}\}$$

Hyperedge Coloring

Let $H = (V;E)$ be a hypergraph, a hyperedge k-coloring of H is a coloring of the hyperedges such that:

(i) A hyperedge has just one color.

(ii) We use k colors to color the hyperedges.

(iii) Two distinct intersecting hyperedges receive two different colors

Chromatic index

The size of a minimum hyperedge k-coloring is the chromatic index of H. We will denote it by $q(H)$.

Lemma

Let H be a hypergraph. We have: $q(H) \geq \Delta_0(H) \geq \Delta(H)$. Where $\Delta_0(H)$ is the maximum cardinality of the intersecting families and $\Delta(H)$ is maximum cardinality of the stars.

Proof

Assume that $\Delta_0(H) = l$. We need l distinct colors to color an intersecting family with at least l hyperedges. Hence

$$q(H) \geq \Delta_o(H) \geq \Delta(H)$$

Hyperedge coloring property

A hypergraph has the hyperedge coloring property if $q(H) = \Delta(H)$. For instance a star has the hyperedge coloring property.

CHAPTER 4

APPLICATION OF HYPERGRAPH THEORY

Like in most fruitful mathematical theories, the theory of hypergraphs has many applications. Hypergraphs model many practical problems in many different sciences. We find this theory in psychology, genetics, etc. but also in various human activities. Hypergraphs have shown their power as a tool to understand problems in a wide variety of scientific field.

Moreover it well known now that hypergraph theory is a very useful tool to resolve optimization problems such as scheduling problems, location problems and so on. This chapter shows some possible uses hypergraphs in Applied Sciences.

Chemical Hypergraph Theory

The graph theory is very useful in chemistry. The representation of molecular structures by graphs is widely used in computational chemistry. But the main drawback of the graph theory is the lack of convenient tools to represent organometallic compounds, benzenoid systems and so on.

A hypergraph $H = (V, E)$ is a molecular hypergraph if it represents molecular structure, where $x \in V$ corresponds to an individual atom, hyperedges with degrees greater than 2 correspond to polycentric bonds and hyperedges with $\deg(x) = 2$ correspond to simple covalent bonds.

Hypergraphs appear to be more convenient to describe some chemical structures. Hence the concept of molecular hypergraph may be seen as a generalization of the concept of molecular graph. Hypergraphs have also shown their interest in biology.

Hypergraph Theory for Telecommunication

A hypergraph theory can be used to model cellular mobile communication systems. A cellular system is a set of cells where two cells can use the same channel if the distance between them is at least some predefined value D. This situation can be represented by a graph where:

(a) Each vertex represents a cell.

(b) An edge exists between two vertices if and only if the distance between the corresponding cells is less than the distance called reuse distance and denoted by D.

A *forbidden set* is a group of cells all of which cannot use a channel simultaneously. A *minimal forbidden set* is a forbidden set which is minimal with respect to this property, i.e. no proper subset of a minimal forbidden set is forbidden. From these definitions it is possible to derive a better modelization using hypergraphs. We proceed in the following way:

(a) Each vertex represents a cell.

(b) A hyperedge is minimal forbidden set.

Hypergraphs and Constraint Satisfaction Problems

A constraint satisfaction problem, P is defined as a tuple: $P = (V, D, R_1(S_1), \ldots, R_k(S_k))$ where:

- V is a finite set of variables.
- D is a finite set of values which is called the *domain* of P.
- Each $R_i(S_i)$ is a constraint.
  - $-S_i$ is an ordered list of n_i variables, called the *constraint scope.*
  - $-R_i$ is a relation over D of arity n_i, called the *constraint relation.*
  -

To a constraint satisfaction problem one can associate a hypergraph in the following way:

(a) The vertices of the hypergraph are the variables of the problem.

(b) There is a hyperedge containing the vertices $v_1, v_2, \ldots, v_{\mathrm{t}}$ when there is some constraint $R_i(S_i)$ with scope $S_i = \{v_1, v_2, \ldots v_t\}$.

Hypergraphs and Database Schemes

Hypergraphs have been introduced in database theory in order to model relational database schemes.

A database can be viewed in the following way:

- A set of attributes.
- A set of relations between these attributes.

Hence we have the following hypergraph:

- The set of vertices is the set of attributes.
- The set of hyperedges is the set of relations between these attributes.

CONCLUSION

A hypergraph is defined to be a family of hyperedges which are sets of vertices of cardinality not necessarily 2 (as for graphs). We have discussed basic definitions, some properties, colorings, and certain applications of hypergraph theory. Hypergraphs model practical situations in different sciences in a much more general setting than graphs do. Hypergraph theory is used in chemistry, the engineering field, telecommunication, database schemes, etc.

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DON BOSCO ARTS & SCIENCE COLLEGE ANGADIKADAVU

DEPARTMENT OF MATHEMATICS

2021-2023

Project Report on

NETS AND FILTERS IN TOPOLOGY

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Project Report on

NETS AND FILTERS IN TOPOLOGY

Dissertation submitted in the partial

Fulfillment of the requirement for the award of

MSc Degree in Mathematics of

Kannur University

Name : RAHINA

Roll No. : C1PSMM1907

Examiners:

1.

2.

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BONAFIDE CERTIFICATE

Certified that this project report “NETS AND FILTERS IN TOPOLOGY” is the bonafide work of RAHINA who carried out the project work under my supervision.

Signature of Head   Signature of Supervisor

DECLARATION

I, RAHINA hereby declare that the Project work entitled NETS AND FILTERS IN TOPOLOGY has been prepared by me and submitted to Kannur University in partial fulfillment of requirement for the award of Master of Science is a record of original work done by me under the supervision of Mr. ANIL M V, Assistant Professor, Department of Mathematics, Don Bosco Arts & Science College, Angadikadavu. I, also declare that this Project work has not been submitted by me fully for the award of any Degree, Diploma, Title or recognition before any authority.

Place: Angadikadavu RAHINA

Date: (C1PSMM1907)

ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also I must express my deepest gratitude to people along the way. No words can adequately express the sense of gratitude, still I try to express my heartfelt thanks through words. The outset, I am deeply indebted to my project supervisor Mr. ANIL M V Assistant Professor, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu, for the invaluable guidance, loving encouragement and meticulous care towards me throughout my career. I express my deep sense of gratitude to all the faculty members of the Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu. I can never forget the support and encouragement rendered by the Principal and the Staff of Don Bosco Arts & Science College, Angadikadavu. I could not name many who sincerely supported and helped for the successful completion of this Project. It is my pleasure and duty to thank each and everyone of them who walked with me.

RAHINA

CONTENTS

INTRODUCTION

When we are working on metric spaces (or first countable topological spaces) sequences are sufficient to describe almost all topological properties. This dissertation mainly deals with the generalisation of sequences called nets and the concept of filters. The goal of the first chapter is to justify via examples the fact that sequences are not enough to describe general topological spaces. Here some basic definitions and results are included which we use further.

The second chapter deals with the definition of nets and basic properties of nets which ensure that nets are the generalisation of sequences. Later we characterise some of the topological properties; such as Hausdorff property, continuity of a function, closure of set, etc. Last part of this chapter gives you an interesting proof of Tychonoff theorem via nets that proves the importance of nets in analysis.

The last chapter describes the use of filters and its interesting applications. Here first section talks about the definition and convergence of filters with some examples while the next section characterise topological properties which is more easier compared to that of nets. Coming to the end, we are introducing the concept of ultrafilters and gives another proof of Tychonoff theorem using ultrafilters.

PRELIMINARY

Definition

A Topological space is a pair (X,τ) where X is a non-empty set and τ is a family of subsets of X satisfying:

(i) $\phi \in \tau$ and X $\in \tau$.

(ii) τ is closed under arbitrary union

(iii)τ is closed under finite intersection.

The members of τ are called open sets and the compliment of open sets are called closed sets.

Definition: Sequences and its convergence

A sequence $\{x_n\}$ in a topological space (X,τ) is a function from $\mathbb{N}$ to X. A sequence $\{x_n\}$ is said to be converge to a point x if for every open set U containing x, there exists a positive N such that for every integer n $\geq$ N, $x_n \in$ U.

Example

Let X be any non-empty set.

1) Indiscrete topology : The collection, τ =$\{\phi$, X$\}$ is the smallest topology on X. This topology is called as Indiscrete topology.

In this topology every sequence converges to every point.

2) Discrete topology : If P(X) is the power set of X, then τ = P(X) is a topology on X and is called as Discrete topology on X.
In this topology only eventually constant sequence is convergent and it converges to the repeating point of the sequence.
3) Co-finite topology : Let

$$\tau_{cf} = \{\text{A} \subseteq \text{X} : A^c \text{ is f inite set}\}$$

is a topology on X. It is called as Co-finite topology.
In this topology only eventually constant sequence and sequence with infinitely many distinct points are convergent. First sequence converges to the repeating point and the second sequence converges to every point of X.
4) Co-countable topology : The collection of subsets A of X such that A^c is countable is a topology on X and it is called as Co-countable topology.

$$\text{ie, } \tau_{cc} = \{\text{A} \subseteq \text{X} : A^c \text{ is countable}\}$$

In this topology, only eventually constant sequence converges and it converges to the repeating point .

Definition: Hausdorff space

A space X is said to be Hausdorff space if for every distinct points x, y $\in$ X there exists disjoint open sets U,V in X such that x $\in$ X and y $\in$ X.

Definition: Limit Point of a Set

Let A be a subset of a topological space X and y $\in$ X. Then y is said to be a limit point of A if every open set U containing y contains atleast one point of A other than y.
i.e, (A / {y}) $\cap$ U $\neq \phi$.

Definition: Closure of a Set

The closure of a subset A of a topological space X is defined as the intersection of all closed sets containing A. It is denoted by $\overline{A}$.
i.e, Closure of a set A is the union of A and its derived set, where derived set of A is the collection of all limit points of A.
Clearly, $\overline{A}$ is the smallest closed set containing A.
Equivalently, a set A is closed if it contains all of its limit points, i.e, if A = $\overline{A}$.

Definition: Function

Let X and Y be two sets. A function from X to Y is a subset of X $\times$ Y with the property that for each x $\in$ X, there is a unique y $\in$

Y such that (x, y) $\in$ X $\times$ Y .The set X, Y and f are called domain, codomain and graph of (X, Y, f) respectively. We also denote (X, Y, f) by f : X $\to$ Y .

Definition: Continuity of a Function

Let X and Y be topological spaces and f : X $\to$ Y be a function. Then f is said to be continuous on X if f^{-1}(U) is open in X, for every open set U in Y.

Theorem: In a Hausdorff space X, every sequence converges to at most one limit

proof

Let X be a Hausdorff space.

Let (x_n) be a any sequence converges to x and y.

Assume x $\neq$ y.

Since X is Hausdorff, $\exists$ opensets U and V $\ni$ x $\in$ U and y $\in$ V and U $\cap$ V = ϕ.

$x_n \to$ x $\Rightarrow \exists\ m_1 \in$ N $\ni x_n \in$ U n $\geq m_1$.

$x_n \to$ y $\Rightarrow \exists\ m_2 \in$ N $\ni x_n \in$ V n $\geq m_2$.

Then $x_n \in$ U $\cap$ V n $\geq$ max$\{m_1, m_2\}$ which is a contradiction to the fact that U $\cap$ U = ϕ.

Hence x = y.

Definition: Cover of a set

A cover of a subset A of X is the collection A = $\{A_i$: i $\in$ I$\}$ of subsets of X such that A $\subseteq \bigcup_{i \in I} A_i$.
Any subfamily of A which covers A is called a subcover of A.

Definition: Compact set

A subset A of a space X is said to be a compact subset if every cover of A by open sets of X has a finite subcover [subcover containing finite number of sets]. A space is said to be compact if it is a compact subset of itself.

Definition: Finite Intersection Property(f.i.p)

A family F of subsets of a set X is said to have the finite intersection property if for any n $\in$ N and F_1,F_2 , ..., F_n $\in$ F, the intersection $\bigcap_{i=1}^{n} F_i$ is non-empty.

Theorem: A topological space is compact if and only if every family of closed subsets of it, which has f.i.p, has non-empty intersection.

proof

Let X be any topological space.

Suppose X is compact.

Let $\mathscr{C}$ be any family of closed sets having f.i.p.

Claim:$\bigcap_{C\in\mathscr{C}}$ C $\neq \phi$

Suppose

$$\bigcap_{C\in\mathscr{C}} \text{C} = \phi$$

$$(\bigcap_{C\in\mathscr{C}} \text{C})^c = X$$

$$\bigcup_{C\in\mathscr{C}} C^c = \text{X}$$

C is closed implies C^c is open.

$\therefore$ $\{C^c\}$ for C $\in \mathscr{C}$,is an open cover for X.

Since X is compact, there exists C_1^c, C_2^c....... C_n^c such that $\bigcup_{i=1}^n C_i^c$ = X

$\Rightarrow \bigcap_{i=1}^n C_i = \phi$, a contradiction to the choice of $\mathscr{C}$.

Hence $\bigcap_{C\in\mathscr{C}}$ C $\neq \phi$

The converse part is almost similar only use Demorgan's law and the definition,complement of open set is closed.

CHAPTER 1

Topological properties and sequences

Example 1

Let X be an uncountable set with co-countable topology.

We know that the only converging sequence are eventually constant sequences and the limit is unique.

But X is not Hausdorff space.

For let x and y be any two distinct points.

Let U be any open set of X containing x.

If we want X to be Hausdorff we need an open set contained in U_c.

ie, an open set containing countable elements, since U is open.

Then V_c will be uncountable, as X is uncountable.

$\therefore \nexists$ any open V $\ni$ y $\in$ V and U $\cap$ V = ϕ.

Hence, X is not a Hausdorff space.

So, **uniqueness of limit of converging sequence is not enough to characterise the Hausdorff property.**

Example 2

Let X be an uncountable set.

Let (X, σ) be the discrete topology and (X, τ) be the co-countable topology.

We know that in both topology, the only convergent sequence are the eventually constant sequences.

Let us define the identity function, I : (X, τ) $\rightarrow$ (X, σ).

Since τ and σ have same convergent sequences, the function, I is sequentially continuous.

However since $\tau \rightarrow \sigma$, the function I is not continuous.

Hence, **sequences do not characterise continuity of a function.**

Example 3

Let X be an uncountable set and fix x0 X.

Define $\omega = \{$A $\subseteq$ X : ($x_0 \notin$ A) or ($x_0 \in$ A and A^c is countable)$\}$.

Then ω is a topology for X.

(a) ϕ, X $\in \omega$.

(b) A, B $\in \omega \Rightarrow$ A $\cap$ B $\in \omega$.

(c) A $\in \omega$, $\in$ J implies that $\bigcup_\alpha$ $(A_\alpha) \in \omega$.

If $x_0 \in A_{\alpha 0}$ then $(\bigcup_\alpha A_\alpha)^c = \bigcap_\alpha A_\alpha^c \subseteq A_{\alpha 0}$, which is countable.Thus $\bigcup_\alpha A_\alpha \in \omega$.

Claim: $x_n \to$ x if $x_n =$ x eventually.

Case 1: x $\neq x_0$,

then {x} $\in \omega$, then by definition of convergence,$x_n \to$ x if and only if v = x eventually.

Case 2: x = x_0.

Assume $x_n \neq x_0$ for infinitely many n.

Define F = $\{x_n : x_n \neq x_0\}$.

Then $x_0 \in F^c$ and $(F^c)^c =$ F, countable $\Rightarrow F^c \in \omega$.

ie, F^c is an open neighbourhood of x_0 which fails to hold $x_n \in F^c$ $\forall$ n $\geq n_0$ for any $n_0 \in$ N.

Hence $x_n \neq x_0$ is wrong.

$\Rightarrow x_n = x_0 =$ x eventually.

Conversely, if $x_n \nrightarrow x_0$, then $\exists$ O $\in \omega$, $x_0 \in$ O $\ni x_n \notin$ O infinitely often.

ie, $x_n \neq x_0$ infinitely often.

Hence $x_n \to x_0$ if and only if $x_n = x_0$ eventually.

Let A = $(\{x_0\})^c$

Then $x_0 \in \overline{A}$, but there does not exist a sequence $\{x_n\}$ in A such that $x_n \to x_0$ in ω because any sequence in A satisfy $x_n \neq x_0$ n ≥ 1.

Hence by above result, x_n does not converge to x_0.

But for any open set O such that $x_0 \in$ O, we have O $\cap$ A $\neq \phi$,

$\therefore O^c$ is countable.

$\therefore$Every neighbourhood of x_0 intersect A implies $x_0 \in$ A.

ie, **Sequences do not characterise points of closure.**

Note : A metric space is compact if and only if it is sequentially compact.

But we cannot generalise compactness using sequential compactness. Moreover there exist compact space which is not sequentially compact and vice versa.

CHAPTER 2

Nets in topology

2.1 Nets and its convergence

Definition 2.1.1.Directed set

A directed set is a pair (D, $\geq$) where D is a non-empty set and $\geq$ is a binary relation on D satisfying:

(1) For all m, n, p $\in$ D, m $\geq$ n and n $\geq$ p imply m $\geq$ p.

(2) For all n $\in$ D, n $\geq$ n.

(3) For all m, n $\in$ D, there exists p $\in$ D suchthat p $\geq$ n and p $\geq$ m.

Example

1) For x $\in$ X,η_x denote the set of all neighbourhood system [Collection of neighbourhood of x] in the topological space, X. We define U $\geq$ V to mean U $\subseteq$V in η_x. Then clearly η_x is directed set.

2) A partition of unit interval [0, 1] is a finite sequence sequence P={a_0,a_1,.....a_n} such that 0 = a_0 < a_1 < ......< a_n =1 If Q is a partition of [0, 1] which contains P, then Q is said to be a refinement of P.

Define P $\geq$ Q if P is refinement of Q. Then the collection of all partitions of [0, 1]is a directed set.

Definition 2.1.2 Nets

A net in a set X is a function S : D $\rightarrow$ X, where D is a directed set

Example

Riemann net: Let f be a bounded real valued function from [0, 1] and D be the set of all pairs (P, ξ) where P is a partition of [0, 1]. Say P = $\{a_0, a_1, ...a_n\}$ and $\xi = (\xi_1, \xi_2, ..., \xi_n)$ is finite sequence such that $\xi_i \in$ [ai1, ai] for i = 1, 2, ..., n. And define (P, ξ) (Q, η) of D if P is a refinement of Q. Now the Riemann net S : D $\rightarrow$ R is defined by,

$$S(P, \xi) = \sum_{i=1}^{n} f(\xi_i)(a_i\text{-}a_{i+1})$$

Definition 2.1.3 Convergence of a net

Let (X,τ) be a topological space and let S : D $\rightarrow$ X be a net. Then S is said to be converging to a point x $\in$ X, if given any open set U containing x, there exists m $\in$ D such that for all n $\in$ D, n $\geq$ m implies that S(n) $\in$ U.

Note: As the partition becomes finer and finer, the limit of Riemann net becomes the Riemann integral.

Definition 2.1.4 Eventual subset

A subset E of a directed set D is said to be eventual if there exists m $\in$ D such that for all n $\in$ D, n $\geq$ m implies that n $\in$ E.

A net S : D $\rightarrow$ X is said to be eventually in a subset A of X if the set S^1(A) is an eventual subset of D.

ie, a subset E is eventually if and only if it contains all elements in D 'after a certain stage' and a net S : D $\rightarrow$ X eventually in A if and only if A contains all its terms after a certain stage

Definition 2.1.5 Cofinal subset

Let (D, $\geq$) be a directed set. A subset F of D is said to be a cofinal subset of D if for every m $\in$ D, there exists n $\in$ F such that n $\geq$ m. A net S : D $\rightarrow$ X is said to be frequently in a subset A of X if S^1(A) is a cofinalsubset of D.

• Every eventual subset is a cofinal subset of D.

• Converse is not true: In $\mathbb{N}$ every infinite subset is cofinal but it is not necessarily eventual.

• Cofinal set in the neighbourhood system of a point in a topological space is a local base at that point.

• In the set of all partitions of the unit interval [0, 1], the set of partitions of the form $\{0,\frac{1}{n},\frac{2}{n}......\frac{n-1}{n},1\}$ for n $\in$ N is not a cofinal subset. If we direct the set of all partitions in terms of norm [the length of the longest subinterval] such that P follows Q if and only if the norm of P is less than or equal to that of Q, then it is a cofinal subset.

Definition 2.1.6 Cluster point

Let S : D $\rightarrow$ X be a net. A point x $\in$ X is said to be a cluster point of S if for every neighbourhood U of x in X, and m $\in$ D, there exists n $\in$ D such that n $\geq$ m and S(n) $\in$ U.
Equivalently, x is a cluster point of S if and only if for every neighbourhood U of x, S is frequently in U.

Proposition 2.1.1 Suppose S : D $\rightarrow$ X is a net and F is a cofinal subset of S.If S|F : F $\rightarrow$ X converges to a point x in X, then x is a cluster point of S.

proof

Let U be a neighbourhood of x in X.
Let m $\in$ D be given.
S|F $\rightarrow$ x $\Rightarrow$ $\exists$ m_1 $\in$ F $\ni$ $\forall$ n $\in$ F, n $\geq$ m, $\Rightarrow$ Sn $\in$ U
Choose n_1 $\in$ D such that $n_1 \geq$ m and $n_1 \geq m_1$.
Since F is cofinal subset we can find n $\in$ F $\ni$ n $\geq m_1$.
Then n $\geq n_1 \geq$ m and S(n) $\in$ U. So S is frequently in U.
Since U was arbitrary, x is a cluster point of S.

Definition 2.1.7 Subnet

Let S : D $\rightarrow$ X and T : E $\rightarrow$ X be nets. Then T is said to be a subnet of S if there exists a function N : E $\rightarrow$ D such that,

(i) T = S $\circ$ N.

(ii) for any n $\in$ D, there exists p $\in$ E such that for all m $\in$ E, m $\geq$ p implies N(m) $\geq$ n in D.

Theorem 2.1.1 Let S : D $\rightarrow$ X be a net in a topological space and let x $\in$ X.Then x is a cluster point of X if and only if there exist a subnet of X which converges to x.

proof

Suppose T : E $\rightarrow$ X is a subnet of S converging to x.

claim: x is a cluster point of S.

Let N : E $\rightarrow$ D be the function such that T = S $\circ$ N.

Let U be any neighbourhood of x in X and let m1 D be given. From the definition of subnet (ii), there exists p $\in$ E such that for all m $\in$ E , m $\geq$ p implies N(m) $\geq$ m_1.

Since T converges to x, there exists q $\in$ E such that for all m $\in$ E , m $\geq$ q implies T(m) $\in$ U, ie, SN(m) $\in$ U.

Choose n_1 $\in$ E such that $n_1 \geq$ p and $n_1 \geq$ q. Let N(n_1) = n_0. Then $n_0 \geq m_1$ and S(n_0) $\in$ U.

Since m_1 and U were arbitrary, we get x is a cluster point of S.

Conversely, suppose x is a cluster point of S

Let η_x be a neighbourhood system of the point x in X. Define,

E = {(n, U) $\in$ D $\times$ η_x : S(n) $\in$ U}

For (n, U) , (m, V) $\in$ E we let (n, U) $\geq$ (m, V) means n $\geq$ m and U $\subseteq$ V .

It is easy to show that E is a directed set with the defined binary relation.

Define T : E $\rightarrow$ X by T(n,U) = S(n) for any (n, U) $\in$ E.

For N : E $\rightarrow$ D suchthat N(n, U) = n we get that T = SN(n, U) = S(n) and for any m $\in$ D, for any U' $\in \eta_x$ $m_1 \in$ D $\ni$ $m_1 \geq$ m and S(m_1) $\in$ U'.

Then (m_1, U') $\in$ E. Now for (n, U) $\geq$ (m_1, U') n $\geq m_2 \geq$ m.

Hence T is a subnet of S.

Now we will prove that T $\rightarrow$ x $\in$ X.

Let G be any open set of x in X. Since x is a cluster point of S, we get S is frequently in G.

Fix $n_0 \in$ D $\ni$ S(n_0) $\in$ G. Then (n_0, G) $\in$ E.

Now (m, G) $\in$ E , (m, U) $\geq$ (n_0, G) $\Rightarrow$ T(m, U) = $S_m \in$ U $\subseteq$ G.

Thus T converges to x in X.

Hence the proof.

2.2 Topological properties and nets

In this section we prove that every topological property can be characterised in terms of convergence of nets. In order to do this it suffices to obtain a characterisation of open sets because everything in topology depends ultimately on open sets.It suffices to characterise closure property in terms of convergence of nets since closed sets and open sets are characterised in terms of closures. Here we begin with the characterisation of Hausdorff property.

Characterisation of hausdorff property

Theorem 2.2.1 A topological space is Hausdorff if and only if limits of all nets in it are unique.

Proof Suppose X is a Hausdorff space, S : D $\rightarrow$ X is a net in X and S $\rightarrow$ x and S $\rightarrow$ y in X.

claim : x = y.

If x $\neq$ y, then there exists open sets U, V $\ni$ x $\in$ U , y $\in$ V and U $\cap$ V = ϕ. But by the definition of convergence,

S $\rightarrow$ x implies $\exists$ m_1 $\in$ D $\ni$ s_n $\in$ U $\forall$ n $\geq$ m_1 and n $\in$ D.

S $\rightarrow$ y implies $\exists$ m_2 $\in$ D $\ni$ s_n $\in$ U $\forall$ n $\geq$ m_2 and n $\in$ D.

m_1, m_2 $\in$ D then by property (iii) of directed set $\exists$ p $\in$ D $\ni$ p $\geq$ m_1 and p $\geq$ m_2.

$$\Rightarrow S_n \in U \cap V \; \forall \; n \geq p \text{ and } n \in D.$$

Since U $\cap$ V = ϕ we are getting a contradiction.

Hence x = y

Conversely, suppose limits of all nets are unique.

Suppose X is not Hausdorff.

Then there exists two points x and y with the property that the open sets of x and y are not disjoint.

Let η_x and η_y be the neighbourhood system of x and y respectively.

Let D = $\eta_x \times \eta_y$ and (U_1, V_1) , $(U_2, V_2) \in$ D.

Define $(U_1, V_1) \geq (U_2, V_2)$ if $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$.

Clearly D is a directed set.

Define S : D $\rightarrow$ X $\ni$ S(U, V) be any point in U $\cap$ V .

We will prove that S is converging to both x and y.

Let G be any open set of x. Then (G, X) $\in$ D.

Now if (U, V) $\geq$ (G, X) in D then U $\subseteq$ G and so S(U, V) $\in$ U $\cap$ V $\subseteq$ U $\subseteq$ G.

Thus S converges to x. Similarly S converges to y.

Hence S converges to both x and y which is a contradiction to our hypothesis.

So X is Hausdorff

Characterisation of limit point

Theorem 2.2.2 Let A be a subset of a space X and let x $\in$ X. Then x $\in \overline{A}$ if and only if there exists a net in A which converges to x in X.

proof

Suppose S : D $\to$ A is a net which, when regarded as a net in X converges to x.

Let U be any neighbourhood of X.

Since S $\to$ x in X $\Rightarrow \exists$n $\in$ D m$\geq$ n and m $\in$ D $\Rightarrow$ S(m) $\in$ U.

But $S_m \in$ A, m $\in$ D. So A $\cap$U $\neq \phi$.

U is arbitrary, by the definition of limit point $x \in \overline{A}$.

Conversely suppose $x \in \overline{A}$. Then every neighbourhood of x meets A.

Let η_x be the neighbourhood system of x in X with the usual binary relation.

Define S : D $\to$ A by S(U) =any point in U $\cap$ A. Then S is a net in A.

claim : S $\to$ x in X.

Let G be any open set in X containing x.

For any U $\in$, U $\geq$ G $\Rightarrow$ U $\subseteq$ G.

Hence S(U) $\in$ U $\subseteq$ G.

So U $\subseteq$ G $\Rightarrow$ S(U) $\in$ G $\Rightarrow$ S $\to$ x in X.

Hence there exists a net in A which converges to x.

Corollary 2.2.1 A subset A of X is closed if and only if limits of nets in A are in A

Corollary 2.2.2 A subset B of a space X is open if and only if no net in the complement X/B converges to a point in X

Corollary 2.2.3 Let τ_1,τ_2 be two topologies on a set X such that a net in X converges to a point with respect to τ_1 if and only if it does so with respect to τ_2. Then $\tau_1 = \tau_2$.

Proof

Let B *in* τ_1 we will prove that B $\in \tau_2$.

By Corollary 2.2.2., B is open $\Rightarrow$ no net in X / B converges to a point in B with respect to τ_1.

Since convergence with respect to $\tau 1$ is identical that with respect to τ_2

$\Rightarrow$ no net in X / B converges to a point in B in τ_2.

$\Rightarrow$ B is open by Corollary 2.2.2.

$\Rightarrow$ B $\in \tau_2$.

So $\tau_1 \subseteq \tau_2$.

Similarly $\tau_2 \subseteq \tau_1$.

So we get $\tau_1 = \tau_2$.

Characterisation of continuity of a function

Theorem 2.2.3: Let X, Y be topological spaces , $x_0 \in$ X and f : X $\rightarrow$ Y a function. Then f is continuous at x_0 if and only if whenever a net S converges to x_0 in X, the net f $\circ$ S converges to f(x_0) in Y.

proof

Suppose f is a continuous at x_0 and S : D $\rightarrow$ X is net converging to x_0.

Let V be any neighbourhood of f(x_0) in Y. Then f^{-1}(V) is a neighbourhood of x_0 in X.

Since S $\rightarrow x_0$, $\exists$ m $\in$ D n $\in$ D with n $\geq$ m $\Rightarrow S_n \in f_{-1}$(V).

$\Rightarrow$ n $\geq$ m , f(S_n) $\in$ V .

Thus f $\circ$ S converges to f(x_0) in Y.

Conversely suppose whenever a net S converges to x_0 in X, the net f $\circ$S converges to f(x_0) in Y.

Suppose f is not continuous at x_0.

Then there exists a neighbourhood V of f(x_0) such that f_1(V) is not a neighbourhood of x_0.

$x_0 \in f^{-1}$(V) is not a neighbourhood of x_0 $\Rightarrow$ every neighbourhood of x_0 intersects X / f^{-1}(V).

Now ηx_0 be the neighbourhood system of x_0 with usual binary operation.

Define S : $\eta_{x0} \rightarrow$ X by S(N) = any point in N $\cap$ (X/f^{-1}(V)).

Then S converges to x_0 in X. Then for any neighbourhood U of x_0, U $\in \eta_{x0}$.

Then N $\geq$ U $\Rightarrow$ N $\subseteq$ U $\Rightarrow$ S(N) $\in$ N $\subseteq$U

But f$\circ$S takes values on Y/V.So V is a neighbourhood of f(x_0) which contains no point of net f∘S.

Thus f∘S $\nrightarrow$ f(x_0),a contradiction.

Hence f is continuous

Characterisation of compactness

Theorem 2.2.4: For a topological space X, the following statements are equivalent:

(1) X is compact.

(2) Every net in X has a cluster point.

(3) every net in X has a convergent subnet in X.

proof

(2) $\longleftrightarrow$ (3) is proved previously. So we will prove that (1) $\longleftrightarrow$ (2).

Assume (1) and let S : D → X be a net in X.

Suppose S has no cluster point in X.

ie, No point x ∈ X is a cluster point of S.

ie, for each x, ∃ a neighbourhood of x, N_x and m_x ∈ D n ∈ D, n ≥ m_x ⇒ S(n) ∈ X / N_x.

Cover X by {N_x} x ∈X.

Since X is compact there exists x_1, x_1, ..., x_k ∈ X X =$\cup_{i=1}^{k} N_{xi}$.

Let m_1, m_2, ..., m_k be corresponding elements in D such that n ≥ m_i

⇒ S(n) ∈ X / N_{xi}.

Since D is a directed set $\exists$ m $\in$ D m $\geq m_i$ i = 1, 2, ..., k.

Now by assumption S(m) $\in \cap_{i=1}^{k}(X/Nxi)$ = X / $\cup_{i=1}^{k}Nxi$ which is not possible.

So S has atleast one cluster point in X.

Conversely assume (2) holds.

We will use Theorem 1.1.2. to prove X is compact.

Let $\mathscr{C}$ be a family of closed sets of X having finite intersection property.

$\mathscr{D}$ be the family of all finite intersections of members of C.

Note that D itself closed under finite intersection property, and C $\subseteq$ D.

Define D $\geq$ E for D, E $\in \mathscr{D}$ if D $\subseteq$ E.

Thus $\mathscr{D}$ is a directed set.

If D, E $\in \mathscr{D}$ then D $\cap$ E $\in \mathscr{D}$ and D $\cap$ E $\geq$ D , D $\cup$ E $\geq$ E.

Clearly $\phi \neq$ D , since $\mathscr{C}$ has finite intersection property

Define S : D $\rightarrow$ X , S(D) =any point in D.

By the hypothesis, S has a cluster point, say x in X.

Claim: x $\in \bigcap_{C\in\mathscr{C}}$ C

If not there exists C $\in$ C such that x $\in$ C. Then X / C is a neighbourhood of x, since C is closed set belongs to C.

By the definition of cluster point, for C $\in$ D there exists D $\in$ D such that D $\geq$ C and S(D) $\in$ X / C.

D $\geq$ C $\Rightarrow$ D $\subseteq$ C so X / C $\subseteq$ X / D.

But S(D) $\in$ X / C $\subseteq$ X / D = D_c, which is a contradiction since S(D) is a point in D.

So x $\in \bigcap_{C \in \mathscr{C}}$ C.

CHAPTER 3

Filters in topology

3.1 Filters and its convergence

Definition 3.1.1.Filters

A filter on (or in) a set X is a non-empty family $\mathscr{F}$ of non-empty subsets of X suchthat:

(i) $\mathscr{F}$ is closed under finite intersections ,

(ii) if B $\in \mathscr{F}$ and B $\subseteq$ A then A $\in \mathscr{F}$ for all A, B $\subseteq$ X.

Clearly, X is an element of filter $\mathscr{F}$ always.

Example

(1)Fix some non-empty subset A of X. Then the collection of all supersets of A (in X) is a filter on X. Such a filter is known as an **atomic filter**, the set A being called the atom of the filter.

(2) In case X is infinite, the family $\mathscr{F}$ of all co-finite subsets of X is a filter on X.Such a filter is called as a **co-finite filter**.

(3)Suppose τ is a topology on X. Then for any x $\in$ X, the neighbourhood system η_x at x is a filter. It is called the τ **-neighbourhood filter** at x. It depends on both x and τ .

Definition 3.1.2.Base of a filter

Let $\mathscr{F}$ be a filter on a set X. Then a sub-family $\mathscr{B}$ of $\mathscr{F}$ is said to be a base of $\mathscr{F}$ (or a filter base) if for any A $\in \mathscr{F}$ there exists B $\in \mathscr{B}$ such that B $\subseteq$ A.

Note: If $\mathscr{B}$ is a base of filter $\mathscr{F}$ then every member of $\mathscr{F}$ is a superset of some members of $\mathscr{B}$.

Proposition 3.1.1. Let $\mathscr{B}$ be a family of non-empty subsets of a set X. Then there exists a filter on X having $\mathscr{B}$ as a base if and only if B has the property that for any B_1, $B_2 \in \mathscr{B}$, there exists $B_3 \in \mathscr{B}$ such that $B_1 \cap B_2 \supset B_3$.

proof

Suppose there exists a filter $\mathscr{F}$ on X having $\mathscr{B}$ as a base.
Then $\mathscr{B} \subset \mathscr{F}$ and $\phi \notin \mathscr{F}$ since $\phi \notin$ B .
Also let B_1, $B_2 \in$.
Then $B_1 \in \mathscr{F}$ (since B $\subseteq \mathscr{F}$) and so $B_1 \cap B_2 \in \mathscr{F}$ as $\mathscr{F}$ is closed under finite intersections.
So by the definition of a base , there exists $B_3 \in B_1 \cap B_2 \supseteq B_3$.
Conversely suppose B satisfies the given condition.
Let $\mathscr{F}$ be the family of all supersets of members of B. The condition (ii) in the definition of a filter automatically holds for $\mathscr{F}$.
The empty set cannot be supersets of any set.

Hence $\phi \notin \mathscr{F}$ as $\phi \notin \mathscr{B}$.

It only remains to show that $\mathscr{F}$ is closed under finite intersections.

So suppose A_1, $A_2 \in \mathscr{F}$.

Then there exists B_1, $B_2 \in \mathscr{B}$ such that $B_1 \subset A_1$ and $B_2 \subset A_2$.

By assumption there exists $B_3 \in \mathscr{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

But $A_1 \cap A_2$ is superset of $B_1 \cap B_2$ hence it is a superset of $B_3 \in \mathscr{B}$.

So $A_1 \cap A_2 \in \mathscr{F}$.

Thus $\mathscr{F}$ is a filter on X and $\mathscr{B}$ is a base for it by the construction.

Corollary 3.1.1. Any family which does not contain the empty set and which is closed under finite intersection is a base for a unique filter.

Definition 3.1.3.Sub-base of a filter

Let $\mathscr{F}$ be a filter on a set X. Then a subfamily $\mathscr{S}$ of $\mathscr{F}$ is said to be a sub-base for $\mathscr{F}$ if the family of all finite intersections of members of $\mathscr{S}$ is a base for $\mathscr{F}$. We also say $\mathscr{S}$ generates $\mathscr{F}$.

Proposition 3.1.2. Let $\mathscr{S}$ be a family of subsets of a set X. Then there exists a filter on X having $\mathscr{S}$ as a sub base if and only if $\mathscr{S}$ has the finite intersection property.

proof

If there exists a filter $\mathscr{F}$ on X, containing S then $\mathscr{F}$ has finite intersection property and so does every subfamily of $\mathscr{F}$.

Conversely suppose $\mathscr{S}$ has the finite intersection property.

Let $\mathscr{B}$ be the family of finite intersections of members of $\mathscr{S}$.

Then $\phi \notin \mathscr{B}$ and $\mathscr{B}$ is closed under finite intersections.

So by the Corollary 3.1.1., $\mathscr{B}$ is a base for a filter $\mathscr{F}$ on X and thus $\mathscr{S}$ is a sub base for $\mathscr{F}$.

Definition 3.1.4.Convergence of filter

A filter $\mathscr{F}$ converges to a point p if $\mathscr{F} \supset \eta_p$. i.e, every neighbourhood of p is contained in $\mathscr{F}$.

Definition 3.1.5.Limit point

A point p is a limit point of a filter $\mathscr{F}$ if for every neighbourhood U of p and every A $\in \mathscr{F}$ we have A $\cap$ U $\neq \phi$. p is also called as cluster point of $\mathscr{F}$.

Definition 3.1.6. Subfilter

Let $\mathscr{F}$ be filter on X. Then a filter $\mathscr{G}$ is said to be a subfilter of $\mathscr{F}$ if $\mathscr{F} \subseteq \mathscr{G}$.

There is a canonical way of converting net to filters and vice versa. Let S : D $\rightarrow$ X be a net. For each m $\in$ D, let $B_m = \{(S_n)$: n $\in$ D, n $\geq$ m$\}$. Let

$\mathscr{F} = \{$A $\subseteq$ X : A $\supseteq B_m$, forsome m $\in$ D$\}$

In other words, $\mathscr{F}$ is the collection of all super sets of sets of the form B_m for m ∈ D.

Using the fact that D is a directed set, we can show that $\mathscr{F}$ is a filter on X.

This filter is called the filter associated with the net S.

Conversely let $\mathscr{F}$ be a filter on X. Let

D = {(x, F) ∈ X × $\mathscr{F}$: x ∈ F}

For (x, F), (y, G) ∈ D define (x, F) ≥ (y, G) if F ⊆ G.

It is clear that D is a directed set because $\mathscr{F}$ is closed under finite intersections.

Define S : D → X by S(x, F) = x.

Now this net is called the net associated with $\mathscr{F}$.

Proposition 3.1.3. Let S : D → X be a net and $\mathscr{F}$ be the filter associated with it. Let x ∈ X. Then S converges to x as a net if and only if $\mathscr{F}$ converges to x as a filter. Also x is a cluster point of the net S if and only if x is a cluster point of the filter $\mathscr{F}$.

proof

Assume S converges to x.

Let U be any neighbourhood of x in X.

Then there exists m ∈ D such that B_m ∈ U, where B_m = {S(n) : n ∈ D, n ≥ m}.

But this means U ∈ F by the definition of $\mathscr{F}$.

So $\eta_x \subset \mathscr{F}$,

i.e, $\mathscr{F}$ converges to x.

Conversely suppose $\mathscr{F}$ converges to x.

Let U be any neighbourhood of x.

Then U $\in \mathscr{F}$. By the construction of $\mathscr{F}$, there exists m $\in$ D such that $B_m \subset$ U, where B_m is defined above.

$\Rightarrow$ S(n) $\in$ U n $\in$ D , n $\geq$ m.

Thus S $\rightarrow$ x in X.

Now for the second part , suppose x is a cluster point of the net S,U be any neighbourhood of x.

Let F $\in \mathscr{F}$.

Then we have to show U $\cap$ F $\neq \phi$.

F $\in \mathscr{F} \Rightarrow \exists$ m $\in$ D $\ni B_m \subset$ F.

U is a neighbourhood of x and m $\in$ D

$\Rightarrow$ n $\in$ D $\ni$ n $\geq$ m and S(n) $\in$ U (by the definition of cluster point of a net)

$\Rightarrow$ S(n) $\in$ U $\cap$ F.

Since U and F are arbitrary, we get x is a cluster point of $\mathscr{F}$.

Conversely, suppose x is a cluster point of $\mathscr{F}$.

Let U be any neighbourhood of x and $m_0 \in$ D.

We have B_mm$\in$D where B_m = S(n) : n $\geq$ m is a base for $\mathscr{F}$.

$\Rightarrow B_m \in \mathscr{F}$.

There fore B$m_0 \in \mathscr{F}$.

Hence Bm_0 U $\neq = \phi$,

x is a cluster point of $\mathscr{F}$.

There fore $\exists$ n $\geq m_0 \ni$ S(n) $\in$ U.

Since neighbourhood N of x and $m_0 \in$ D are arbitrary, we get x is a cluster point of S.

Proposition 3.1.4. Let $\mathscr{F}$ be a filter in a space X and S be the associated net in X. Let x $\in$ X. Let $\mathscr{F}$ converges to x as a filter if and only if S converges to x as a net. Moreover, x is a cluster point of the filter $\mathscr{F}$ if and only if it is a cluster point of the net S

3.2 Characterisation of filters

Hausdorff property

Theorem 3.2.1: A topological space is a Hausdorff if and only if no filter can converge to more than one point in it.

proof

Suppose X is a Hausdorff space and $\mathscr{F}$ is a filter on X converging to x and y.

This means $\eta_x \subset \mathscr{F}$ and $\eta_y \subset \mathscr{F}$, where η_x and η_y are the neighbourhood system of x and y respectively.

Now if x $\neq$ y $\exists$ U $\in \eta_x$ and V $\in \eta_y \ni$ U $\cap$ V $= \phi$, which contradict that $\mathscr{F}$ has finite intersection property.

So x = y.

Conversely assume that no filter in X converges to more than one limit in X.

If X is not Hausdorff , there exists x, y $\in$ X, x $\neq$ y such that every neighbourhood of x intersects every neighbourhood of y.

$\therefore$ $\eta_x \cup \eta_y$ has finite intersection property,so there exists a filter $\mathscr{F}$ on X containing $\eta_x \cup \eta_x$.

Then clearly $\mathscr{F}$ converges to both x and y which contradict our hypothesis.

So X is a hausdorff space.

Theorem 3.2.2: For a topological space X, the following statements are equivalent:

(1) X is compact.

(2) Every filter on X has a cluster point.

(3) Every filter on X has a convergent subfilter.

Proof is followed from the equivalence condition of compactness using nets by Theorem 2.2.4.

Definition 3.2.1.Image Filter

Suppose X,Y are topological spaces and f : X $\rightarrow$ Y is a function. Then the image filter on Y is defined as the filter generated by the base $\{f(A) : A \in \mathscr{F}\}$. The image filter is denoted by f($\mathscr{F}$).

Claim: $\mathscr{B} = \{f(A) : A \in \mathscr{F}\}$ is a base for a filter.

Evidently $\phi \notin$ f($\mathscr{F}$).

Also for B_1, $B_2 \in \mathscr{B}$ there exists A_1, $A_2 \in \mathscr{F} \ni$ f(A_1) =B_1 and f(A_2) = B_2 .

We know that $A_1 \cap A_2 \in \mathscr{F}$ and $B_1 \cap B_2 \supset$ f($A_1 \cap A_2$).

Also f($A_1 \cap A_2$) $\in \mathscr{B}$.

i.e, for any B_1, $B_2 \in \mathscr{B}$ we are able to find B_3= f($A_1 \cap A_2$) $\in \mathscr{B} \ni$ $B_3 \subset B_1 \cap B_2$.

Therefore $\mathscr{B}$ is a base for a filter.

Continuity of a function

Proposition 3.2.1: Let X, Y be a topological space, x X, and f : X $\to$ Y a function. Then f is continuous at x if and only if whenever a filter $\mathscr{F}$ converges to x in X, the image filter f($\mathscr{F}$) of $\mathscr{F}$ converges to f(x) in Y.

proof

Assume first that f is continuous at x and $\mathscr{F}$ is a filter which converges to x in X.

We have to show f($\mathscr{F}$) converges to f(x) in Y.

Let N be any neighbourhood of f(x) in Y.

Then by continuity f^1(N) is a neighbourhood of x in X.

By convergence of $\mathscr{F}$ to x f^1(N) $\in \mathscr{F}$.

So N $\supseteq f^1$(N) $\in$ f($\mathscr{F}$) .

N is an arbitrary neighbourhood of f(x), implies f($\mathscr{F}$) converges to f(x).

Conversely suppose the condition about filter is satisfied.

We have to show that f is continuous at x. If not there exists a neighbourhood N of f(x) such that f^1(N)is not a neighbourhood of x.

So every neighbourhood of x intersects X / f^1(N) and hence the family, S = $\eta_x \cup$ X / f^1(N)has finite intersection property.

Therefore S generates a filter $\mathscr{F}$ and converges to x.

However f($\mathscr{F}$) is not converging to f(x).

Since X / f^1(N) $\in \mathscr{F}$ we get f(X /f^1(N)) $\in$ f($\mathscr{F}$) .

But Y / N contains f(X / f^1(N)) Y / N $\in$ f($\mathscr{F}$).

Therefore N $\notin$ f($\mathscr{F}$) [no filter can contain both a set and its compliment].

Thus f($\mathscr{F}$) does not converge to f(x). This contradiction proves that f is continuous at x.

Theorem 3.2.3: Let X be a topological product of an indexed family of spaces $\{X_i$: i $\in$ I$\}$. Let F be a filter on X and let x $\in$ X. Then F converges to a point x in X if and only if for each i $\in$ I, the filter π_i(F) converges to π_i(x) in X_i.

proof

The necessity condition holds using the continuity of π_i and proposition 3.2.1.

Conversely suppose that π_i(F) converges to π_i(x) in X_i for all i.

Claim: F $\rightarrow$ x in X.

Let N be any neighbourhood of x.

Then N contains basic open set V of x.

Let V = $\Pi_{i \in I}$ where $V_i = X_i$ for all i except for some finite i_1, i_2, ..., i_n.

Without loss of generality let it be 1,2,...,n.

Now π_i(F) $\rightarrow$ π_i(x) $\forall$ i = 1, 2, ..., n.

So $\exists$ $F_i \in$ F $\ni$ Vi $\supset$ $\pi_i(F_i)$ for i = 1, 2, ..., n .

But π_i^{-1}(V$_i$)$\supset$ $\pi_i^{-1}(\pi_i(F_i))\supset$ $_i$ for i=1,2,.......n.

So N $\supset$ V = $\bigcap_{i=1}^{n}$ π_i^{-1}(V$_i$)$\supset$ $\bigcap_{i=1}^{n}$ F_i

But $\bigcap_{i=1}^{n}$ F_i is in $\mathscr{F}$ since $\mathscr{F}$ is closed under finite intersection.

N $\in \mathscr{F} \Rightarrow \eta_x \supset \mathscr{F}$.

$\therefore$ $\mathscr{F}$ converges to x.

3.3 Ultrafilters

Definition 3.3.1. Ultrafilter

An ultrafilter is a maximal filter on X. i.e, a filter $\mathscr{F}$ is an ultrafilter on X if $\mathscr{G}$ is any other filter on X with $\mathscr{F} \subset \mathscr{G}$, then $\mathscr{F} = \mathscr{G}$.

Theorem 3.3.1. Let $\mathscr{F}$ be a filter on a set X,then there is an ultrafilter $\overline{\mathscr{F}}$ on X with $\overline{\mathscr{F}} \supset \mathscr{F}$.

proof

(a) Let $\{\mathscr{F}_i\}_{i\in I}$ be a chain (i.e, totally ordered set of filters on X) of collection of filters containing $\mathscr{F}$ with the inclusion ordering.
For the chain, take $\cup_{i\in I} \mathscr{F}_i = \mathscr{G}$ and $\mathscr{F} \subset \mathscr{F}_i$ $\forall$i $\Rightarrow \mathscr{F} \subseteq \mathscr{G}$ is a filter which is an upper bound for the chain.
So there exist a maximal filter$\overline{\mathscr{F}}$ which contains $\mathscr{F}$ by Zorn's lemma, as we required.

Proposition 3.3.2. For a filter $\mathscr{F}$ on X, the following statements are equivalent:
(i) For every subset Y $\subset$ X we have Y $\in \mathscr{F}$ or X / Y $\in \mathscr{F}$.
(ii) $\mathscr{F}$ is an ultrafilter.

proof

(i) $\Rightarrow$ (ii)
For all Y $\subset$ X either Y $\in \mathscr{F}$ or X / Y $\in \mathscr{F}$.
If there exist a larger filter then it contains both Y and X/Y which is not possible as Y $\cap$ (X / Y) = ϕ.
(ii) $\Rightarrow$ (i)
Let Y $\subset$ X. If there exist A_1, $A_2 \in \mathscr{F}$ such that Y $\cap$ A_1 = (X / Y) $\cap$ $A_2 = \phi \Rightarrow A_1 \cap A_2 = \phi$ which is a contradiction.

So we must have

Y $\cap$ A = ϕ A $\in \mathscr{F}$ or (X / Y) $\cap$ A $\neq \phi$ $\forall$ A $\in \mathscr{F}$.

If Y $\cap$ A $\neq \phi$ $\forall$ A $\in \mathscr{F}$, then by (part b) there exist a filter $\mathscr{G}$ containing $\mathscr{F} \cup \{$Y ,since $\mathscr{F}$ is an ultrafilter.

So we must have $\mathscr{G} = \mathscr{F}$.

If (X / Y) $\cap$ A $\neq \phi$ A $\in \mathscr{F}$, similarly we will get $\mathscr{G} \supseteq \mathscr{F} \cup \{$X / Y $\} \Rightarrow \mathscr{G} = \mathscr{F}$.

$\mathscr{G} = \mathscr{F}$ implies either Y $\in \mathscr{G}$ or X / Y $\in \mathscr{G}$.

Proposition 3.3.3. Let $\mathscr{F}$ be an ultrafilter on X, let f : X $\to$ Y be a function and let f($\mathscr{F}$) = {Y $\supset$ f(A): A $\in \mathscr{F}$} is an ultrafilter of Y.

proof

We will use the characterisation of Proposition 3.3.2.

Let W $\subset$ Y then f^{-1}(Y / W) = X / f^{-1}(W).

Since $\mathscr{F}$ is an ultrafilter either f^{-1}(W) $\in \mathscr{F}$ or f^{-1}(Y /W) $\in \mathscr{F}$.

$$W \supset f(\ f^{-1}(W)) \in f(\mathscr{F}).$$

where as,

$$Y\ /\ W \supset f(\ f^{-1}(Y\ \ W)) \in f(\mathscr{F}).$$

So f($\mathscr{F}$) is an ultrafilter on Y .

Proposition 3.3.4. An ultrafilter $\mathscr{F}$ converges to a point iff that point is a limit point of it.

proof Suppose x is a limit point.

Then if U is a neighbourhood of x, U intersects A $\forall$ A $\in \mathscr{F}$.
By part (b) of Proposition 3.1.2. there exist $\mathscr{G}$ containing $\mathscr{F} \cup \{U\}$ but $\mathscr{F}$ is maximal which implies that U $\in \mathscr{F}$.
i.e, U $\in \mathscr{F}$ for every neighbourhood U of x implies $\eta_x \subseteq \mathscr{F} \Rightarrow \mathscr{F}$ converges to x.
Conversely suppose $\mathscr{F}$ converges to x.
Then any neighbourhood U $\in \mathscr{F} \Rightarrow$ U $\cap$ A $\neq \phi$ $\forall$ A $\in \mathscr{F} \Rightarrow$ x is a limit point of $\mathscr{F}$.

Proposition 3.3.5. A topological space is compact iff every ultrafilter in it is convergent.

proof Suppose X is compact.

Suppose $\mathscr{F}$ is an ultrafilter and it has no limit point.
Let x $\in$ X and U_x be a neighbourhood of x.
We know either U_x or $(U_x)^c \in \mathscr{F}$ (by the characterisation of ultrafilter).
Since there exist A $\in \mathscr{F}$ such that A $\cap U_x = \phi \Rightarrow (U_x)^c \in \mathscr{F}$
$\{U_x\}_{x \in X}$ is a cover for X.
Hence it has finite subcover.
Let it be U_{x1}, U_{x2}, ..., U_{xn} such that $\bigcup_{i=1}^{n} U_{xi} =$ X.
Correspondingly there exist A_1, A_2, ..., A_n such that $A_i \cap U_{xi} = \phi$.

Clearly A = $A_1 \cap A_2 \cap ... \cap A_n \in \mathscr{F}$.

$(U_{x1})^c \in$ F such that $(U_{x1})^c = \bigcup_{i=2}^{n} U_{xi} \in$ F $\mathscr{F}$.

Then,

$$A \cap (U_{x1})^c = A \cap (\bigcup_{i=2}^{n} U_{xi}\ ) = \phi.$$

which is a contradiction.

Hence there exist a limit point for $\mathscr{F} \Rightarrow$ every ultrafilter is convergent.

Conversely suppose every ultrafilter in it is convergent by proposition 3.3.3.

Suppose X is not compact.

Then there exist collection $\mathscr{C}$ of closed sets of X having finite intersection property and $\bigcap_{C \in \mathscr{C}}$ C = ϕ ($\Rightarrow \{C^c$: C $\in \mathscr{C}$ } is an open cover for X which has no finite sub-cover.)

Any collection of sets having finite intersection property is a sub-base for a filter.

Hence $\exists$ a ultrafilter $\mathscr{G}$ containing $\mathscr{C}$.

Let x be a limit point of $\mathscr{G}$ then there exists X / C $\ni$ x $\in$ X / C ($\therefore$ $\{C^c$: C $\in \mathscr{C}$ } is an open cover).

As x is a limit point X/C intersect all elements of $\mathscr{G}$ but C $\in \mathscr{C} \subseteq \mathscr{G}$. So C $\cap$ X / C = ϕ which is not possible.

Therefore $\nexists$ any limit point for $\mathscr{G}$, which is a contradiction.

Hence X is Compact.

CONCLUSION

The importance of nets and filters in general topology, especially where compactness is involved, is established here. The nets were introduced by E H Moore in the early 20th century while the idea of filters and its applications are primarily developed by European mathematicians, beginning with the work of H Cartan which is followed by Bourbaki. Some topological properties which are not easy to characterise using sequences are also characterised with the help of nets and filters. However, the importance of convergence theory of nets and filters extends beyond this project.

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DON BOSCO ARTS & SCIENCE COLLEGE ANGADIKADAVU

DEPARTMENT OF MATHEMATICS

2021-2023

Project Report on

PECULIAR FORMS OF NUMBERS AND RELATED THEOREMS

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BONAFIDE CERTIFICATE

Certified that this project report "PECULIAR FORMS OF NUMBERS AND RELATED THEOREMS" is the bonafide work of ANAGHA. C who carried out the project work under my supervision.

Signature of Head   Signature of Supervisor

DECLARATION

I, ANAGHA.C hereby declare that the Project work entitled PECULIAR FORMS OF NUMBERS AND RELATED THEOREMS has been prepared by me and submitted to Kannur University in partial fulfillment of requirement for the award of Master of Science is a record of original work done by me under the supervision of Mrs. NAJUMUNNISA K, Assistant Professor, Department of Mathematics, Don Bosco Arts & Science College, Angadikadavu. I, also declare that this Project work has not been submitted by me fully for the award of any Degree, Diploma, Title or recognition before any authority.

Place: Angadikadavu ANAGHA.C

Date: (C1PSMM1903)

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ANAGHA.C

CONTENTS

INTRODUCTION

Number theory an old branch of mathematics is the science of studying the properties of numbers. In accordance with research methods and objective, we briefly divided Number theory into four classes:Elementary Number theory, Analytic Number theory,Geometric Number theory.

The theory of numbers has always occupied a unique position in the World of mathematics. Number theory, by natural a discipline that demanded a high standard of rigor. The aim of the elementary theory of numbers is to investigate the properties of integers.

There are plenty of numbers with some interesting and magical properties in the history of the numbers. The numbers with some unique and fascinating properties have always received a great attention in the world of mathematics. There are so many kinds of numbers like $Perfect$ number, $Mersenne$ numbers , $Fermat$ numbers ,$fibanocci$ etc which shows some special characteristics. it was well said by great mathematician Kronecker that "God created the natural numbers,and all rest is the work of man".

The second chapter deals with perfect numbers and related theorems of perfect numbers. The third chapter consists of study on $Mersenne$ numbers with some theorem relating $perfect\ and\ mersenne$ numbers. In fourth chapter we discus about $Fermat$ numbers with some theorems.Finally in the last chapter we discus about a sequence called $fibanocci$ and its terms are called $fibanocci$ numbers

The main aim of this project is to provide introduction to some "Numbers of peculiar form" and some of its basic properties and characteristics with some theorems.

CHAPTER - 1

PRELIMINARY RESULTS

Definition 1.1

Let n be a fixed integer. Two integers a and b are said to be congruent module n,

$$a \equiv b(modn)$$

If n divides the difference $a - b$; that is, provided that $a - b = kn$ for some integer k.

Example

(i) $15 \equiv 3(mod4)$

(ii) $3^2 \equiv -1(mod5)$

Theorem 1.2

Assume $a \equiv b(modm)$ if d/m and d/a then d/b.

Definition 1.3

Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of n and $\sigma(n)$ denoted the sum of these divisors.

Theorem 1.4

If $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then

$$\tau(a)=(k_1 + 1)(k_2 + 1) \cdots (k_r + 1)$$

$$\sigma(n)= \frac{P_1^{k_1+1}-1}{p_1-1} \frac{P_2^{k_2+1}-1}{p_2-1}\cdots \frac{P_r^{k_r+1}-1}{p_r-1}$$

Definition 1.5

For a real or complex α and any integer $n \geq 1$ we define

$$\sigma_\alpha(n) = \sum_{d/n}$$

the sum of the divisors of n.

Definition 1.6

An arithmetic function f is called multiplicative if f is not identically zero and if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$

Theorem 1.7

$FERMAT'S$ theorem

Let p be a prime and suppose that $p \not| a$ then $a^{p-1} \equiv 1(modp)$

Theorem 1.8

If p is a prime and $p\Big|ab$ then $p\Big|a$ or $p\Big|b$.

Theorem 1.9

If p is a prime ,then

$$(2\Big|p) = \begin{cases} 1 & \text{if} \quad p \equiv 1(mod8)orp \equiv 7(mod8) \\ -1 & \text{if} \quad p \equiv 3(mod8)orp \equiv 5(mod8). \end{cases}$$

Theorem 1.10

Let the integer a have order k modulo n. Then $a^h \equiv 1(modn)$ if and only if $k\Big|h$; in particular, $k\Big|\phi(n)$.

Theorem 1.11

Euler's criterion : Let p be an odd prime and $gcd(a, p) = 1$. Then a is a quadratic residue of p if and only if $a^{(p-l)/2} \equiv 1(modp)$.

Definition 1.12

For $n \geq 1$, let $\phi(n)$ denote the number of positive integers not exceeding n that are relatively prime to n.

Theorem 1.13

Let the integer a have order k modulo n. Then $a^h \equiv 1(modn)$ if and only if k/h; in particular, $k/\phi(n)$.

Definition 1.14

An integer q is called a quadratic residue modulo n if it is congruent to a perfect square modulo n.
ie, $x^2 \equiv q(modn)$
Otherwise, q is called a quadratic non-residue modulo n.

Definition 1.15

Legendre symbol is a function of

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p}, \\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } p, \\ 0 & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

Legendre's original definition was by means of the explicit formula

$\left(\dfrac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$ and $\left(\dfrac{a}{p}\right) \in \{-1, 0, 1\}$.

Definition 1.17

The **Euclidean Algorithm** for finding $gcd(A, B)$ is as follows:

(1) If A $= 0$ then $gcd(A, B) = B$, since the $gcd(0, B) = B$, and we can stop.

(2) If B $= 0$ then $gcd(A, B) = A$, since the $gcd(A, 0) = A$, and we can stop.

(3) Write A in quotient remainder form $(A = B \cdot Q + R)$

(4) Find $gcd(B, R)$ using the Euclidean Algorithm since $gcd(A, B) = gcd(B, R)$

Lemma :18

Suppose a and b are not both zero.

a) (a,b)=(b,a)

b) if $a > 0$ and a/b then $(a, b) = a$

c) if $a \equiv$ c(mod b), then $(a, b) = (c, b)$

Example :

$A = 78,\ B = 36$

(a) $A \neq 0$

(b) $B \neq 0$

(c) Use long division to find that $78/36 = 2$ with a remainder of 6.

We can write this as:

(d) $78 = 36 \cdot 2 + 6$

(e) Find $gcd(36, 6)$, since $gcd(78, 36) = gcd(36, 6)$

Theorem 1.19

$a^r = (a - 1)(a^{r-1} - a^{r-2} \cdots a + 1)$

Theorem 1.20

if d/a and d/b then $d/a - b$

Theorem 1.21

$\frac{a^x - 1}{a+1} = a^{2x-1} - a^{2x-2} + \cdots + a - 1$

NOTATION

P_n	:	Perfect Numbers
M_n	:	Mersenne Numbers
M_p	:	Mersenne Prime Numbers
u_n	:	Fibonacci Numbers
F_n	:	Fermat Numbers

CHAPTER - 2

PERFECT NUMBERS

The history of the theory of numbers abounds with famous conjectures and open question. The present chapter focuses on some of the intriguing conjectures associated with perfect numbers. A few of these have been satisfactorily answered, but most remain unresolved; all have stimulated the development of the subject as a whole.

The $Pythagorean's$ considered it rather remarkable that the number 6is equal to the sum of its positive divisor, other than it self:

$$6 = 1 + 2 + 3$$

The next number after 6 having this feature is 28; for the positive divisors of 28 are found to be1,2,4,7,14,and 28, and

28 =1+2+4+7+14

In line with their philosophy of attributing mystical qualities to numbers, the $Pythagorean's$ called such numbers "perfect". We state this precisely in definitions.

Definition 2.1

A positive integer $n$is said to be perfect if n is equal to the sum of all its positive divisors, excluding n itself.

The sum of the positive divisors of an integer n ,each of them less than n, is given by $\sigma(n)$ - n.Thus, the condition "n is perfect" amounts to asking that $\sigma(n)$ - $n = n$ or equivalently, that

$$\sigma((n) = 2n$$

For example, we have

$$\sigma(6) = 1 + 2 + 3 + 6 = 2 \cdot 6$$

and

$$\sigma(28) = 1 + 2 + 4 + 7 + 14 + 28 = 2 \cdot 28$$

so that 6 an 28 are both perfect numbers.

For many centuries , philosophers were more concerned with the mystical or religious significance of perfect numbers than with their mathematical properties.

Only four perfect numbers were known to the ancient Greeks. *Nicomachus* in his *Introductio Arithmeticae* (circa 100 A.D) lists

$$P_1 = 6 \qquad P_2 = 28 \qquad P_3 = 496 \qquad P_4 = 8128$$

He says that they are formed in an "orderly" fashion,one among the units, one among the tens, one among the hundreds, and one among the thousands (this is, less than 10,000). Based on this meager evidence, it was conjectured that

1. The n^{th} perfect number P_n contains exactly n digits; and

2. The even perfect numbers end , alternately, in 6 and 8.

Both assertions are wrong. There is no perfect number with 5 digits; the next perfect number (first given correctly in an anonymous 15^{th} century manuscript) is

$$P_5 = 33550336$$

While the final digit of P_5 is 6, the succeeding perfect number namely,

$$P_6 = 8589869056$$

Also ends in 6 ,not 8 as conjectured. To salvage something positive direction, we shall show later that the even perfect number do always end in 6 or 8-but not necessarily alternately.

If nothing else, the magnitude of P_6 should convince the reader of the rarity of perfect number. It is not yet known whether there are finitely many or infinitely many of them.

The problem of determining the general form of all perfect numbers dates back almost t the beginning of mathematical time. It was partially solved by Euclid of the elements he proved that if the sum

$$1+2+2^2+2^3+....+ 2^{k-^1} =\text{p}$$

is a prime number , then $2^{k-}1p$ is a perfect number (of necessity even). For instance, 1+2+4=7 is a prime; hence, $4 \cdot 7 = 28$ is a perfect number. Euclid's argument makes use of the formula for the sum of a geometric progression.

$$1 + 2 + 2^2 + 2^3 + ... + 2^{k-1} = 2^k - 1$$

which is found in various *Pythagoreans* texts. In this notation, the result read as follow : If $2^k - 1$, is prime (k>1) , then n=$(2^{k-1})(2^k$ -1) is perfect number. About 2000 years after Euclid, Euler took a decisive step in proving that all even numbers must be of this type. We incorporate both these statements in our first theorem.

Theorem - 2.2

If 2^k -1 is prime (k>1), then n=$(2^{k-1})(2^k$ -1) is perfect and every even perfect number is of this form.

Proof :

Let 2^k -1 $=P$, a prime, and consider the integer n=$(2^{k-1})P$. Inasmuch as $\gcd(2^k - 1, P) = 1$, the multiplicativity of σ (as well as Theorem 1.6) entails that

$$\sigma\ (n) = \sigma(2^{k-1})P)$$

$$=(2^k -1)(P+1)$$

$$=(2^k -1)2^k$$

$$=2n$$

making n a perfect number. For the converse, assume that n is an even perfect number. We may write n as n=$(2^{k-1})m$, where m is an odd integer and $k \geq 2$. It follows from $\gcd(2^k - 1, m) = 1$ that

$$\sigma\ (\text{n})\ =\ \sigma\ ((2^{k-1})m)$$

$$=\sigma\ (2^{k-1})\ \sigma\ (m)$$

$$=(2^k-1)\ \sigma\ (m)$$

whereas the requirement for a number to be perfect gives

$$\sigma\ (\text{n})\ \ =2n\ \ =2^k m$$

Together, these relations yield

$$2^k m{=}2^k-1)\sigma(m)$$

which is simply to say that $(2^k-1)/2^k m$. But 2^k-1 and 2^k are relatively prime , whence $2^k-1)/m$; say, $m=2^k-1M$. Now the result of substituting this value of m into the last-displayed equation and canceling 2^k-1 is that $\sigma(m)=2^k m$. Because m and M are both divisors of m (with $M<m$), we have

$$2^k m=\sigma(m)\geq m+M=2^k M$$

leading to $\sigma(m)=m+M$. The implication of this equality is that m has only two positive divisors, to wit, M and m itself. It must be that m is prime and $M=1$; in other words, $m=(2^k-1)M=(2^k-1)$ is a prime number,completing the present proof.

Because the problem of finding even perfect numbers is reduced to the search for primes of the form (2^k-1), a closer look at these integers might be fruitful. One thing that can be proved is that if

$2^k - 1$ is a prime number, then the exponent k must itself be prime. More generally, we have the following lemma.

Lemma - 2.3

If $a^k - 1$ is prime (a > 0, $k \geq 2$), then $a = 2$ and k is also prime

Proof :

It can be verified without difficulty that

$$a^k - 1 = (a - 1)(a^{k-1} + a^{k-2} + \ + a + 1)$$

where, in the present setting,

$$a^{k-1} + a^{k-2} + ... + a + 1 \geq \text{a} + 1 > 1$$

Because by hypothesis $a^k - 1$ is prime, the other factor must be 1; that is,a - 1 = 1 so that a= 2.
If k were composite, then we could write $k = r$s, with $1 < r$ and $1 < s$. Thus,

$$a^k - 1 = (a^r)^s - 1$$

$$=(a^r - l)(a^{r(s-1)} + a^{r^{(s-2)}} + ... + a^r + 1)$$

and each factor on the right is plainly greater than 1. But this violates the primality of $a^k - 1$, so that by contradiction k must be prime

For p = 2, 3, 5, 7, the values 3, 7, 31, 127 of $2^P - 1$ are primes, so that are all perfect numbers.

$$2(2^2 - 1) = 6$$

$$2^2(2^3 - 1) = 28$$

$$2^4(2^5 - 1) = 496$$

$$2^6(2^7 - 1) = 8128$$

Many early writers erroneously believed that $2^P - 1$
is prime for every choice of the prime number p. But in 1536, Hudalrichus Regius in a work entitled *Utriusque Arithmetices* exhibits the correct factorization

$$2^{11} - 1 = 2047 = 23.89$$

If this seems a small accomplishment, it should be realized that his calculations were in all likelihood carried out in Roman numerals, with the aid of an abacus (not until the late 16^{th} century did the Arabic numeral system win complete ascendancy over the Roman one). Regius also gave $p = 13$ as the next value of p for which the expression $2^P - 1$ is a prime. From this, we obtain the fifth perfect number.

$$2^{12}(2^{13} - 1) = 33550336$$

One of the difficulties in finding further perfect numbers was the unavailability of tables of primes. In 1603, Pietro Cataldi, who is remembered chiefly for his invention of the notation for continued fractions, published a list of all primes less than 5150. By the direct procedure of dividing by all primes not exceeding the square root of a number, Cataldi determined that $2^{17} - 1$ was prime and, in consequence, that

$$2^{16}(2^{17} - 1) = 8589869056$$

is the sixth perfect number. A question that immediately springs to mind is whether there are infinitely many primes of the type $2^P - 1$, with p a prime. If the answer were in the affirmative, then there would exist an infinitude of (even) perfect numbers. Unfortunately, this is another famous unresolved problem.
This appears to be as good a place as any at which to prove our theorem on the final digits of even perfect numbers.

Theorem - 2.4

An even perfect number n ends in the digit 6 or 8; equivalently, either $n \equiv 6(mod 10)$ or $n \equiv 8(mod 10)$.

Proof :

Being an even perfect number, n may be represented as
$n = 2^{k-l}(2^k - 1)$, where $2^k - 1$ is a prime. According to the last lemma, the exponent k must also be prime. If$k = 2$, then$n = 6$, and the asserted result holds. We may therefore confine our attention to the case $k > 2$. The proof falls into two parts, according as k takes the form $4m + 1$ or $4m + 3$
If k is of the form $4m + 1$, then

$$\begin{aligned} n &= 2^{4m}(2^{4m+1} - 1) \\ &= 2^{8m+1} - 2^{4m} \end{aligned}$$

$$= 2.16^{2m} - 16^m$$

A straightforward induction argument will make it clear that $16^t \equiv 6(mod 10)$ for any positive integer t. Utilizing this congruence, we get

$$n \equiv 26 - 6 \equiv 6(mod 10)$$

Now, in the case in which $k = 4m + 3$,

$$n = 2^{4m+2}(2^{4m+3} - 1)$$
$$= 2^{8m+5} - 2^{4m+2}$$
$$= 2^{8m}.2^5 - 4.16^m$$
$$= 2.16^{2m}.16 - 4.16^m$$
$$= 2.16^{2m+1} - 4.16^m$$

Falling back on the fact that $16^t \equiv 6(mod 10)$, we see that

$$n \equiv 26 - 46 \equiv -12 \equiv 8(mod 10)$$

Consequently, every even perfect number has a last digit equal to 6 or to 8.
Hence the proof.

A little more argument establishes a sharper result, namely, that any even perfect number $n = 2^{k-1}(2^k - 1)$ always ends in the digits 6 or 28. Because an integer is congruent modulo 100 to its last two digits, it suffices to prove that, if k is of the form $4m + 3$, then $n \equiv 28(mod 100)$. To see this, note that

$$2^{k-1} = 2^{4m+2} = 16^m 4 \equiv 64 \equiv 4(mod 10)$$

Moreover, for $k > 2$, we have $4/2^{k-1}$, and therefore the number formed by the last two digits of 2^{k-1} is divisible by 4. The situation is this: The last digit of 2^{k-1} is 4, and 4 divides the last two digits. Modulo 100, the various possibilities are

$$2^{k-1} \equiv 4,\ 24,\ 44,\ 64,\ or\ \ 84$$

But this implies that

$$2^k - 1 = 2.2^{k-1} - 1 \equiv\ 7,\ 47,\ 87,\ 27, or\, 67\, (mod 100)$$

whence

$$n = 2^{k-1}(2^k - 1)$$

$$\equiv 47,\ 2447,\ 4487,\ 6427,\ or\, 84.67(mod 100)$$

It is a modest exercise, which we bequeath to the reader, to verify that each of the products on the right-hand side of the last congruence is congruent to 28 modulo 100.

CHAPTER - 3

MERSENNE PRIMES

It has become traditional to call numbers of the form

$$M_n = 2^n - 1 \qquad \text{n} > 1$$

Mersenne numbers after Father Marin Mersenne who made an incorrect but provocative assertion concerning their primality.

Now let us consider some Mersenne numbers

M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
1	3	7	15	31	63	127	255

Those Mersenne numbers that happen to be prime are said to be *Mersenne primes*. By what we proved in above, the determination of Mersenne primes Mn—and, in turn, of even perfect numbers—is narrowed down to the case in which n is itself prime.

Now let us see the first 5 Mersenne primes numbers.

M_2	M_3	M_5	M_7	M_{13}
3	7	31	127	8191

In the preface of his Cogitata Physica-Mathematica (1644), Mersenne stated that M_p is prime for $p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127,$ 257 and composite for all other primes

$p < 257$. It was obvious to other mathematicians that Mersenne could not have tested for primality all the numbers he had announced; but neither could they. Euler verified (1772) that M_{31} was prime by examining all primes up to 46339 as possible divisors, but M_{67} , M_{127}, and M_{257} were beyond his technique; in any event, this yielded the eighth perfect number

$$2^{30}(2^{31}-1) = 2305843008139952128$$

It was not until 1947, after tremendous labor caused by unreliable desk calculators, that the examination of the prime or composite character of M_P for the 55 primes in the range $p < 257$ was completed. We know now that $Mersenne$ made five mistakes. He erroneously concluded that $M_{67}$and M_{257} are prime and excluded M_{61}, M_{89}, and M_{107} from his predicted list of primes. It is rather astonishing that over 300 years were required to set the good friar straight. All the composite numbers M_n with $n \leq 257$ have now been completely factored.

In the study of Mersenne numbers, we come upon a strange fact: when each of the first four Mersenne primes (namely, 3, 7, 31, and 127) is substituted for n in the formula $2^n - 1$, a higher $Mersenne$ prime is obtained. Mathematicians had hoped that this procedure would give rise to an infinite set of $Mersenne$ primes; in other words, the conjecture was that if the number M_n is prime, then M_{M_n} is also a prime.

Alas, in 1953 a high-speed computer found the next possibility

$$M_{M_{13}} = 2^{M_{13}} - 1 = 2^{8191} - 1$$

to be composite.

There are various methods for determining whether certain special types of *Mersenne* numbers are prime or composite. One such test is presented next.

Theorem - 3.1

If p and $q = 2p + 1$ are primes, then either $q \Big| M_p$ or $q \Big| M_p + 2$, but not both.

Proof :

With reference to Fermat's theorem, we know that

$$2^{q-1} - 1 \equiv 0 (mod q)$$

and, factoring the left-hand side, that

$$(2^{(q-1)/2} - 1)(2^{(q-1)/2)} + 1) = (2^P - 1)(2^P + 1)$$
$$\equiv 0 (mod q)$$

What amounts to the same thing:

$$M_p(M_p + 2) \equiv 0 (mod q)$$

The stated conclusion now follows directly from Theorem 1.8. We cannot have both $q \Big| M_p$ and $q \Big| M_p + 2$, for then $q \Big| 2$, which is impossible.

EXAMPLE 1

Let us consider $p = 23$,then $q = 2p + 1$ =47 is also a prime. so that we may consider the case of M_{23}. The question reduces to one of whether $47 \big| M_{23}$ or, to put it differently, whether $2^{23} \equiv 1(mod 47)$. Now, we have

$$\begin{aligned} 2^{23} &= 2^3 2^{20} \\ &= 2^3(2^5)^4 \\ &\equiv 2^3(-15)^4(mod 47) \end{aligned}$$

Also we get

$$(-15)^4 = (225)^2 \equiv (-10)^2 \equiv 6(mod 47)$$

From these two congruence's we get

$$2^{23} \equiv 2^3 \cdot 6 \equiv 1(mod 47)$$

that is M_{23}.

EXAMPLE 2

Let $p = 29$ then $q = 2 \cdot 29 + 1 = 59$

$$\begin{aligned} M_p = M_{29} = 2^{29} - 1 &= 53, 68, 70, 912 - 1 \\ &= 53, 68, 70, 911 \end{aligned}$$

here neither $q \big| M_p$ nor $q \big| M_p + 2$. It reasonable to ask: What conditions on q will ensure that $q \big| M_P$? The answer is to be found in Theorem 3.2.

Theorem 3.2

If $q = 2n + 1$ is prime, then we have the following:

(a) $q \big| M_n$, provided that $q \equiv 1(mod 8)$ or $q \equiv 7(mod 8)$.

(b) $q \big| M_n + 2$, provided that $q \equiv 3(mod 8)$ or $q \equiv 5(mod 8)$.

Proof :

(a) To say that $q \big| M_n$ is equivalent to asserting that

$$2^{(q-l)/2} = 2^n \equiv 1(modq)$$

In terms of the Legendre symbol, the latter condition becomes the requirement that, $(2/q) = 1$. But according to Theorem 1.9 $(2/q) = 1$ when we have $q = 1(mod 8)$ or $q = 7(mod 8)$.

(b) Similerly we can say that $q/M_n + 2$ by

$$2^{(p-1)/2} = 2^n \equiv -1(modq)$$

In terms of Legende symbols the requeired condition becomes

$(2/q) = -1$

Also we have $(2/q) = -1$ when we have $q \equiv 3(mod 8)$ or $q \equiv 5(mod 8)$ from theorem 1.9.

Corollary :3.3

If p and $q = 2p + 1$ are both odd primes, with $p \equiv 3(mod 4)$, then q/M_p.

Proof :

We know that an odd prime is either of the form $4k + 1$ or $4k + 3$.

If p = 4k+3 then q = 2p+1 = 2(4k+3)+1 = 8k+7 then buy above theorem

$q \equiv 7(mod 8)$.

$\Rightarrow q/m_p$

If $p = 4k + 1$ then $q = 2p + 1 = 2(4k + 1) + 1 = 8k + 3$ then buy above theorem $q \equiv 3(mod 8)$. $\Rightarrow q/M_n + 2$ then by using theorem 3.1 $q \not| M_n$

Theorem 3.4

If p is an odd prime, then any prime divisor of M_p is of the form $2kp + 1$.

Proof :

We have,

$$M_p = 2^p - 1$$

Let q be any prime devisor of M_p then,

$$2^p \equiv 1(mod q)$$

If k is the smallest positive integer that satisfies $2^k \equiv 1(mod q)$ then by theorem 1.10 we get k/p.

The case $k = 1$ cannot arise; for this would imply that $q/1$, an impossible situation. Therefore, because both k/p and $k > 1$, the primality of p forces $k = p$.

In compliance with Fermat's theorem, we have $2q - 1 \equiv 1(mod q)$, and therefore, thanks to Theorem 1.10 again, $k/q - 1$. Knowing that $k = p$, the net result is $p/q - 1$. To be definite, let us put $q - 1 = pt$; then $q = pt + 1$. The proof is completed by noting that if t were an odd integer, then q would be even and a contradiction occurs. Hence, we must have $q = 2kp + 1$ for some choice of k, which gives q the required form.

Theorem 3.5

If p is an odd prime, then any prime divisor q of M_p is of the form $q \equiv \pm 1(mod 8)$.

Proof:

Suppose that q is a prime divisor of M_p, so that $2^P \equiv 1(modq)$. According to Theorem 3.4, q is of the form $q = 2kp + 1$ for some integer k. Thus, using *Euler's criterion*, $(2/q) \equiv 2^{(q-l)/2} \equiv 1(modq)$, whence $(2/q) = 1$. Theorem 1.9 can now be brought into play again to conclude that $q \equiv \pm 1(mod 8)$

An algorithm frequently used for **testing the primality** of M_p is the *Lucas – Lehmer* test. It relies on the inductively defined sequence

$$S_1 = 4 \qquad S_{k+1} = S_k^2 - 2 \qquad k \geq 1$$

Thus , the sequence begins with the values 4 , 14 , 194 , 37634 , · The basic theorem, as perfected by *Derrick Lehmer* in 1930 from the pioneering results of *Lucas*, is this : For $p > 2$, M_p is prime if and only if $S_{p-1} \equiv 0(modM_p)$. An equivalent formulation is that M_P is prime if and only if $S_{p-2} \equiv \pm 2^{(p+l)/2}(modM_p)$. A simple example is provided by the *Mersenne* number $M_7 = 2_7 - 1 = 127$.

Working modulo 127, the computation runs as follows:

S_1	S_2	S_3	S_4	S_5	S_6
4	14	67	42	-16	0

This establishes that M_7 is prime.

Theorem 3.6 : Euler

If n is an odd perfect number, then

$$n = p_1^{k_1} \cdot p_2^{2_{j_2}} \cdot p_3^{2_{j_3}} \cdots p_r^{2_{j_r}}$$

where the Pi 's are distinct odd primes and $P1 \equiv k_1 \equiv 1(mod 4)$.

Proof:

Let $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_r^{k_r}$ be the prime factorization of n. Because n is perfect, we can write

$$2n = \sigma(n) = \sigma(p_1^{k_1}) \cdot \sigma(p_2^{k_2}) \ldots \sigma(p_r^{k_r})$$

Being an odd integer, either $n \equiv 1(mod 4)$ or $n \equiv 3(mod 4)$; in any event, $2n \equiv 2(mod 4)$. Thus, $\sigma(n) = 2n$ is divisible by 2, but not by 4. The implication is that one of the $\sigma(p_i^{k_i})$, say $\sigma(p_1^{k_1})$, must be an even integer (but not divisible by 4), and all the remaining $\sigma(p_i^{k_i})$'s are odd integers.

For a given P_i, there are two cases to be considered: $P_i \equiv 1(mod 4)$ and $P_i \equiv 3(mod 4)$. If $P_i \equiv 3 equiv - 1(mod 4)$, we would have

$$\begin{aligned} \sigma(p_i^{k_i}) &= 1 + P_i + p_i^2 + \cdots + p_i^{k_i} \\ &\equiv 1 + (-1) + (-1)^2 + \cdots + (-1)_i^{k_i} (mod 4) \end{aligned}$$

$$\equiv \begin{cases} 0(mod4) & \text{if} \quad k_i \; is \; odd \\ 1(mod4) & \text{if} \quad k_i \; is \; even \end{cases}$$

Because $\sigma(p_1^{k_1}) \equiv 2(mod4)$, this tells us that $P_1 \not\equiv .3(mod4)$ or, to put it affirmatively, $p_1 \equiv 1(mod4)$. Furthermore, the congruence $\sigma(p_i^{k_i}) \equiv 0(mod4)$ signifies that 4 divides $\sigma(p_i^{k_i})$, which is not possible. The conclusion: if $P_i \equiv 3(mod4)$, where $i = 2, \cdots, r$, then its exponent k_i is an even integer.

Should it happen that $P_i \equiv 1(mod4)$—which is certainly true for $i = 1$—then

$$\sigma(p_i^{k_i}) = 1 + P_i + p_i^2 + \cdots + p_i^{k_i}$$

$$\equiv 1 + 1^1 + 1^2 + \;\; + 1^{k_i}; (mod4)$$

$$\equiv k_i + 1(mod4)$$

The condition $\sigma(p_1^{k_1}) \equiv 2(mod4)$ forces $k_1 \equiv 1(mod4)$. For the other values of i, we know that $\sigma(p_i^{k_i}) \equiv 1or3(mod4)$, and therefore $k_i \equiv 0or2(mod4)$; in any case, k_i is an even integer. The crucial point is that, regardless of whether $P_i \equiv 1(mod4)$ or $P_i \equiv 3(mod4)$, k_i is always even for $i \neq 1$. Our proof is now complete.
In view of the preceding theorem, any odd perfect number n can be expressed as

$$n = p_1^{k_1} \cdot p_2^{2_{j_2}} \cdot p_3^{2_{j_3}} \cdots p_r^{2_{j_r}}$$

$$= p_1^{k_1} (p_2^{j_2} \cdot p_3^{j_3} \cdots p_r^{j_r})^2$$

$$= p_1^{k_1} m^2$$

This leads directly to the following corollary

Corollary

If n is an odd perfect number, then n is of the form

$$n = p^k m^2$$

where p is a prime, $p \nmid m$, and $p \equiv k \equiv 1(mod4)$; in particular, $n \equiv 1(mod4)$.

Proof.

The last assertion is the only non-obvious one. Because $p \equiv 1(mod4)$, we have $p^k \equiv 1(mod4)$. Notice that m must be odd; hence, $m \equiv 1$ or $3(mod4)$, and therefore upon squaring, $m^2 \equiv 1(mod4)$. It follows that

$$n = p^k m^2 = 11 \equiv 1(mod4)$$

establishing our corollary.

CHAPTER - 4

FERMAT NUMBERS

Let us mention another class of numbers that provides a rich source of conjectures, the Fermat numbers. These may be considered as a special case of the integers of the form $2^m + 1$. We observe that if $2^m + 1$ is an odd prime, then $m = 2^n$ for some $n \geq 0$. Assume to the contrary that m had an odd divisor $2k + 1 > 1$, say $m = (2k + 1)r$; then $2^m + 1$ would admit the nontrivial factorization

$$2^m + 1 = 2^{(2k+1)r} + 1 = (2^r)^{2k+1} + 1$$
$$= (2^r + 1)(2^{2kr} - 2^{(2k-1)r} + ... + 2^{2r} - 2^r + 1)$$

which is impossible. In brief, $2^m + 1$ can be prime only if m is a power of 2.

Definition 4.1.

A Fermat number is an integer of the form

$$F_n = 2^{2^n} + 1, \qquad n \geq 0$$

If F_n is prime, it is said to be a Fermat prime.
Let us see first 5 prime Fermat numbers,

$F_0 = 3$ $\quad F_1 = 5$ $\quad F_2 = 17$ $\quad F_3 = 257$ $\quad F_4 = 65537$

Fermat announced that: "I have found that numbers of the form $2^{2^n} + 1$ are always prime numbers and has long since signified to analysts the truth of this theorem".But he fails to prove this.Later Euler resolved that Fermat's assumption was wrong,by founding

$$F_5 = 2^{2^5} + 1 = 424967297$$

to be divisible by 641.Now consider the theorem that shows $641/F_5$

,

Theorem 4.2

The Fermat number F_5 is divisible by 641.

Proof

We begin by putting $a = 2^7$ and b = 5, so that

$$1 + ab = 1 + 2^7 \cdot 5 = 641$$

It is easily seen that

$$\begin{aligned} 1 + ab - b^4 &= 1 + (a - b^3)b \\ &= 1 + 640 - 625 \\ &= 1 + 15 \\ &= 1 + 3b = 2^4 \end{aligned}$$

But this implies that

$$\begin{aligned} F_5 = 2^{2^5} + 1 &= 2^{32} + 1 \\ &= 2^4 a^4 + 1 \\ &= (1 + ab - b^4)a^4 + 1 \\ &= (1 + ab)a^4 + (1 - a^4 b^4) \end{aligned}$$

$$= (1 + ab)[a^4 + (1 - ab)(1 + a^2 b^2)]$$

which gives $641/F_n$.

To this day it is not known whether there are infinitely many Fermat primes or, for that matter, whether there is at least one Fermat prime beyond F_4. The best guess is that all Fermat numbers $F_n > F_4$ are composite.
A useful property of Fermat numbers is that they are relatively prime to each other. Or "No two Fermat numbers have a common divisor grater than 1".

Theorem 4.3

No two Fermat numbers have a common divisor grater than 1

Proof

suppose that F_n and F_{n+k},where k>0, are two Fermat numbers , and that m/F_n, m/F_{n+k}
If $x = 2^{2^n}$, we have

$$\frac{F_{n+k} - 2}{F_n} = \frac{2^{2^{n+k}} + 1 - 2}{2^{2^n} + 1}$$

$$= \frac{x^k - 1}{x + 1}$$

$$= x^{2k-1} - x^{2k-2} + \cdots - 1$$

and so F_n /F_{n+k}-2 . Hence m/F_{n+k}, it follows that $m/(F_{n+k} - 2)$. Therefore m/2 .Since F_n is odd, m=1 which proves the theorem.

It follows that each of the number F_1 ,F_2 ,F_3 ,F_4 $\cdots$ F_n is divisible by an odd prime which does not divided any of the other. It shows that there are infinitely many Fermat numbers, which leads to the number of Fermat prime is also infinite.

Later Jesuit priest T.Pepin devised the practical test (Pepin's test) for determining the primality of F_n that is embodied in the following theorem.

Theorem 4.4 Pepin's test

For n $2 \geq 1$, the Fermat number $F_n = 2^{2^n} + 1$ is prime if and only if

$$3^{(F_n-1)/2} \equiv -1 (mod F_n)$$

Proof First let us assume that

$$3^{(F_n-1)/2} \equiv -1 (mod F_n)$$

Upon squaring both sides, we get

$$3^{F_n-1} \equiv 1 (mod F_n)$$

The same congruence holds for any prime p that divides F_n:

$$3^{F_n-1} \equiv 1 (mod p)$$

Now let k be the order of 3 modulo p. Theorem 1.13 indicates that $k \big| F_n - 1$, or in other words, that $k \ \big| \ 22"$; therefore k must be a power of 2.

It is not possible that $k = 2^r$ for any $r \leq 2^n - 1$. If this were so, repeated squaring of the congruence $3^k \equiv 1 (mod p)$ would yield

$$3^{2^{2^n-1}} \equiv 1 (mod p)$$

or

$$3^{(F_n-1)/2} \equiv 1(modp)$$

We would then arrive at $1 \equiv -1(modp)$, resulting in $p = 2$, which is a contradiction. Thus the only possibility open to us is that

$$k = 2^{2^n} = F_n - 1$$

Fermat's theorem tells us that $k \leq -1$, which means, in turn, that $F_n = k + 1 \leq p$. Because $p \Big| F_n$, we also have $p \leq F_n$.Together these inequalities mean that $F_n = p$, so that F_n is a prime.

On the other hand, suppose that F_n, $n \geq 1$, is prime. The Quadratic Reciprocity Law gives

$$(3/F_n) = (F_n/3) = (2/3) = -1$$

when we use the fact that $F_n = (-1)^{2n} + 1 = 2(mod3)$. Applying Euler's Criterion,we end up with

$$3^{(F_n-1)/2} \equiv -1(modF_n)$$

Let us demonstrate the primality of $F_3 = 257$ using Pepin's test. Working modulo 257, we have

$$F_n = 2^{2^n} + 1$$
$$F_3 = 2^{2^3} + 1$$
$$F_3 = 256 + 1$$
$$F_3 = 257$$
$$3^{(F_3-l)/2} = 3^{256/2}$$

$$
\begin{aligned}
&= 3^{128} \\
&= 3^3(3^5)^{25} \\
&\equiv 27(-14)^{25} \\
&\equiv 27.14^{24}(-14) \\
&\equiv 27(17)(-14) \\
&\equiv 2719 \equiv 513 \equiv -1(mod 257)
\end{aligned}
$$

so that F_3 is prime.

Theorem 4.5

Any prime divisor p of the Fermat number $F_n = 2^{2^n}+1$, where $n \geq 2$, is of the form $p = k \cdot 2^{n+2} + 1$.

Proof

For a prime divisor p of F_n,

$$2^{2^4} \equiv -1(modp)$$

which is to say, upon squaring, that

$$2^{2^{n+1}} \equiv 1(modp)$$

If h is the order of 2 modulo p, this congruence tells us that

$$h \big| 2^{n+1}$$

We cannot have $h = 2^r$ where $1 \leq r \leq n$, for this would lead to

$$2^{2^n} \equiv 1(modp)$$

and, in tum, to the contradiction that $p = 2$. This lets us conclude that $h = 2^{n+1}$.Because the order of 2 modulo p divides $\phi(p) = p-1$, we may further conclude that $2^{n+1}\Big|p-1$. The point is that for $n \geq 2, p \equiv 1(mod 8)$, and therefore, by Theorem 1.9 , the Legendre symbol (2/p) = 1. Using Euler's criterion, we immediately pass to

$$2^{(p-1)/2} \equiv (2/p) = 1(modp)$$

An appeal to Theorem 1.10 finishes the proof. It asserts that $h\Big|(p-1)/2$, or equivalently, $2n+1\Big|(p-1)/2$. This forces $2^{n+2}\Big|p-1$, and we obtain $p = k \cdot 2^{n+2} + 1$ for some integer k.

A resume of the current primality status for the Fermat numbers F_n, where $0 \leq n \leq 35$, is given below.

n	**C**haracter of F_n
0, 1, 2, 3, 4	prime
5, 6, 7, 8, 9, 10, 11	completely factored
12, 13, 15, 16, 18, 19,25,27,30	two or more prime factors known
17,21,23,26,28,29,31,32	only one prime factor known
14,20,22,24	composite, but no factor known
33,34 ,35	character unknown

The case for F_{16} was settled in 1953 and lays to rest the tantalizing conjecture that all the terms of the sequence

$$2+1, \qquad 2^2+1, \qquad 2^{2^2}+1, \qquad 2^{2^{2^2}}+1 \qquad 2^{2^{2^{2^2}}}+1, \quad \cdots$$

are prime numbers.

CHAPTER - 5

FIBONACCI

The Fibonacci numbers may be defined by the recurrence relation

$u_0 = 0, u_1 = 1,$

and

$u_n = u_{n-1} + u_{n-2}$ for n > 1.

1, 1,2,3,5,8, 13,21,34,55,89, 144,233,377, ...

is called the Fibonacci sequence and its terms the Fibonacci numbers.

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}
0	1	1	2	3	5	8	13	21	34	55	89	144	233

Theorem:5.1

For the Fibonacci sequence, $gcd(u_n, u_{n+1}) = 1$ for every $n \geq 1$.

Proof

Let us suppose that the integer $d > 1$ divides both u_n and u_{n+1}. Then their difference $u_{n+1} - u_n = u_{n-1}$ is also divisible by d. From this and from the relation $u_n - u_{n-1} = un - 2$, it may be concluded that $d \big| u_{n-2}$. Working backward continuously like this we can say that $d \big| u_{n-3}$,$d \big| u_{n-4}$, ... , and finally that $d \big| u_1$. But $u_1 = 1$, which is certainly not divisible by any $d > 1$. This contradiction ends our proof.

We can see that $u_3 = 2$, $u_5 = 5$, $u_7 = 13$, and $u_{11} = 89$ are all prime numbers,which might lead us to guess that u_n is prime whenever the subscript $n > 2$ is a prime.

But the term u_{19} shows that the assumption is false. Since the term

$$u_{19} = 4181 = 37 \cdot 113$$

As we know, the greatest common divisor of two positive integers can be found from the Euclidean Algorithm after finitely many divisions. we know "Given $n > 0$, there exist positive integers a and b such that to calculate $gcd(a, b)$ by means of the Euclidean Algorithm exactly n divisions are needed".

To verify this ,let $a = u_{n+2}$ and $b = u_{n+l}$. The Euclidean Algorithm for obtaining $gcd(u_{n+2}, u_{n+1})$ leads to the system of equations;

$$u_{n+2} = 1 \cdot u_{n+l} + u_n$$

$$u_{n+l} = 1 \cdot u_n + u_{n-1}$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$u_4 = 1 \cdot u_3 + u_2$$

$$u_3 = 2 \cdot u_2 + 0$$

the number of divisions necessary here is n. Also we get;

$$gcd(u_{n+2}, u_{n+1}) = u_2 = 1$$

which confirms anew that successive Fibonacci numbers are relatively prime.

For example consider; n = 6. The following calculations show that we need 6 divisions to find the greatest common divisor of the integers $u_8 = 21$ and $u_7 = 13$:

$$21 = 1 \cdot 13 + 8$$

$$13 = 1.8 + 5$$

$$8 = 1 \cdot 5 + 3$$

$$5 = 1 \cdot 3 + 2$$

$$3 = 1 \cdot 2 + 1$$

$$2 = 2 \cdot 1 + 0$$

Gabriel Lame observed in 1844 that if n division steps are required in the Euclidean Algorithm to compute $gcd(a, b)$, where$a > b > 0$, then $a \geq u_{n+2}$, $b \geq u_{n+l}$. Consequently, it was common at one time to call the sequence u_n the Lame sequence.

Also we can see that one of the features of the Fibonacci sequence is that the greatest common divisor of two Fibonacci numbers is itself a Fibonacci number.The equation

$$u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$$

bring out this fact.

Now let us prove the equation holds for all $m \geq 2$ and $n \geq 1$.

For this fix $m \geq 2$, this identity (equation) is established by induction on n.

Let $n = 1$, then the equation becomes

$$u_{m+1} = u_{m-1}u_1 + u_m u_2 = u_{m-1} + u_m$$

it is obviously true. Let us assume that it is true for $n = 1, 2, ..., k$. Now we have to show that the equation holds, when $n = k + 1$.

The equation holds for n=k, implies

$$u_{m+k} = u_{m-1}u_k + u_m u_{k+1}$$

The equation holds for n=k-1, implies

$$u_{m+(k-1)} = u_{m-1}u_{k-1} + u_m u_k$$

Now add this two equation we get;

$$u_{m+(k-1)} + u_{m+k} = u_{m-1}u_{k-1} + u_m u_k + u_{m-1}u_k + u_m u_{k+1}$$

$$\Rightarrow u_{m+(k-1)} + u_{m+k} = u_{m-1}(u_k + u_{k-1}) + u_m(u_{k+1} + u_k)$$

By the definition of e Fibonacci numbers,we get;

$$u_{m+(k+1)} = u_{m-1}u_{k+1} + u_m u_{k+2}$$

which is of the form of the required equation ,that is the equation holds for $n = k + 1$.Thus the induction step is complete.Then the equation holds for all $m \geq 2$ and $n \geq 1$.

Theorem 5.2 :

For $m \geq 1$, $n \geq 1 u_{mn}$ is divisible by u_m.

Proof :

We argue by induction on n, the result being certainly true when $n = 1$. For our induction hypothesis, let us assume that u_{mn} is divisible by u_m for $n = 1, 2, ..., k$. The transition to the case $u_{m(k+l)} = u_{mk+m}$ is realized using the equation;

$$u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$$

Because u_m divides u_{mk} by supposition, the right-hand side of this expression must be divisible by $u_m\cdot$ Accordingly, $u_m/u_{m(k+l)}$. That is u_{mn} is divisible by u_m This is what we have to prove.

Lemma:5.3

If $m = qn + r$, then $gcd(u_m, u_n) = gcd(u_r, u_n)$.

Proof:

From the equation

$$u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$$

we can wright

$$\begin{aligned} gcd(u_m, u_n) &= gcd(u_{qn+r}, u_n) \\ &= gcd(u_{qn-1}u_r + u_{qn}u_{r+1}, u_n) \end{aligned}$$

From theorem 5.2 we can have u_{mn} is divisible by u_m and also we know the fact that

$gcd(a + c, b) = gcd(a, b)$, whenever b/c,

From theorem 5.2 we can say u_{qn} is divisible by u_n that is $u_{qn}u_{r+1}$ is divisible by u_n and now use the fact $gcd(a + c, b) = gcd(a, b)$, whenever b/c and write

$$gcd(u_{qn-1}u_r + u_{qn}u_{r+1}\ ,\ u_n) = gcd(u_{qn-1}u_r\ ,\ u_n)$$

Claim : $gcd(u_{qn-1}, u_n) = 1$

Let $d = gcd(u_{qn-1}, u_n)$.The relations d/u_n and u_n/u_{qn} imply that

d/u_{qn},and therefore d is a positive common divisor of the successive Fibonacci numbers u_{qn-1} and u_{qn}. Because successive Fibonacci numbers are relatively prime, the effect of this is that $d = 1$.

We have $gcd(a, be) = gcd(a, b)$ when $gcd(a, c) = 1$

$$gcd(u_m, u_n) = gcd(u_{qn-1}u_r, u_n) = gcd(u_r, u_n)$$

hence the lemma.

Theorem:5.4

The greatest common divisor of two Fibonacci numbers is again a Fibonacci number; specifically,

$$gcd(u_m, u_n) = u_d \qquad \text{where } d = gcd(m, n)$$

Proof:

Assume that $m \geq n$ Applying the Euclidean Algorithm torn and n, we get the following system of equations:

$$m = q_1 \cdot n + r1 \qquad 0 < r_1 < n$$
$$n = q_1 \cdot r_1 + r2 \qquad 0 < r_2 < r_1$$
$$r1 = q_3 \cdot r_2 + r_3 \qquad 0 < r_3 < r_2$$
$$.$$
$$.$$
$$.$$
$$r_{n-2} = q_n \cdot r_{n-1} + r_n \qquad 0 < r_n < r_{n-1}$$
$$r_{n-1} = q_{n+1} \cdot r_n + 0$$

From the previous lemma,

$$gcd(u_m, u_n) = gcd(u_{r_1}, u_n) = gcd(u_{r_1}, u_{r_2}) = \cdots = gcd(u_{r_{n-1}}, u_{r_n})$$

Because r_n/r_{n-1}, Theorem 14.2 tells us that $u_{r_n}/u_{r_{n-1}}$ whence $gcd(u_{r_{n-1}}, u_{r_n}) = u_{r_n}$. But r_n, being the last nonzero remainder in the Euclidean Algorithm for r_n and n, is equal to $gcd(m, n)$.We get,

$$\begin{aligned} gcd(u_m, u_n) &= u_{gcd(m,n)} \\ &= u_d \end{aligned}$$

Hence the theorem.

Corollary:5.5

In the Fibonacci sequence, u_m/u_n if and only if m/n for $n \geq m \geq 3$

Proof:

A good illustration of Theorem 5.4 is provided by calculating $gcd(u_{16}, u_{12}) = gcd(987, 144)$. From the Euclidean Algorithm,

$$987 = 6.\ 144 + 123$$

$$144 = 1\ .\ 123 + 21$$

$$123 = 5\ .\ 21 + 18$$

$$2\ 1 = 1\ .\ 18 + 3$$

$$18 = 6.\ 3 + 0$$

and therefore $gcd(987, 144) = 3$. The net result is that

$$gcd(u_{16}, u_{12}) = 3 = u_4 = u_{gcd(l6,12)}$$

as asserted by Theorem 5.4

CONCLUSION

The purpose of the project is to give a simple account of some spacial form of numbers and their speciality.The project gives the more details about $perfect\ numbers$, $mersenne\ prime$, $fermat$, $nummber$, and $fibanocci\ numbers$

In first section we consider a number form called $Perfect\ number$, and its properties.In second section we consider a number form called $mersenne\ number$, and its properties. In third section we consider a number form called $fermat\ number$, and its properties.In fourth section we consider a series called $fibonacci\ series$, and its properties.

It is a section in the subject called NUMBER THEORY also some time known as “HIGHER MATHEMATICS”.

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DON BOSCO ARTS & SCIENCE COLLEGE ANGADIKADAVU

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2021-2023

Project Report on

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TOPOLOGICAL VECTOR SPACE

Dissertation submitted in the partial

Fulfillment of the requirement for the award of

MSc Degree in Mathematics of

Kannur University

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KANNUR UNIVERSITY

BONAFIDE CERTIFICATE

Certified that this project report " TOPOLOGICAL VECTOR SPACE" is the bonafide work of PRANAV K P who carried out the project work under my supervision.

Signature of Head

Signature of Supervisor

DECLARATION

I, PRANAV K P hereby declare that the Project work entitled TOPOLOGICAL VECTOR SPACE has been prepared by me and submitted to Kannur University in partial fulfillment of requirement for the award of Master of Science is a record of original work done by me under the supervision of Mrs. AJEENA JOSEPH, Assistant Professor, Department of Mathematics, Don Bosco Arts & Science College, Angadikadavu. I, also declare that this Project work has not been submitted by me fully for the award of any Degree, Diploma, Title or recognition before any authority.

Place: Angadikadavu PRANAV K P

Date: (C1PSMM1901)

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Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also I must express my deepest gratitude to people along the way. No words can adequately express the sense of gratitude, still I try to express my heartfelt thanks through words. The outset, I am deeply indebted to my project supervisor Mrs. AJEENA JOSEPH Assistant Professor, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu, for the invaluable guidance, loving encouragement and meticulous care towards me throughout my career. I express my deep sense of gratitude to all the faculty members of the Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu. I can never forget the support and encouragement rendered by the Principal and the Staff of Don Bosco Arts & Science College, Angadikadavu. I could not name many who sincerely supported and helped for the successful completion of this Project. It is my pleasure and duty to thank each and everyone of them who walked with me.

PRANAV K P

CONTENTS

INTRODUCTION

Functional Analysis plays an increasing role in applied science as well as in Mathematics itself. Consequently, it becomes more and more desirable to introduce the student to the field at an early stage of study. Functional Analysis is the study of certain Topological-Algebraic structures and of these structures and of the methods by which knowledge of these structures can be applied to analytic problems. A good introduction on this subject should include a presentation of its axiomatic. That is of the general theory of Topological vector spaces. Also, it contains some interesting applications to the branches of Mathematics.

In Mathematics, if a set is endowed with Algebraic and Topological structures, then it always fascinating to prove relationship between these to structures. A Topological Vector Space (also called Linear Topological Space) is a basic structure on Topology on which a vector space X over a field F. As the name suggests the space blends a topological structures with the algebraic concept of a vector space. The elements of topological vector spaces are typically functions or linear operators acting on topological vector spaces, and the topology is often defined so as to capture a particular notion of convergence of sequences of functions. We know that Normed space (linear space) means a vector space together with a norm on it. Also we know that every normed space is a metric space. Hilbert space and Banach spaces are well known examples for this. Also, there is no discussion of uniform spaces, of completeness occurs only in the context of metric spaces. One can study vector spaces over the field of real or complex numbers along with a suitable topology. They are known as Topological vector spaces.

This topic Topological vector spaces give a minimum introduction to topological vector spaces and it deal with some of the basic theories of topological vector spaces. Notion of vector topology, boundedness, compactness, local convexity are introduced. In this present dissertation our aim is to generalize the concept of normed space to topological vector space so that normed space is a special case of topological vector spaces.

PRELIMINARIES

Definition 0.1(Vector spaces): A vector space over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is a set X, whose elements are called vectors, and in which two operations, vector addition and scalar multiplication, are defined with following algebraic properties:

1. For every pair of vectors x and y corresponds a vector $x+y$,in such a way that $x + y = y + x$ and $x + (y + z) = (x + y) + z$.

2. X contains the unique vector zero such that $x+0 = x$ for every $x \in X$.

3. For $x \in X$, there corresponds to a unique vector $-x$ in X such that $x + (-x) = 0$.

4. To every pair (α, x) with $\alpha \in \mathbb{K}$ and $x \in X$ corresponds a vector αx, in such a way that

$$1 \times x = x,$$

$$\alpha(\beta x) = (\alpha\beta)x,$$

and such that the two distributive laws

$$\alpha(x + y) = \alpha x + \alpha y$$

$$(\alpha + \beta)x = \alpha x + \beta x$$

hold.

Definition 0.2.(Topological space): A Topological space is a pair (X, τ) where X is a set and τ is a family of subsets of X satisfying:

1.$\phi \in \tau$ and $X \in \tau$.

2.τ is closed under arbitrary unions.

3.τ is closed under finite intersections.

The family τ is said to be a topology on the set. Members of τ are said to be open sets in X.

Definition 0.3. (Normed spaces): Let X be a linear space over K. A norm on X is a function $||\cdot||$ from $X \times X$ to $\mathbb{R}$ such that for all $x, y \in X$ and $k \in K$:

1. $||x|| \geq 0$ and $||x|| = 0$ if and only if $x = 0$
2. $||x + y|| \leq ||x|| + ||y||$
3. $||kx|| = |k| \cdot ||x||$

A normed space is a linear space with a norm on it.

Definition 0.4. (Discrete and indiscrete topological space): Let X be a set and let τ be the collection of all subsets of X. Then τ is the topology on X and is known as the discrete topology.

Also, the set $U = \{\phi, X\}$ is a topology on X called the indiscrete topology.

Definition 0.5. (Banach Spaces): A complete normed space is called a Banach space.

Definition 0.6. (Homeomorphism): A Homeomorphism from a space X to a space Y is a bijective function f from X to Y such that both f and f^{-1} are continuous.

Definition 0.7. (Open set): A subset $A \subseteq X$ is said to be open if for every $x \in A$, there exists some open ball around x which is contained in A.

Definition 0.8. (Local base): Let X be a space, $x \in X$. Then a local base at x is a collection $\mathcal{L}$ of neighbourhoods of x such that given any neighbourhood N of x, there exists $L \in \mathcal{L}$ satisfying $L \subset N$.

Definition 0.9. (Continuous function): Let (X, τ) and $(Y,\mathcal{U})$ be topological spaces. A function $f : X \to Y$ is continuous at a point x in X provided that for each neighbourhood V of $f(x)$ there is a neighbourhood U of x such that $f(U) \subset V$. A function $f : X \to Y$ is continuous provided it is continuous at each point of x.

Definition 0.10. (Compact subset): A subset A of a space X is said to be a compact subset of X if for every cover of A by open subsets of X has finite sub cover. A space X is said to be compact if X is a compact subset of itself.

Definition 0.11. (Directed set): A directed set is a pair $(D, \geq)$ where D is a non-empty set and $\geq$ a binary relation on D satisfying:

1. For all $m, n, p \in D$, if $m \geq n$ and $n \geq p$, then $m \geq p$
2. For all $n \in D$, $n \geq n$
3. For all $m, n, p \in D$, there exists $p \in D$ such that $p \geq m$ and $p \geq n$.

Definition 0.12. (Net): A net in a set X is a function $f : D \to X$ where D is a directed set.

Definition 0.13. (Filter): A filter on a set X is a non-empty family $\mathcal{F}$ of subsets of X such that:

1. $\phi \notin \mathcal{F}$
2. $\mathcal{F}$ is closed under finite intersections
3. If $B \in \mathcal{F}$ and $B \subset A$, then $A \in \mathcal{F}$ for all subsets A, B of X.

Definition 0.14. (Ultra filter): A filter $\mathcal{F}$ on a set X is said to be an ultra filter if it is a maximal element in the collection of all filters on X, partially ordered by inclusion, that is, $\mathcal{F}$ is an ultra-filter if it is not properly contained in any filter on X.

Definition 0.15. (Metrizable): A topological space X is said to be metrizable if its topology is generated by a metric.

Definition 0.16. (Filter base): Let $\mathcal{F}$ be a filter on a set. Then a sub-family $\mathbf{B}$ of $\mathcal{F}$ is said to be a base for $\mathcal{F}$ (filter base) if for any $A \in \mathcal{F}$ there exists $B \in \mathbf{B}$ such that $B \in A$.

Definition 0.17. (Locally compact): A topological space X is said to be locally compact at a point $x \in X$ if x has a compact neighbourhood in X. X is called locally compact if it is locally compact at every point.

Definition 0.18. (Hausdorff space): A space X is said to be a Hausdorff space if for every distinct points $x, y \in X$ there exist disjoint open sets U, V in X such that $x \in U$ and $y \in V$.

Theorem 0.19. (The Hahn-Banach Theorem): If M is a linear subspace of a normed linear space X and if f is a bounded linear functional on M, then f_0 can be extended to a bounded linear functional f defined on the whole space such that $||f|| = ||f_0||$.

Definition 0.20. **(Tottaly bounded):** Let A be a subset of a metric space (X, d). Let $\epsilon > 0$. A finite set $\{x_1, x_2, ..., x_n\}$ of points of X is said to form a finite ϵ-net if $A \subseteq B_\epsilon(x_1) \cup ... \cup B_\epsilon(x_n)$. The set A is totally bounded if A has a finite ϵ-net for every $\epsilon > 0$.

CHAPTER 1

VECTOR TOPOLOGY AND LOCAL BASE

Definition 1.1 A topological vector space X is a vector space with a topology τ under which the mappings

- $(x, y) \to x + y$ of $X \times X$ into X
- $(\alpha, x) \to \alpha x$ of $\mathbb{K} \times X$ into X, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$

are continuous. This topology τ is known as linear or vector topology.

Example 1.2. Let $X \neq \{0\}$ be a vector space and let τ be the indiscrete topology in X. That is, $\tau = \{\phi, X\}$. Then with this topology, X is a topological vector space.

Proof. To prove that '+' is continuous at (x, y), recall that f is continuous at x if and only if for every neighbourhood V of $f(x)$ there exists a neighbourhood U of x such that $f(U) \subseteq V$.

The only neighbourhood of $+(x, y) = x + y$ is X. Also, $X \times X$ is a neighbourhood of (x, y) and $+(X \times X) = X$ and $\cdot(\alpha, x) = \alpha x, \cdot(\mathbb{F} \times X) = X$. We get addition and scalar multiplication as continuous at (x,y).

Example 1.3. A normed linear space is a topological vector space.

Proof. To prove this, we have a normed linear space is a vector space with a norm. Every normed space may be regarded as a metric space in which the distance $d(x, y)$ between x and y is $\|x - y\|$.

Next, to verify that the algebraic operations addition and scalar multiplications are continuous, we have to show that given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|(x+y)-(a+b)\| < \epsilon, \quad \forall x, y \in X$$

with $\|x-a\| < \delta$ and $\|y-b\| < \delta$.
Consider

$$\|x+y-(a+b)\| = \|(x-a)+(y-b)\|$$

$$\leq \|x-a\| + \|y-b\|$$

$$\leq \delta + \delta$$

$$= 2\delta$$

Take $\delta = \epsilon/2$, we get $\|x+y-(a+b)\| < \epsilon$. Therefore, vector addition is continuous.

Now, to prove scalar multiplication is continuous, we have the identity

$$\begin{aligned} \lambda x - \alpha a &= \lambda x - \alpha a + \lambda a - \lambda a \\ &= \lambda(x-a) + a(\lambda - \alpha). \end{aligned}$$

$$\|\lambda x - \alpha a\| = \|\lambda(x-a) + a(\lambda-\alpha)\|$$

$$\leq \|\lambda(x-a)\| + \|a(\lambda-\alpha)\|$$

$$\leq |\lambda|\|x-a\| + |\lambda-\alpha|\|a\|$$

Hence, $\|\lambda x - \alpha a\| < \epsilon$, whenever $\|x-a\| < \frac{\epsilon}{2|\lambda|}$ and $|\lambda-\alpha| < \frac{\epsilon}{2\|a\|}$.

Hence, the result.

Here, a normed linear space satisfies the conditions of a topological vector space.

Example 1.4 The real line with discrete topology is not a topological vector space.

Proof. To show that the real line $\mathbb{R}$ with discrete topology is not a topological vector space, we will show that there is a point $(\lambda, x) \in \mathbb{F} \times X$ at which the function '$\cdot$' is not continuous.
Let $\lambda_x = z \neq 0$.Then U_z is a neighborhood of z.
Since $U_\lambda = \{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}) : n = 1, 2, \ldots\}$ is a basis basis of the neighborhoods of λ, there exists $n \in \mathbb{N}$ such that

$$(\lambda - \frac{1}{n}, \lambda + \frac{1}{n})\{x\} \subset U_\lambda U_x \subset U_z = \{z\} = \{\lambda x\}$$
$$\Rightarrow (\lambda - \frac{1}{n}, \lambda + \frac{1}{n})\{x\} \subseteq \{\lambda x\}$$

This is not possible. Since interval $\not\subset$ singleton set

Definition 1.5. A topological vector space is said to be Hausdorff or separated if $x \neq 0 \implies 0 \in U$ and $x \notin U$ for some U.

Example 1.6 The empty space is a Hausdorff space.

Example 1.7 Any discrete space is a Hausdorff space. That is, a topological space with the discrete topology.

Example 1.8. Any metric space is Hausdorff in the induced topology. That is, any metrizable space is Hausdorff.

Proposition 1.9. Let X be a topological vector space. Then for any $x \in X$, the mapping $f_x : X \to X$ given by $f_x(y) = x + y$ is a homeomorphism.

Proof Suppose that X is a topological vector space and for any

$x \in X$, $f_x : X \to X$ is defined by $f_x(y) = x + y$.

To prove that f_x is a homeomorphism
suppose $f_x(y) = f_x(z)$
$\implies x + y = x + z$
$\implies y = z$

(1.9.1) Therefore,f_x is injective.

Now to show that f_x is surjective.
Gen $y \in X$, there is a $y - x \in X$, with
$f_x(y - x) = x + y - x = y$.

(1.9.2) Therefore, f_x is surjective.
It remains to show that $f(x)$ is continuous.
Consider the function $g : X \to X \times X$ defined by $g(x) = (x, y)$.
Given $U \times \{y\} = W$, a neighborhood of $g(x)$, we have a neighborhood of U of x with

$$g(U) = U \times \{y\} \subset W$$

Hence, g is continuous.
By the definition of a topological vector space, the function $h : X \times X \to X$ defined by $h[(x, y)] = x + y$.
Then h is continuous.
Therefore, the composition $h \circ g$ is continuous.

But, $h \circ g(x) = h(g(x)) = h[(x, y)] = x + y = f_x(y)$, i.e., $h \circ g(x) = f_x(y)$.

(1.9.3) Therefore, f_x is continuous.

Similarly, we can show that $f_x^{-1} : X \to X$ defined by

(1.9.4) $$f_x^{-1}(y) = y - x \text{ is continuous .}$$

From equations (1.9.1), (1.9.2), (1.9.3), and (1.9.4), it follows that f_x is a homeomorphism.

Definition 1.10 Let X be a vector space over the field K, where $K = \mathbb{R}$ or $\mathbb{C}$. Then a non-empty set A in X is said to be *convex* if

$$\lambda x + (1 - \lambda)y \in A$$

whenever $x, y \in A$ and $0 \leq \lambda \leq 1$, or equivalently A is convex if $x, y \in A$, $\lambda \geq 0$, $\mu \geq 0$, and $\lambda + \mu = 1$. Then $\lambda x + \mu y \in A$.

Example 1.11. In a normed linear space X, the closed unit ball $B = \{x \in X : |x| \leq 1\}$ is convex.
Proof. For if $x, y \in B$ and $0 \leq \lambda \leq 1$, then $\|x\| \leq 1$ and $\|y\| \leq 1$. Hence,

$$\|\lambda x + (1 - \lambda)y\| \leq \lambda\|x\| + (1 - \lambda)\|y\| \leq \lambda + (1 - \lambda)$$

Therefore, $\lambda x + (1 - \lambda)y \in B$ and B is convex.

Definition 1.12. A non-empty set A of a vector space X is said to be balanced or critical if $x \in A$, $|\lambda| \leq 1$ imply $\lambda x \in A$. That is, $\lambda A \subset A$ for all λ with $|\lambda| \leq 1$.

Example 1.13. In $\mathbb{R}^n$, the closed unit ball $B(0, 1) = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is a balanced set

Proof Let $x \in B(0, 1), and\ |\lambda| \leq 1$.
Now $x \in B(0, 1) \implies \|x\| \leq 1$,then

$\|\lambda x\| = |\lambda|\|x\| \leq 1$

$\implies \lambda x \in B(0,1)$.
This show that B(0,1) is balanced.

Definition 1.14. A nonemty set A of a vector space X is said to be absorbing or absorbent if for each $x \in X$,there exist some $\rho > 0$ such that $\lambda x \in$ A for all $|\lambda| \leq \rho$

Exambple 1.15. In $\mathbb{R}^n$,the closed unit ball B(0,1) = $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is an absorbent set

Proof. Let $x \in B(0,1) \implies \|x\| \leq 1$ and $|\lambda| \leq 1$.
Choose $\rho = 1$ >0 such that $|\lambda| \leq \rho$.
That is,$\lambda x \in B(0,1)$ for all $|\lambda| \leq \rho$.
i.e., for each $x \in X$, there exist some $\rho > 0$ such that $\lambda x \in B(0,1)$ $\forall |\lambda| \leq \rho$.

Hence B(0,1) is absorbent.

Remark 1.16. An absorbing set contains 0.

Defintion 1.17. A non-empty set A of a vector space X is said to be absolutely convex if

$$\lambda A + \mu A \subset A$$

whenever $|\lambda| + |\mu| \leq 1$.

Definition 1.18. Let V be an absorbing subset of a vector space X. For each $x \in X$ let

$$p_V(x) = inf\{\lambda > 0 : x \in \lambda V\}.$$

Then p_V is called guage or Minkowski functional of V.

Theorem 1.19. A set A is absolutely convex if and only if A is both balanced and convex.

Proof. Suppose that A is absolutely convex.
i.e., if $\lambda A + \mu A \subset A$ whenever $|\lambda| + |\mu| \leq 1$.
If $\lambda \geq 0, \mu \geq 0$ and $\lambda + \mu = 1$ and $x, y \in A$,then we have

$$(\lambda x + \mu y) \in A. \tag{1.19.1}$$

Since A is absolutely convex, clearly A is convex.
Next we want to show that A is balanced.
For this take $\mu = 0$ in (1.19.1), we get

$$x \in A, |\lambda| \leq 1 \implies \lambda x \in A.$$

Hence A is balanced.
Conversely suppose that A is convex and balanced.
We want to show that A is absolutely convex.
Let $x, y \in A$ and $|\lambda| + |\mu| \leq 1$.

- Case 1. If $\lambda = 0$ or $\mu = 0$
  then $\lambda x + \mu y \in A$, because A is balanced.

- Case 2. If $\lambda \neq 0$ and $\mu \neq 0$
  then $\left(\frac{\lambda}{|\lambda|}\right) x \in A$, because $\left|\frac{\lambda}{|\lambda|}\right| = 1$ and A is balanced.
  similarly we get $\left(\frac{\mu}{|\mu|}\right) y \in A$.
  Also,

  $$\frac{|\lambda|}{|\lambda|+|\mu|} + \frac{|\mu|}{|\lambda|+|\mu|} = \frac{|\lambda|+|\mu|}{|\lambda|+|\mu|} = 1$$

  But A is convex. Hence

  $$\left(\frac{|\lambda|}{|\lambda|+|\mu|}\right) \frac{\lambda}{|\lambda|} x + \left(\frac{|\mu|}{|\lambda|+|\mu|}\right) \frac{\mu}{|\mu|} y \in A \tag{1.19.2}$$

[Since $\left(\frac{\lambda}{|\lambda|}\right) x \in A$,$\left(\frac{\mu}{|\mu|}\right) y \in A$,and $\frac{|\lambda|}{|\lambda|+|\mu|} + \frac{|\mu|}{|\lambda|+|\mu|} = 1$,$A$ is convex.]

Therefore by equation (1.19.2) we get, $\frac{\lambda x}{|\lambda|+|\mu|} + \frac{\mu y}{|\lambda|+|\mu|} \in A$
Now A is balanced and $|\lambda| + |\mu| \leq 1$ we have

$$(|\lambda| + |\mu|)\left(\frac{\lambda x}{|\lambda|+|\mu|} + \frac{\mu y}{|\lambda|+|\mu|}\right) \in A$$

$$\implies \lambda x + \mu y \in A$$

Therefore A is absolutely convex.

Theorem 1.20. Let X be topological vector space and let U be the neighbourhood of 0. Then λU is a neighbourhood of 0 for every non-zero real number λ.

Proof. Given that $\lambda \neq 0$
Consider the mapping $f : X \to X$ defined by $f(x) = \lambda x$.
Then f is continuous.
[since by the definition of a topological vector space scalar multiplication is continuous.]
Also, inverse of f is, $f^{-1} : X \to X$ which is given by $f^{-1}(x) = (\frac{1}{\lambda})x$ is also continuous.
Furthermore, f is a bijection.
Hence f is a homeomorphism.
Thus f is an open map.
[since we have the result that f is open if and only if f is a homeomorphism and continuous.]
Since U is a neighbourhood of 0, there exists an open neighbourhood G of 0 such that $G \subset U$.
Since f is an open map, $f(G)$ is an open neighbourhood of 0, such that
$f(G) \subset \lambda U$ [Since$f(G) = \lambda G \subset \lambda U \implies f(G) \subset \lambda U$.]

So, λU is a neighbourhood of 0.

Theorem 1.21. Every neighbourhood of 0 in a topological vector space X is absorbing.

Proof. Let U be any neighbourhood of 0. Fix $x \in X$. Consider the map f on the scalar field into X defined by, $f(\lambda) = \lambda x$.
This map f is continuous.
Hence there exist $\rho > 0$ such that $|\lambda| < \rho \implies f(\lambda) \in U$.
That is $|\lambda| < \rho \implies \lambda x \in U$.
Hence U is absorbing.
That is, every neighbourhood of 0 in a topological vector space X is absorbing.

Theorem 1.22. Every neighbourhood of 0 in a topological vector space X includes a balanced neighbourhood of 0.

Proof. Let U be any neighbourhood of 0 in X. Fix $x \in X$. Consider the function $f : \mathbb{K} \times X \to X$ defined by, $f(\lambda, x) = \lambda x$.
Then f is continuous, by the definition of a topological vector space.
Hence there exists a neighbourhood V of (0, 0) such that

$$f(V) \subset U \tag{1.22.1}$$

Take $V = V_1 \times V_2$ where V_1 is a neighbourhood of '0 ' in $\mathbb{K}$ and V_2 is a neighbourhood of '0 ' in X.
But then for some $r > 0$, V_1 contains the closed ball $\bar{S}(0, r) = \{\mu : |\mu| \leq r\}$
Now, for this fixed r, let

$$W = \cup\{\mu V_2 : |\mu| \leq r\}.$$

Then $rV_2 \subset W$ and rV_2 is a neighbourhood of 0 in X, by the previous theorem(1.20).Consequently, W is a neighbourhood of 0 in X.

Also, by (1.22.1), $W \subset U$.
Finally, we shall show that W is balanced.
Let $x \in W$ and $0 < |\lambda| \leq 1$.

$$\begin{aligned} x \in W &\implies x \in \mu V_2 \text{ with } |\mu| \leq r \\ &\implies \lambda x \in \lambda\mu V_2 \text{ with } |\lambda\mu| \leq r \\ &\implies \lambda x \in W. \end{aligned}$$

Hence W is balanced.
i.e., every neighbourhood of 0 in a topological vector space X includes a balanced neighbourhood of 0.

Theorem 1.23. Let X be a topological vector space and $A \subset X$. Then

$$\bar{A} = \cap\{A + U : U \text{ is a neighbourhood of } 0\}.$$

In particular,$\bar{A} \subset A + U$ for every neighbourhood U of 0.

Proof. First let $x \in \bar{A}$.
Let U be any neighbourhood of 0. We have by the above theorem "Every neighbourhood of 0 in a topological vector space X includes a balanced neighbourhood of 0", we may assume that U is balanced.
Then $x + U$ is a neighbourhood of x.
Since $x \in \bar{A}$, we have,

$$(x + U) \cap A \neq \phi$$

[we have the result that "Let A be a subset of a topological space, and let $x \in X$. Then $x \in \bar{A}$ if and only if every neighbourhood of x has a nonempty intersection with A"].
And there exists $x + u \in A$ for some u in U. Then, $x \in A - U$.
Since U is balanced, we have $A - U = A + U$.

Hence, $x \in A + U$.

Therefore $x \in \cap\{A + U : U$ is a neighborhood of $0\}$

Thus

$$\bar{A} \subset \cap\{A + U\}. \tag{1.23.1}$$

To prove, $\cap\{A + U : U$ is a neighbourhood of $0\} \in \bar{A}$.
For this, let $x \in \cap\{A + U : U$ is a neighbourhood of $0\}$.
Assume that $x \notin \bar{A}$.
Then there exists a balanced neighbourhood U of 0 such that $(x + U) \cap A = \phi$
Hence, $x \notin A - U = A + U$,
and thus $x \notin \cap\{A + U : U$ is a neighbourhood of $0\}$,
a contradiction to our assumption.
Thus $x \in \bar{A}$. Arbitrariness of x gives

$$\cap\{A + U : U \text{ is a neighbourhood of } 0\} \subset \bar{A}. \tag{1.23.2}$$

Thus from equations (1.23.1) and (1.23.2) we get,

$$\bar{A} = \cap\{A + U : U \text{ is a neighbourhood of } 0\}.$$

Theorem 1.24. Let X be a topological vector space. Then the

1. Closure of a balanced set is balanced.

2. Closure of a convex set is convex.

3. Interior of a convex set is convex.

Proof. 1. Assume that X is a topological vector space and let A be a balanced set.
We have to prove that $\bar{A}$ is balanced.

For this, let $x \in \bar{A}$. Then for any neighbourhood U of 0, we have U is balanced and $x + U$ is a neighbourhood of x. As $x \in \bar{A}$

$$(x + U) \cap A \neq \phi$$

Hence, there exists $x_0 \in (x + U) \cap A$. Hence

$$\lambda x_0 \in \lambda[(x + U) \cap A], \forall |\lambda| \leq 1.$$

Since U and A are balanced, we have

$$\lambda[(x + U) \cap A] \subset (\lambda x + U) \cap A.$$

Hence,

$$\lambda x_0 \in (\lambda x + U) \cap A,$$

for all λ with $|\lambda| \leq 1$.
Hence $(\lambda x + U) \cap A \neq \phi$ and $\lambda x \in \bar{A}$ for all λ with $|\lambda| \leq 1$.
Therefore, $\bar{A}$ is balanced.
Hence (1) is proved.

2. Let A be a convex set. We have to prove that $\bar{A}$ is convex.
Let $x, y \in \bar{A}$. Now given any neighbourhood U of 0, there exists a balanced neighbourhood V of 0 such that

$$V + V \subset U.$$

Now

$$x \in \bar{A} \implies (x + V) \cap A \neq \phi$$

$$y \in \bar{A} \implies (y + V) \cap A \neq \phi$$

And hence there exists $x_0 \in (x + V) \cap A$ and there exists $y_0 \in (y + V) \cap A$
Let λ be such that $0 \leq \lambda \leq 1$.
Now, since A is convex, we have

$\lambda x_0 + (1-\lambda)y_0 \in (\lambda x + V) \cap \lambda A + [(1-\lambda)y + V] \cap (1-\lambda)A$

$\subset A \cap [\lambda x + (1-\lambda)y + U]$
Therefore,

$$\lambda x + (1-\lambda)y + U \cap A \neq \phi.$$

Hence,

$$\lambda x + (1-\lambda)y \in \bar{A}$$

This proves that $\bar{A}$ is convex.

3. Let A be any convex set. We have to prove that interior of A is convex. For this, let $x, y \in int(A)$. Fix λ with $0 \leq \lambda \leq 1$. Let

$$z = \lambda x + (1-\lambda)y$$

Let U and V be neighbourhoods of x and y respectively in A. Write

$$W = \lambda U + (1-\lambda)V$$

Then

$$W = \cup\{\lambda u + (1-\lambda)V : u \in U\}$$

is a neighbourhood of z. Also, $W \in A$. Hence $z \in Int(A)$.

Therefore, $Int(A)$ is also convex.

Theorem 1.25. Every neighbourhood of 0 in a topological vector space X includes a closed balanced neighbourhood of 0.

Proof. We know that the function

$f : X \times X \to X$ defined by, $f[(x,y)] = x + y$, is continuous

In particular, f is continuous at $0 = (0, 0)$ [since f is continuous at each point]
Hence, given a neighbourhood U of 0, there exists a neighbourhood N of $(0, 0)$ with $f(N) \subset U$.
[since we have by definition of continuity, "Let X and Y be two topological spaces, a function f from X to Y is said to be continuous at $x \in X$ if for each neighbourhood U of $f(x)$ there exist a neighbourhood V of x such that $f(V) \subset U$ ".]
Take $W = V_1 \times V_2$ Where V_1 and V_2 are neighbourhoods of 0.
Let

$$V = V_1 \cap V_2.$$

Then

$$V + V \subset V_1 + V_2 = f(v_1, V_2) \subset f(N) \subset U.$$

Thus,

$$V + V \subset U. \tag{1.25.1}$$

theorem 1.22, we can choose a balanced neighbourhood W of 0 such that

$$\overline{W} \subset V. \tag{1.25.2}$$

Then by theorem 1.23,

$$\overline{W} \subset V + W \subset V + V$$

and hence, $\overline{W} \subset V + V$, by (1.25.2).
That is, $\overline{W} \subset U$ by using equation (1.25.1).
That $\overline{W}$ is balanced follows from part (1) of theorem 1.24
Also we know that $\overline{W}$ is closed.
That is, $\overline{W}$ is a closed balanced neighbourhood of 0.

That is, Every neighbourhood of 0 in a topological vector space X includes a closed and balanced neighbourhood of 0.

CHAPTER 2

NORMABLE SPACES

Defintion 2.1. Let X be a vector space. A Semi-norm on X is a function $p : X \to \mathbb{R}$ satisfying

1. $p(x) \geq 0$

2. $p(\lambda x) = |\lambda| p(x)$

3. $p(x + y) \leq p(x) + p(y)$ $\forall x, y \in X$ and $\lambda \in \mathbb{F}$

   A semi-norm is a norm if it satisfies the further condition:

4. $p(x) = 0 \iff x = 0$

   From (2) it follows that $p(0) = 0$.

Example 2.2. Let X be a vector space and let f be a linear functional on X. Define

$$p(x) = |f(x)| \ \forall x \in X.$$

Then p is a semi-norm on X.

Proof. We have, $p(x) = |f(x)|$ $\forall x \in X$.
Clearly, $p(x) \geq 0$, because $|f(x)| \geq 0$. Also,

$$p(\lambda x) = |f(\lambda x)|$$

$$= |\lambda f(x)| \text{ , since } f \text{ is linear}$$

$$= |\lambda||f(x)|$$

$$= |\lambda|p(x).$$

and

$$p(x+y) = |f(x+y)|$$

$$= |f(x) + f(y)|$$

$$\leq |f(x)| + |f(y)|$$

$$= p(x) + p(y).$$

Hence p is a semi-norm on X.

Example 2.3. A norm on a vector space X is always a semi-norm.

Proof. Let X be a vector space. We have norm on X is a real-valued function which satisfying the following conditions

1. $\|x\| \geq 0 \forall x \in X$
2. $\|x\| = 0 \iff x = 0$, the zero element in X
3. $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in X$
4. $\|\alpha x\| = |\alpha|\|x\|, \forall x \in X$ and for all scalars α

By the definition norm it is clear that it satisfies the conditions of semi-norm. i.e., Norm on a vector space is always a semi-norm.

Proposition 2.4. If p is a semi-norm on X, then

$$|p(x) - p(y)| \leq p(x-y) \text{ for all } x, y \text{ in } X$$

Proof. Assume that X be vector space and p is a semi-norm on X. We want to show that $|p(x) - p(y)| \leq p(x-y)$ for all $x, y \in X$.
For that, we have, $x = y + (x - y)$ and so

$$p(x) = p[y + (x - y)]$$

$$\leq p(y) + p(x - y)$$

That is,

(2.4.1) $$p(x) \leq p(y) + p(x - y)$$

[by the definition of semi-norm $p(x + y) \leq p(x) + p(y)$]
By symmetry

$$p(y) \leq p(x) + p(y - x)$$

$$= p(x) + p[(-1)(x - y)]$$

$$= p(x) + |-1|p(x - y)$$

That is,

(2.4.2) $$p(y) \leq p(x) + p(x - y)$$

From equations (2.4.1) and (2.4.2) we get

$$p(x) - p(y) \leq p(x - y) \text{ and } p(y) - p(x) \leq p(x - y)$$

$$\implies p(x) - p(y) \leq p(x - y) \text{ and } -(p(x) - p(y)) \leq p(x - y)$$

$$\implies p(x) - p(y) \leq p(x - y) \text{ and } -p(x - y) \leq p(x) - p(y)$$

$$\implies -p(x - y) \leq p(x) - p(y) \leq p(x - y)$$

$$\implies |p(x) - p(y)| \leq p(x - y)$$

Since x and y arbitrary, we get if p is a semi-norm on X, then

$|p(x) - p(y)| \leq p(x - y)$ for all $x, y \in X$.

Definition 2.5. A set in a topological vector space X is said to be bounded if it is absorbed by every neighbourhood of 0. In other words, a set A is bounded if and only if for every neighbourhood U of 0 there exist $\rho > 0$ such that $\lambda A \subset U$ whenever $|\lambda| < \rho$.

Example 2.6. Every singleton set in a topological vector space is a bounded set.

Theorem 2.7. Let A be a set in a topological vector space X. Then the following assertions are equivalent:

1. A is bounded.

2. For every sequence (x_n) in A and for every sequence (ϵ_n) of scalars with $\epsilon_n \to 0$ it is true that $\epsilon_n x_n \to 0$ as $n \to \infty$.

3. For every sequence (x_n) in A, it is true that $(\frac{1}{n})x_n \to 0$ as $n \to \infty$.

Proof. Let X be a topological vector space and A be a set in X.
Step 1 : Suppose that A is bounded. We want to show that, for every sequence (x_n) in A and for every sequence (ϵ_n) of scalars with $\epsilon_n \to 0$ it is true that $\epsilon_n x_n \to 0$ as $n \to \infty$.
Let (x_n) be a sequence in A. Let U be any neighbourhood of 0. Since A is bounded there exist $\rho > 0$ such that $\lambda A \subset U$ whenever $|\lambda| < \rho$. Let $\epsilon_n \to 0$. But then $|\epsilon_n| < \rho$ eventually.Hence

$$\epsilon_n x_n \in \epsilon_n A \subset U$$

(Since A is bounded) eventually. Hence

$$\epsilon_n x_n \to 0 \text{ as } n \to \infty$$

(Since U is a neighbourhood of 0)
Since (x_n) is arbitrary, for every sequence (x_n) in A and for every

sequence (ϵ_n) of scalars with $\epsilon_n \to 0$ it is true that $\epsilon_n x_n \to 0$ as $n \to \infty$.

i.e., (1) $\implies$ (2)

Step 2 : Suppose that for every sequence (x_n) in A and for every sequence (ϵ_n) of scalars with $\epsilon_n \to 0$ it is true that $\epsilon_n x_n \to 0$ as $n \to \infty$.
We want to prove that for every sequence (x_n) in A, it is true that $(\frac{1}{n})x_n \to 0$ as $n \to \infty$.
Let (x_n) be a sequence in A, and take $\epsilon_n = \frac{1}{n}$, where $n = 1, 2, ...$
So that $\epsilon_n \to 0$.
Then from (2), we have $\epsilon_n x_n \to 0$.
i.e.,

$$(\frac{1}{n})x_n \to 0 \text{ as } n \to \infty$$

Thus (2) $\implies$ (3)

Step 3 : Suppose that every sequence (x_n) in A, it is true that $(\frac{1}{n})x_n \to 0$ as $n \to \infty$.We want to show that A is bounded.
Assume that A is not bounded. Then there exist a balanced neighbourhood U of 0 such that $\lambda A \not\subset U$ for some λ with $|\lambda| < \rho$ where $\rho > 0$ is arbitrary. Now, U is balanced, and so

$$\lambda A \not\subset U \text{ for all } \rho > 0.$$

Take $\rho = \frac{1}{n}$ so that we have

$$\frac{1}{n}A \not\subset U.$$

Choose

$$x_n \in A - nU \text{ for } n = 1, 2, ...$$

Then

$$\frac{1}{n}x_n \notin U \text{ for } n = 1, 2, ...$$

Hence $\frac{1}{n}x_n$ does not converges to zero.

Therefore (3) $\implies$ (1)

Definition 2.8. Let X be a topological vector space. A subset A of X is called totally bounded, if for each neighbourhood U of 0 there is a finite subset F of X such that $A \subset F + U$.

Theorem 2.9. 1. Every compact set is totally bounded.

2. Every totally bounded set is bounded.

Proof. 1. Let A be any compact set. Let U be any open neighbourhood of 0.
Then

$$\{a + U : a \in A\}$$

is an open cover of A. Since A is compact we have every open cover of A has a finite subcover. Therefore there exist a finite number of points $a_1, a_2, ..., a_n$ in A such that

$$A \subset \cup\{a_i + U\}$$

This shows that A is totally bounded.

2. Let A be any totally bounded set. We want to show that A is bounded. Let U be any balanced neighbourhood of 0. Pick a balanced neighbourhood V of 0 such that

$$V + V \subset U \tag{2.9.1}$$

Since A is totally bounded, there is a finite subset F of X such that

(2.9.2) $A \subset F + V$

But F is bounded. Hence

(2.9 .3) $F \subset nV$ for some $n \geq 1$

From equations (2.9.2) and (2.9.3) we get

$$A \subset nV + V \subset nV + nV \subset n(V + V),$$

That is $A \subset n(V + V) \subset nU$ by equation (2.9.1). Thus $A \subset nU$ which implies

$$tA \subset U \text{ for } |t| < \tfrac{1}{n}$$

Hence A is bounded.

Theorem 2.10. A subset of a topological vector space X is compact if and only if it is totally bounded and complete.

Proof. Let X be a topological vector space and suppose that A is a compact subset of X.
We want to show that A is totally bounded and complete.
We have by theorem 2.9 " Every compact set is totally bounded "
Therefore A is Totally bounded.
Now we want to show that A is complete.
For that it is enough to show that every cauchy sequence in A is converges to some element in A.
Let $[x_\alpha : \alpha \in D]$ be any cauchy set in A. Let

$$T_z = \{x_\alpha : \alpha \geq z\} \text{ for each } \alpha \in D$$

But D is a directed set. So, $T_z = \{x_\alpha : \alpha \geq z\}$ has Finite Intersection Property. But then,

$$\cap\{\overline{T}_z : z \in D\} \neq \phi,$$

by compactness of A. Consequently, there exist an element $a \in A$ such that

$$a \in \cap\{\overline{T_z} : z \in D\}.$$

We shall show that $\{x_\alpha\}$ converges to a.
Let $U \in B$. Choose $V \in B$ such that $V + V \subset U$.
Since $\{x_\alpha\}$ is Cauchy, there exist α_0 such that $\alpha \geq \alpha_0, \alpha' \geq \alpha_0$ implies

$$x_\alpha - x_{\alpha_0} \in V \tag{2.10.1}$$

But $a \in \overline{T_z} \forall z \in D$.
Thus, $(a + V) \cap T_z \neq \phi$ and accordingly, there exist

$$x_\alpha \in a + V \ \forall \alpha \geq z. \tag{2.10.2}$$

Hence for $\alpha' \geq z$, we have from equations (2.10.1.) and (2.10.2)

$$x_{\alpha'} - a = x_{\alpha'} - x_\alpha + x_\alpha - a \in V + V \subset U.$$

Thus $\{x_\alpha\}$ converges to a. Thus A is complete.
Conversely suppose that A is totally bounded and complete.
We want to show that A is compact.
Let C be any collection of closed sets in A with the finite interaction property
Then there exists an ultra filter F on A with $F \supset C$. But then F is Cauchy, because A is totally bounded, since $\mathbb{K}$ is complete. $F \to x \in A$.
Let $S \in C$, let V be a neighbourhood of 0. Then,

$$x + V \supset B$$

for some $B \in F$ because F is a filter base.
But, $S \in F$.
Hence $B \cap S \neq \phi$ and hence

$$(x + V) \cap S \neq \phi.$$

Thus, $x \in S$. But S is closed in A and therefore, $x \in S$.
That is, $\cap C \neq \phi$.
This proves that A is compact.

Lemma 2.11. Let A be a convex set. If $\alpha \geq 0, \beta \geq 0$, then

$$(\alpha + \beta)A = \alpha A + \beta A$$

Proof. Let A be a convex set and suppose that $\alpha \geq 0, \beta \geq 0$. We want to show that $(\alpha + \beta)A = \alpha A + \beta A$.

- Case 1: If $\alpha = 0, \beta = 0$.

  Then the relation $(\alpha + \beta)A = 0 = \alpha A + \beta A \implies (\alpha + \beta)A = \alpha A + \beta A$

- Case 2: Let $\alpha > 0, \beta > 0$.

  If $z \in A$ then

  $$\alpha z + \beta z \in \alpha A + \beta A.$$

  But

  $$\alpha z + \beta z = (\alpha + \beta)z.$$

  Therefore,

  $$(\alpha + \beta)z \in \alpha A + \beta A, \forall z \in A,$$

that is,

$$(\alpha+\beta)A \subset \alpha A + \beta A \tag{2.11.1}$$

If $x, y \in A$

$$\alpha x + \beta y = (\alpha+\beta)(\alpha_1 x + \beta_1 y), \tag{2.11.2}$$

where $\alpha_1 = \frac{\alpha}{\alpha+\beta}$ and $\beta_1 = \frac{\beta}{\alpha+\beta}$.

But $\alpha_1 > 0$, $\beta_1 > 0$ and $\alpha_1 + \beta_1 = 1$. Thus, since A is convex, it follows that

$$\alpha_1 x + \beta_1 y \in A \tag{2.11.3}$$

From equations (2.11.2) and (2.11.3) we have

$$\alpha x + \beta y \in (\alpha+\beta)A$$

hence

$$\alpha A + \beta A \subset (\alpha+\beta)A \tag{2.11.4}$$

From equations (2.11.1) and (2.11.4) we observe that

$$\alpha A + \beta A = (\alpha+\beta)A.$$

Definition 2.12. A topological vector space is said to be normable if its topology is given by a norm.

Example 2.13. Finite dimensional Hausdorff spaces are normable.

Theorem 2.14. (Kolmogoroff 's Criterion for Normability) Let X be a topological vector space. Then X is normable if and only if X is Hausdorff and X contains a bounded convex open neighbourhood of 0.

Proof. Suppose that X is normable.
Let $V = \{x \in X : \|x\| < 1\}$.
Then V is a non-empty open subset of X.
Let G be any neighbourhood of 0. Then $W \subset G$, where for some $r \geq 0$

$$W = \{x \in X : \|x\| < r\}.$$

Also

$$W = \{x \in X : \|x\| \leq r\} \implies \tfrac{1}{r}\{x \in X : \|x\| \leq r\} = V$$
$$\text{i.e.,} \qquad V = \tfrac{1}{r}W,$$

and $\frac{1}{r}W \subset \frac{1}{r}G$
i.e., $V \subset \frac{1}{r}G$
Hence V is a bounded set in X.
Next we would show that V is convex.
Let $x, y \in V$ and let $0 \leq t \leq 1$ we have the following implications:

$$x \in V \implies \|x\| \leq 1$$
$$y \in V \implies \|y\| \leq 1.$$

and hence

$$\begin{aligned}\|tx + (1-t)y\| &\leq t\|x\| + (1-t)\|y\| \\ &\leq t + (1-t) \\ &= 1\end{aligned}$$

That is, $\|tx + (1-t)y\| \leq 1.$

Thus $tx + (1-t)y \in V$.
Thus we established that there exists a non empty, open, bounded, convex set namely V in X. And also we have the result that " Every Topological vector space is a Hausdorff space. "
i.e., X is Hausdorff and X contains a bounded convex open neighbourhood V of 0.
Conversely suppose that X is Hausdorff and X contains a bounded convex open neighbourhood U of 0.
Then we can find an absolutely convex open neighborhood V of 0 such that $V \subset U$.
Boundedness of $U \implies$ Boundedness of V . Let $\|\cdot\|$ be the Minkowski's functional of V , that is,

$$\|x\| = \inf\{\lambda > 0 : x \in \lambda V \text{ if } x \neq 0\} \tag{2.14.1}$$

But then $\|x\| = 0$ if $x = 0$. Write

$$A(x) = \{\lambda > 0 : x \in \lambda V\} \text{ , for each } x \in X$$

Since V is absorbing, there exists $\rho = \rho(x) > 0$ Such that, $\alpha x \in A$ if $|\alpha| \leq \rho$. Take, $\alpha = \rho$. Then $\rho x \in V$. Therefore $x \in \frac{1}{\rho}V$.
This means that $\frac{1}{\rho} \in A(x)$ and hence $A(x) \neq \phi$. But $A(x)$ is bounded below by 0. Thus

$$0 \leq \|x\| \leq \tfrac{1}{\rho}.$$

Hence $0 \leq \|x\| < \infty \; \forall x \in X$.
Next we shall prove that,

$$\|x + y\| \leq \|x\| + \|y\| \text{ .}$$

Take any $x, y \in X$. Let $\epsilon > 0$. By the definition of infimum, there exist $\lambda \in A(x)$ satisfying

$$\lambda < \|x\| + \epsilon.$$

Similarly, there exist $\mu \in A(x)$ satisfying

$$\mu < \|y\| + \epsilon.$$

Hence

$$\lambda + \mu < \|x\| + \|y\| + 2\epsilon. \tag{2.14.2}$$

Also, $\lambda \in A(x)\mu \in A(x)$ imply $x \in \lambda V$ and $y \in \mu V$. Accordingly, by using Lemma 2.11, we get

$$x + y \in \lambda V + \mu V = (\lambda + \mu)V$$

and hence $\|x + y\| \leq \lambda + \mu \leq \|x\| + \|y\| + 2\epsilon$ by (2.14.2) But $\epsilon > 0$ is arbitrary. Hence

$$\|x + y\| \leq \|x\| + \|y\|.$$

Next, if $\mu > 0$ then

$$\begin{aligned} \|\mu x\| &= \inf\{\lambda > 0 : \mu x \in \lambda V\} \\ &= \inf\{\mu \tfrac{\lambda}{\mu} : x \in \tfrac{\lambda}{\mu} V\} \\ &= \mu \inf\{\tfrac{\lambda}{\mu} : x \in \tfrac{\lambda}{\mu} V\} \\ &= \mu\|x\| \end{aligned}$$

$$\|\mu x\| = \mu\|x\| \tag{2.14.3}$$

Also, since V is balanced, λV is also balanced. Therefore,

$$\mu x \in \lambda V \iff |\mu| x \in \lambda V$$

and consequently,

$$\|\mu x\| = \||\lambda| x\| = |\mu|\|x\| \qquad \text{by (2.14.3)}$$

Thus

$$\|\mu x\| = |\mu|\|x\| \ \forall \text{ scalars } \mu$$

Further, we claim that

$$x \neq 0 \implies \|x\| > 0$$

Suppose $x \neq 0$ in X. Since X is Hausdorff, there exists a

(2.14.4) balanced neighbourhood W of 0 such that $x \in W$.

Since V is balanced, we can find $t > 0$ such that $V \subset tW$. Given any $\epsilon > 0$, by the infimum property, choose $\lambda \in A(x)$ such that, $\lambda < \|x\| + \epsilon$.

Now we have the following implications:

$\lambda \in A(x) \implies x \in \lambda V$

$\implies \frac{x}{\lambda}V \subset tW$

$\implies x \in \lambda tW$

$\implies \lambda t > 1$ by (2.14.4) and by using the fact that W is balanced

$\implies \lambda > \frac{1}{t}$

$\implies \frac{1}{t} < \lambda < \|x\| + \epsilon$

$\implies \frac{1}{t} < \|x\| + \epsilon$

$\implies 0 < \frac{1}{t} < \|x\|$ since ϵ is arbitrary

Thus $x \neq 0 \implies \|x\| > 0$. Or equivalently, $\|x\| = 0 \implies x = 0$.

Thus $\|.\|$ is a norm on X.

Let τ original topology on X and ξ is the norm topology on X. Let

W be any τ - neighbourhood of 0. Since V is bounded, we have $rV \subset W$ for some $r > 0$

Hence $\{rV : r > 0\}$ is a base at 0 for τ . But $V = \{x : \|x\| < 1\}$. So $\{rV : r > 0\}$ is also a base at 0 for ξ.

Example 2.15. **(Non-normable space)**: Let $\wp$ be the vector space of all complex sequence, (x_k).Define

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{|x_n - y_n|}{1 + |x_n - y_n|} \right)$$

where $x = (x_k)$ and $y = (y_k) \in \wp$.
Then d is a metric for $\wp$.
Consider $B(0, r) = \{x \in \wp : d(x, 0) < r\}$, where $0 < r < 1$.
Given $r > 0$, choose n such that $12n < r$. Take $x = \delta^k$
Let λ be an arbitrary complex number.Then

$$d(\lambda x, 0) = \frac{1}{2^n} \left(\frac{\lambda}{1 + |\lambda|} \right) \leq \frac{1}{2^n} < r.$$

Hence $\lambda x \in B(0, r)$. Since λ is arbitrary in C, we have $Cx \subset B(0, r)$.

Therefore, $\wp$ has no bounded neighborhood of 0. Consequently, $\wp$ is not normable.

APPLICATIONS OF TOPOLOGICAL VECTOR SPACES

Topological vector spaces have many important applications in various fields of mathematics, physics, and engineering. Here are a few examples:

- **Quantum mechanics:** Topological vector spaces are used extensively in the mathematical foundations of quantum mechanics, which is the study of the behavior of particles on a microscopic scale. In this context, topological vector spaces provide a framework for describing the properties of quantum states and the operators that act on them.

- **Functional programming:** Topological vector spaces are also used in functional programming, which is a programming paradigm that emphasizes the use of functions as the primary building blocks of programs. In this context, topological vector spaces provide a framework for studying the behavior of higher-order functions, which are functions that take other functions as input or output.

- **Partial differential equations:** Topological vector spaces are used to study partial differential equations, which arise in many areas of science and engineering. For example, the space of solutions to a partial differential equation can be viewed as a topological vector space, and the topology on that space can give information about the behavior of the solutions.

- **Differential geometry:** Topological vector spaces are important in differential geometry, which is the study of geometric structures on manifolds. For example, the tangent space to a manifold at a point is a topological vector space, and the space of vector fields on a manifold is a topological vector space as well.

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DON BOSCO ARTS & SCIENCE COLLEGE ANGADIKADAVU

DEPARTMENT OF MATHEMATICS

2021-2023

Project Report on

ALGEBRAIC GRAPH THEORY

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ANGADIKADAVU

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Project Report on

ALGEBRAIC GRAPH THEORY

Dissertation submitted in the partial

Fulfillment of the requirement for the award of

MSc Degree in Mathematics of

Kannur University

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KANNUR UNIVERSITY

BONAFIDE CERTIFICATE

Certified that this project report "ALGEBRAIC GRAPH THEORY" is the bonafide work of GIFTY P GEORGE who carried out the project work under my supervision.

Signature of Head | Signature of Supervisor

DECLARATION

I, GIFTY P GEORGE hereby declare that the Project work entitled ALGEBRAIC GRAPH THEORY has been prepared by me and submitted to Kannur University in partial fulfillment of requirement for the award of Master of Science is a record of original work done by me under the supervision of Mrs. PRIJA V, Assistant Professor, Department of Mathematics, Don Bosco Arts & Science College, Angadikadavu. I, also declare that this Project work has not been submitted by me fully for the award of any Degree, Diploma, Title or recognition before any authority.

Place: Angadikadavu GIFTY P GEORGE

Date: (C1PSMM1906)

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GIFTY P GEORGE

CONTENTS

INTRODUCTION

Graph theory has become a very popular and rapidly growing area of discrete mathematics for its numerous theoretical development and countless applications to practical problems. As a research area, graph theory is still relatively young, but it is maturing rapidly with many deep results having been discovered over the last couple of decades.The meaning of the term graph in graph theory is not same as the term graph that used to represent a function or statistical data. The study of graph theory was introduced by Euler in 1736. For the contribution of Euler towards graph theory, he is known as the father of graph theory. But the term graph was introduced by Sylvester in a paper published in 1878. It took 200 years since Euler's publication to revive graph theory and the first book on graph theory was published in 1936 by Denes Konig.

In the last decade of 19th century the literature of Algebraic Graph Theory has grown enormously and many research papers have appeared in this area of graph theory.Algebraic graph theory is a fascinating subject concerned with the interplay between Algebra and Graph Theory. Algebraic tools can be used to give surprising and elegant proofs of graph theoretic facts and there are many interesting algebraic objects associated with graphs. In recent years Algebraic Graph T heory has become an interesting topic of research where properties of graphs are studied by translating them into algebraic structures and the results and methods of algebra are used to deduce theorems about the graphs. On the other hand, many algebraic structures can also be studied by translating them into graph and using the properties of graphs. Algebraic graph theory includes the study of symmetry and regularity properties of graphs which can be studied using group theory. Asymmetry property of a graph is related to the existence of automorphism. The concept of automorphism of a graph, which is the permutation of vertices that preserves adjacency, played an important role in the characterization of graphs. The existence of automorphism led to the idea of permutation group of graphs. In recent times many researches are showing their interest towards the relation of graphs with rings. In this project we mention some significant results in this field which have been proved in the last century.

PRELIMINARIES

Like all other areas of mathematics, there is a certain amount of terminology with which we must be familiar in order to discuss the graphs and their properties. In this chapter we discuss some definitions and properties related to graph and algebra.

0.1 Graph Theoretic Definitions

Definition 0.1.1.

A graph is an ordered triplet (V (G),E(G), I_G) where V (G) is an empty set, E(G) is a set disjoint from V (G) and I_G is an incidence relation that associates with each element of E(G) and ordered pair of element of V (G).

Elements of V (G) are called vertices or points or nodes and elements of E(G) are called edges or links of G.V (G), and E(G) are called the vertex set and edge set respectively.

Definition 0.1.2.

Simple Graph:A graph is simple if it has no loops and multiple edges.

Definition 0.1.3.

A simple graph G is complete if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n.

Note 0.1.4.

1. A simple graph with n vertices can have at most $\frac{n(n-1)}{2}$edges.
2. The cardinality of V (G) is called the order of G and the cardi-

nality of E(G) is called the size of G.

Definition 0.1.5.

The set of all neighbors of vertex u is the open neighborhood of u or the neighborhood set of u and is denoted by N(u).

N[u] = N(u) $\cup$ {u} is the closed neighborhood of u.

Definition 0.1.6.

Isomorphism of graphs:If G and H are simple graphs, an isomorphism from G to H is a bijection ϕ : V(G) $\longrightarrow$ V(H) such that u and v are adjacent if and only if ϕ(u) and ϕ(v) are adjacent in H.

Definition 0.1.7.

Bipartite graph:A graph is bipartite if its vertex set can be partitioned into two non empty subsets X and Y such that each edge of G has one end in X and the other end in Y .The pair (X,Y) is called a bipartition of the bipartite graph.

Definition 0.1.8.

Complete bipartite graph:A simple bipartite graph is complete if each vertex of X is adjacent to all the vertices of Y .

If |X| = p and |Y| = q then the complete bipartite graph is denoted by $K_{p,q}$.

Definition 0.1.9.

A complete bipartite graph of the form $K_{1,n}$ is called a star graph.

Definition 0.1.10. Subgraph:A graph H is called a subgraph of G if V(H)$\subseteq$ V(G) or E(H) $\subseteq$ E(G).

Definition 0.1.11.

A digraph is a graph with directed edges. If G is a digraph then the elements of the set E(G) are ordered pairs and are called arcs.

For a digraph, there are two types of degree defined. For a vertex v of a digraph G, the number of edges incident from it is called out-degree of v and is denoted by $deg^{-}(v)$ and the number of edges incident to it is called in-degree of v and is denoted by $deg^{+}(v)$.

0.2 Definitions related to Algebra

Definition 0.2.1.

A ring R is a set together with two binary operations(addition and scalar multiplication) such that

1. R is an abelian group with respect to addition(so that R has a zero element denoted by 0, and every $x \in A$ has an additive inverse -x).

2. Multiplication is associative and distributive over addition.

3. $xy = yx \ \forall \ x, y \in R$, and have an identity element(denoted by 1).

4. There exists $1 \in R$ such that $x1 = 1x = x \ \forall \ x \in R$. The identity element is unique.

Here the word ring we shall mean a commutative ring with an identity element, that is, a ring satisfying above axioms.

Definition 2.2.2.

A ideal r of a ring R is a subset of R which is an additive subgroup and is such that $Rr \subset r$(i.e.,$x \in R$, and $y \in r$ imply $xy \in r$).

Definition 0.2.3.

If a and b are two non zero elements of a ring R such that ab = 0, then a and b are divisors of zero or zero-divisors.In particular a is left zero-divisor and b is a right zero-divisor.

Remark 0.2.4.

There is no distinction between left and right zero-divisors in a commutative ring.

Definition 0.2.5.

An integral domain is a commutative ring with unity containing no zero-divisors. For example, $\mathbb{Z}$ and K$[x_1, x_2, ..., x_n]$ (K a field, x_i indeterminates) are integral domains.

Definition 0.2.6.

An element x ∈ R is nilpotent if $x^n = 0$ for some n>0.

A nilpotent element is a zero-divisor but not conversely.

Definition 0.2.7. A unit in R is an element x which divides 1, that is an element x such that xy = 1 for some y ∈ R.

Definition 0.2.8.

A field is a ring R in which 1 0 and every non zero element is a unit.Every field is an integral domain but not conversely.

Definition 0.2.9.

A ring R is said to be Noetherian if it satisfies the following three conditions.

1. Every non empty set of ideals in R has a maximal element.

2. Every ascending chain of ideals in R is stationary.

3. Every ideal in R finitely generated.

Definition 0.2.10.

An Artin ring is one which satisfies the descending chain condition on ideals.

Definition 0.2.11.

A ring is quasi local if it contains a unique maximal ideal.

Definition 0.2.12.

In a commutative ring R an annihilator(ideal) of an element x, denoted ann(x), is the set of those elements y for which xy = 0.In terms of the zero divisor graph this would be the set of vertices adjacent to x. Note that x itself may be an element of ann(x).

Chapter 1

THE ZERO-DIVISOR GRAPH OF A COMMUTATIVE RING

1.1 Zero-Divisor Graph

Definition 1.1.1.

Let R be a commutative ring (with unity 1) and let Z(R) be its set of zero divisors.The zero divisor graph of a ring R is a simple graph (that is with no loops and multiple edges) whose set of vertices consists of all non zero zero divisors of R , with an undirected edge between two distinct vertices x and y if and only if xy = 0. The zero divisor graph of$R$ is denoted byΓ(R).

Exercise 1.1.2.

Let us draw a graph for the ring $R = \mathbb{Z}_{12}$, where

$\mathbb{Z}_{12} = \{0, 1, 2, ..., 11\}$. The zero-divisors of $\mathbb{Z}_{12}$ are {2, 3, 4, 6, 8, 9, 10} which is the vertex set of the zero divisor graph $\Gamma(\mathbb{Z}_{12})$. Edge set of this graph contains pair of vertices. For example, {3, 4} is an edge because $3 \times 4 = 0$ in $\mathbb{Z}_{12}$.Note that we do not consider the loops, so even though $6 \times 6 = 0$ in $\mathbb{Z}_{12}$. Therefore {6, 6} is not an edge. Now we list all edges in the complete edge set E as follows.

$E = \{\{2, 6\}, \{3, 4\}, \{3, 8\}, \{4, 6\}, \{4, 9\}, \{6, 8\}, , 9\}, \{6, 10\}\}.$

The graph $\Gamma(\mathbb{Z}_{12})$ is,

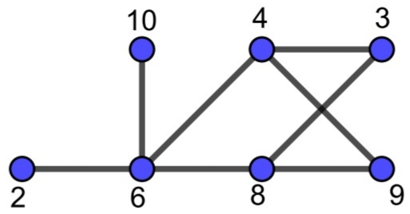

Figure 1.1

The main objective of this topic is to study the interplay of ring theoretic properties of R with graph theoretic properties of $\Gamma(R)$. This study helps to illuminate the structure of $\mathbb{Z}(R)$.

The idea of a zero-divisor graph of a commutative ring was introduced by I.Beck,where he was mainly interested in colorings.This investigation of colorings of a commutative ring was then continued by D.D.Anderson and M. Naseer. Their definition was slightly different than ours; they let all elements of R be vertices and had distinct x and y adjacent if and only if xy = 0. We will denote their zero-divisor graph of R by $\Gamma_0(R)$. In $\Gamma_0(R)$, the vertex 0 is adjacent to every other vertex, but non-zero-divisors are adjacent only to 0. Note that $\Gamma(R)$ is a (induced) subgraph of $\Gamma_0(R)$. Our results for $\Gamma(R)$ have natural analogs to $\Gamma_0(R)$; however we feel our definition better illustrates the zero-divisor structure of R.

Remark 1.1.3.

The zero divisor graph $\Gamma(R)$ is the empty graph if and only if R is an integral domain.

1.2 Properties of $\Gamma(R)$

In this section we show that $\Gamma(R)$ is always connected and has small girth and diameter and we determine which complete graphs and star graphs may be realized as $\Gamma(R)$.

Theorem 1.2.1.

Let R_1 and R_2 be two finite commutative rings. If $R_1 \cong R_2$, then $\Gamma R_1 \cong \Gamma R_2$.

Proof.

Assume $R_1 \cong R_2$. Then there exists an isomorphic map
$\phi : R_1 \to R_2$, Since ϕ is a ring isomorphism that preserves zero-divisor. That is ϕ maps zero-divisor of R_1 to a zero-divisor of R_2. Let a_i and b_i be the zero divisor of R_1 and R_2 respectively, where i = 1, 2, ..., n. Then ϕ $(a_i) = b_i$ for some i. Also if a_i is adjacent to a_j , then ϕ (a_i) is adjacent to $\phi(a_j$), for all i j. Then ϕ is a graph isomorphism. Hence $\Gamma(R_1) \cong \Gamma$ (R_2).

Remark 1.2.2.

Converse of the above theorem is not true.For this consider the example given below.

Exercise 1.2.3.

Below are the zero-divisor graphs for several rings. These examples show that non isomorphic rings may have the same zero divisor graph and that the zero divisor graph does not detect nilpotent elements.

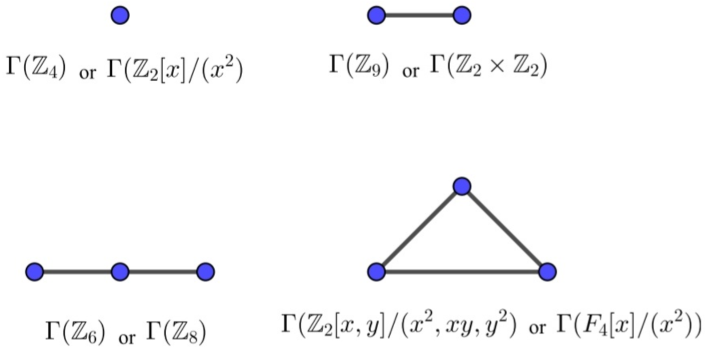

Figure1.2

Exercise 1.2.4.

Path graph (Linear graph) cannot always be realised as a zero divisor graph. It is not possible to find a ring which gives $\Gamma(R)$ with vertices {a, b, c, d} and edges a - b, b - c, c - d. That is there is no ring R such that $\Gamma(R)$ is,

Figure 1.3

We can give a proof for it as follows:

Suppose R is a ring with $\Gamma(R)$ as shown in the figure above.

Then $Z(R)$= {0, a, b, c, d}.

Since {a, b} and {b, c} are edges, so ab = bc = 0.

Therefore b(a + c) = 0 → a + c = 0 or a + c ∈ (R) . Since b makes an edge with a and c only, a + c = 0, a, b, c.We will show that the possibilities gives a contradiction. Suppose a + c = 0. Then ad = (-c)d = 0.But there is no edge between a and d. So a + c ≠ 0. Also a + c cannot be a or c either because then c = 0 or a = 0 by the cancellation property of rings. Now let us assume that a + c = b. By the symmetry of graph we can assume that b + d = c. Then

$$(a + c)(b + d) = bc$$
$$\Rightarrow (a + c)b + (a + c)d = bc$$
$$\Rightarrow ad = bc = 0.$$

This is a contradiction.Hence the proof.

Exercise 1.2.5.

We have seen above that $\Gamma(R)$ can be a triangle or square. But $\Gamma(R)$ cannot be an n-gon for n ≥ 5.

It can be proved as in the above case.

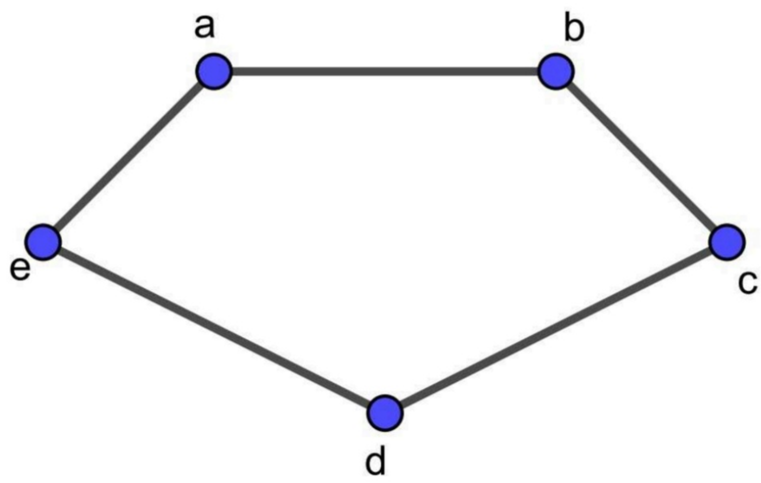

Figure 1.4

First consider the case n = 5.

Let $Z(R)$ = {0, a, b, c, d, e} with ab = bc = cd = de = ae = 0 and no other zerodivisor relation.Then (-a)b = 0 and (-a)e = 0.Thus -a=a. Similarly -x=x for every x $\in Z(R)$.Also (b + e)a = 0 or b + e = 0, a, b, e.Clearly we cannot have b + e = b and b + e = e. Also b + e = 0 is not possible.Hence

b + e = a and thus $a^2 = 0$. Since we consider simple graph there is no loops.Hence $x^2 = 0$ for x $\in Z(R)$ is not possible. The case for n $\geq$ 5 is similar.

Remark 3.2.6.

For each n $\geq$ 3 there is a zero divisor graph with n-cycle.

Remark 3.2.7.

Let A and B be integral domain and let R = A $\times$ B. Then $\Gamma(R)$ is a complete bipartite graph with $|\Gamma R| = |A| \times |B| - 2$.

Exercise 1.2.8.

If $A = \mathbb{Z}_{12}$, then $\Gamma(R)$ is a star graph with $|\Gamma R|$=|B|.

For example, $\Gamma(\mathbb{F}_p \times \mathbb{F}_q) = K^{p-1,q-1}$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{F}_q) = K^{1,q-1}$.We give two specific examples.

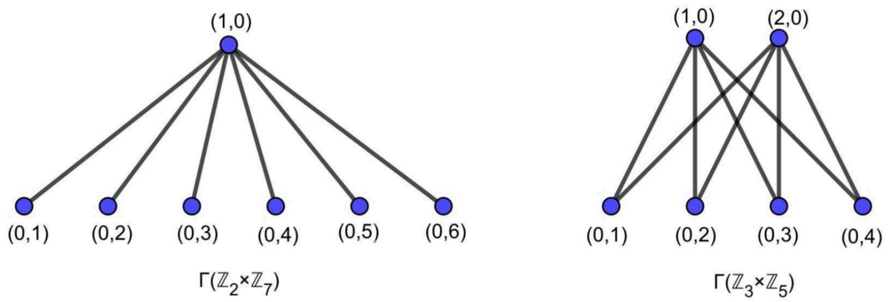

Figure 1.5

Remark 1.2.9.

ΓR may be infinite.That is a ring may have an infinite number of zero divisors. But ΓR is probably of most interest when it is finite,for then one can draw $\Gamma(R)$.

We will show that $\Gamma(R)$ is finite only when R itself finite.

Theorem 1.2.10.

Let R be a commutative ring. Then $\Gamma(R)$ is finite if and only if either R is finite or an integral domain.In particular, if $1 \leq |\Gamma\ R| < \infty$, then R is finite and not a field.

Proof.

Suppose that $\Gamma(R)$ is finite and non empty. Then there are non zero x, y $\in R$ with xy = 0. Let I = ann(x). Then y $\in$ I and in fact,ry $\in$ I for all r $\in$ R and I $\subseteq Z(R)$. Thus R must be finite.Suppose R is infinite with finitely many zero divisors. Since I is a subset of the zero divisors of R, it is finite.Thus there exists an i $\in$ I such that J = {r $\in R:$ ry = i} is infinite.For r, s $\in$ J, (r s)y = 0. Thus, ann(y) is infinite, contradicting the fact that there are only finitely many zero divisors.

Definition 1.2.11.

A graph Γ is connected if there is a path between any two distinct vertices.

For distinct vertices x and y of Γ, let d(x, y) be the length of the shortest path from x to y (d(x, y) = ∞ if there is no such path).

Definition 1.2.12.

The diameter of Γ is diam(Γ) = sup {d(x, y) : x, y $\in \Gamma$}.

The girth of Γ, denoted by g(Γ), is defined as the length of the shortest cycle in Γ (g(Γ)=∞ if Γ contains no cycle). Note that if Γ contains a cycle, then g(Γ) $\leq$ 2diam(Γ)+1.

We next show that the zero divisor graphs are all connected and have exceedingly small($\leq$ 3) diameter and girth. Thus, for distinct x, y $\in Z$(R)*, either xy = 0, xz = zy = 0 for some z $\in Z$(R)* - {x, y}, or xz_1 = $z_1 z_2$ = z_2y = 0 for some distinct $z_1, z_2 \in Z$(R)* -{x, y}.

Theorem 1.2.13.

Let R be a commutative ring (not necessarily finite). Then Γ R is connected. Moreover diam (Γ R)$\leq$ 3.

Proof.

Let x, y $\in \Gamma(R)$, with x $\neq$ y. If xy = 0, then d(x, y) = 1. Suppose now that xy $\neq$ 0. If x^2 = 0 =y^2 , then x - xy - y is a path of length two and d(x, y) = 2. Suppose x^2 = 0 and $y^2 \neq$ 0. There exists an element b $\in \Gamma(R)$ with b $\in$ y such that by = 0. If bx = 0, then x - b - y is a path of length two between x and y. In either case, d(x, y) = 2. A symmetric argument holds if y^2 = 0 and $x^2 \neq$ 0. Thus we may suppose that neither x^2 and y^2 is zero. Then there exists non zero zero divisors a, b $\in \Gamma(R)$ (not necessarily distinct) with ax = 0 = by. If a = b, then x - a - y is a path of length two, and hence d(x, y) = 2. Thus we may assume that a $\neq$ b. Consider the element ab. If ab = 0, then x - a - b - y is a path of length 3, and hence d(x, y) $\leq$ 3. If ab $\neq$ 0, then x - ab - y is a path of length 2, and hence d(x, y) = 2. In all of the cases, there is a path between x and y of length less than or equal to 3, and since x and y were arbitrary it follows that the diameter of $\Gamma(R)$ is less than or equal to 3.

Remark 1.2.14.

It is clear that the zero divisor graphs of $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$and $\mathbb{Z}_6$ have diameter 0,1 and 2 respectively. We can show that the diameter 3 is also achieved. For this consider the ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The distance between the elements (1,1,0) and (0,1,1) is three. In fact, a shortest path is (1,1,0) - (0,0,1) - (1,0,0) - (0,1,1).

The fact that the distance between points is small also constraints the length of the shortest cycle, that is to say, the girth of the graph. The following corollary make use of the previous theorem to establish a bound for the girth of a zero divisor graph.

Corollary 1.2.15.

If R is a ring, then the girth of $\Gamma(R)$ is less than eight.

Proof.

It is enough to suppose to the contrary that we could find a ring R such that $\Gamma(R)$ has a smallest cycle C of length exactly 8, say, v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - $v_8 = v_0$. Let P_1 denote the path v_0 - v_1 - v_2 - v_3 - v_4 and P_2 denote the path v_0 - v_7 - v_6 - v_5 - v_4.To help in visualizing the proof the figure below provides a hypothetical representation of each of the two main considerations which follow.First observe that for 0 < i < 4 < j < 8, v_i and v_j are not connected. Assume the case is otherwise. Then v_0 - v_1...- v_i - v_j...- v_0 is a cycle of length less than eight contradicting the assumption that the girth is eight. Now assume there is a path v_0- x - y - v_4

by previous theorem(If not, then there is a path v_0 - x -v_4. The proof goes through if in this case we just identify x and y. The impossibility of v_0 and v_4 being adjacent is apparent). The fact just proved implies that the path v_0- x - y - v_4 intersects P_1, or perhaps P_2, but not both.Thus, by symmetry we may as well assume that the path v_0- x - v_4 does not intersect P_2. This assumption yields a cycle $v_0 - x - y - v_4 - v_5 - v_6 - v_7 - v_8 = v_0$ of length seven, the final contradiction.

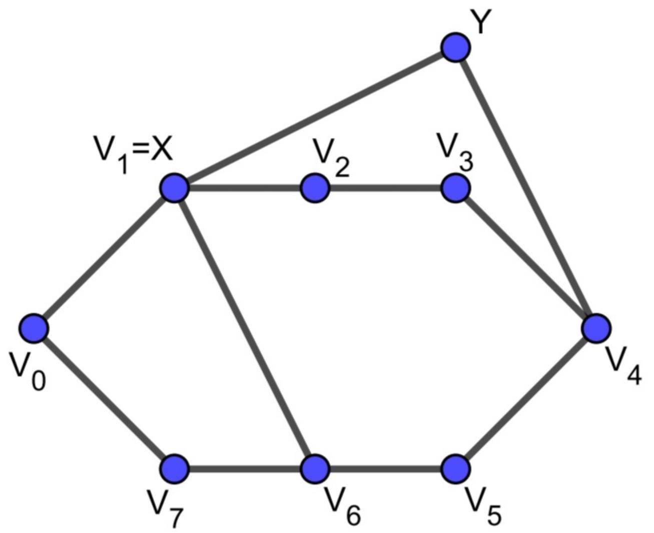

Figure 1.6

Exercise 1.2.16.

Let T be a an integral domain, and n ≥ 3 an integer. Define

$$R = \text{T}[X_1,X_2,..., X_n] \ / \ (X_1X_2,X_2X_3,...,X_nX_1)$$

and let x_i be the coset of X_i in R. Then x_1-x_2...-x_n-x_1- is a cycle of length n.

Chapter 2

CAYLEY GRAPHS

2.1 Cayley Graphs

Cayley graphs, named after mathematician Arthur Cayley, is an important concept relating group theory and graph theory. Cayley graphs are frequently used to render the abstract structure of a group easily visible by a way of representing this structure in graph form. Properties of a group G, such as its size or number of generators, become much easier to examine when G is rendered as a Cayley graph. Though also referred to as group diagrams, the definition of Cayley graphs is suggested by Cayley's theorem, which states that every group G is isomorphic to a subgroup of the symmetric group acting on G.

Definition 2.1.1.

Let G be a finite group with identity 1. Let S be a subset of G satisfying $1 \notin$ S and S = S^{-1}. That is, s $\in$ S if and only if $s^{-1} \in$ S. The Cayley graph Cay (G, S) on G with connection set S is defined as follows:

(a) the vertices are the elements of G;

(b) there is an edge joining g and h if and only if h = sg for some s

$\in$ S.

The set of all Cayley graphs on G is denoted by Cay (G).

Note 2.1.2.

When G is an abelian group, we use additive notation for the group operation. Hence we write S = -S for the connection set and h = s + g(for some s $\in$ S).

For example consider Cay ($\mathbb{Z}_9$, $\{1, 3, 6, 8\}$).

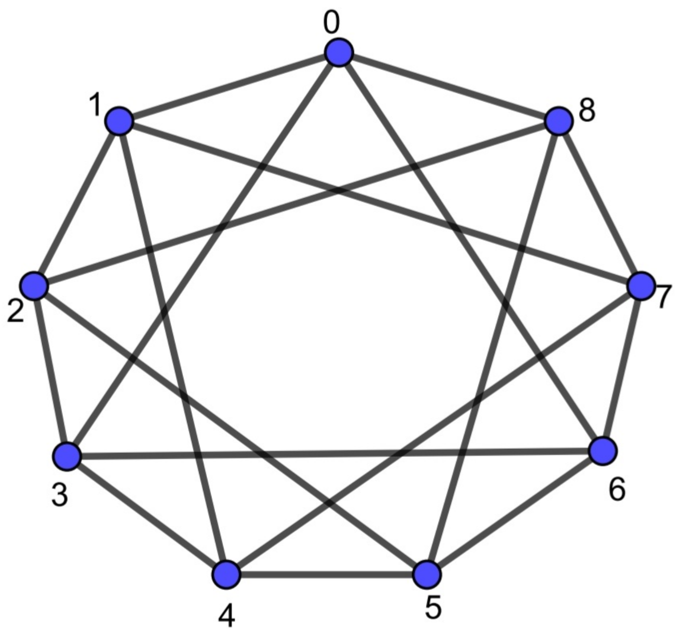

Figure 2.1

2.2 Types of Cayley Graphs

There are three main types of Cayley graphs:

1. **Cayley Digraphs**

2. **Cayley Color Graphs**

3. **Simple Cayley Graphs**

(1)**Cayley Digraphs:** Let G be a finite group and let S $\subseteq$G. The Cayley digraph on G with connection set S, denoted D(G, S), is the digraph with vertex set G and with (g, h) being an arc if and only if g$h^{-1} \in$ S.

Consider the example: Cay ($\mathbb{Z}_{10}$, {1, 3, 7,9}) given below

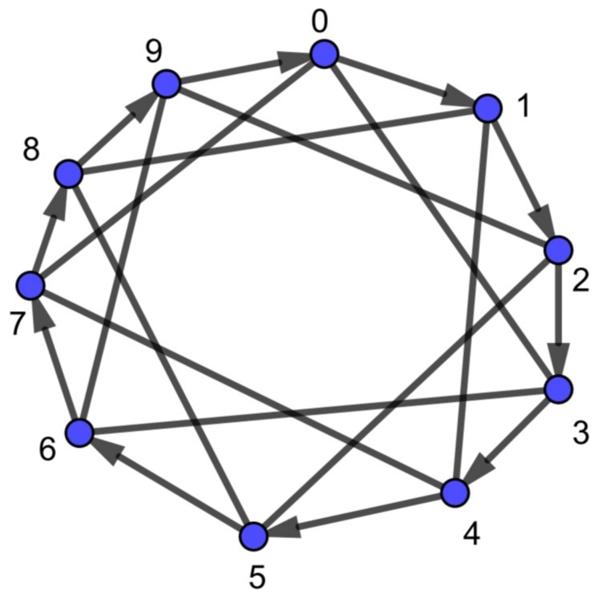

Figure 2.2

(2) **Cayley Color Graphs:** We can extend the idea of Cayley digraph to a Cayley color graph, where S is a generating set for G, each $s_i \in$ S is assigned a color, and if g = s_ih, then the arc connecting them is colored s_i.

Example given below shows the Cayley color graph of s_3 with connection set S = {a, b}, where a = (1 2 3) and b = (1 2).

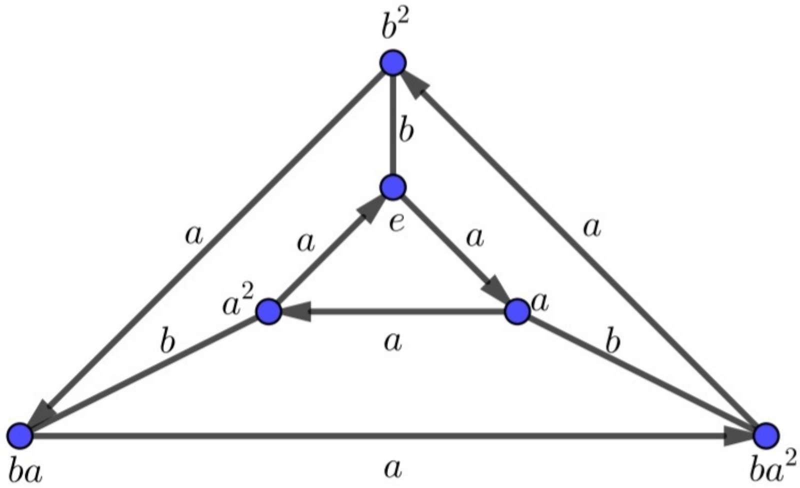

Figure 2.3

Notice that it is often helpful to represent each group element in terms of group presentations, thus displaying the arcs more clearly: Now let us examine the automorphism groups of these Cayley color graphs. First, let us present another definition.

Definition 2.2.1.

An automorphism $\phi \in$ Aut (D(G, S)) is color preserving if given an arbitrary arc {g, h}, {g, h} and {ϕ (g) , ϕ (h)} have the same color.

Before presenting the main theorem, we must first present an intermediate result.

Proposition 2.2.3.

Let G be a group with generating set S and let ϕ be a color preserving permutation on V (D(G, S)). Then ϕ is a color preserving automorphism of D(G, S) if $\phi\ (gh) = \phi\ (g)h$.

Proof.

Suppose that $\phi\ (gh) = \phi\ (g)h$. To show that ϕ is color preserving, we need to show that if gh^{-1} = s then $\phi(gh^{-1})$=s.

Suppose gh^{-1} = s. Then,

$$\begin{aligned} \phi(gh^{-1}) &= \phi(g)h^{-1} \\ &= \phi(g)g^{-1}\text{s} \\ &= \phi(gg^{-1}\text{s}) \\ &= \text{s} \end{aligned}$$

Theorem 2.2.4.

Let G be a nontrivial group with generating set S. Then the group of color preserving automorphisms of D(G, S) is isomorphic to G.

Proof.

Let G be a group of order n and $g_i \in$ G for 1 $\leq$ i $\leq$ n. Define the map ϕ_i : V (D(G, S)) $\longrightarrow$ V (D(G, S)) by ϕ_i (g) = $g_i g$. This map is surjective, since given any $g \in$ V (D(G, S)), $g = \phi_i\ (g_i^{-1}\ g)$. This map is also injective since if

$$\begin{gathered} \phi_i\ (g_1) = \phi_i\ (g_2) \\ \Rightarrow g_i g_1 = g_i g_2 \\ \Rightarrow g_1 = g_2. \end{gathered}$$

Now let g_1,$g_2 \in$ G. By above proposition, ϕ_i is a color preserving automorphism since,

$$\begin{aligned}\phi_i\ (g_1 g_2) &= g_i\ (g_1\ g_2)\\ &= (g_i g_1)\mathrm{g}_2\\ &= \phi_i\ (g_1)\ g_2.\end{aligned}$$

Now let A = $\{\phi_i : 1 \leq \mathrm{i} \leq \mathrm{n}\}$ and define a map α : G $\longrightarrow$ A by $\alpha(g_i) = \phi_i$. We must verify this map is an isomorphism from G to the groups of color preserving automorphisms. This map is injective since $\phi_i \neq \phi_j$ when i $\neq$ j. We will show that the map is surjective by proving that for any color preserving automorphism ϕ, $\phi = \phi_i$ for some $\phi_i \in$ A. Let e be an identity element of G and let ϕ(e) = g_i. Given an arbitrary element $g_j \in$ G, we can write this element as a product of generators from our generating set s. Let $g_j = {s_1}^{r_1}\ {s_2}^{r_2}$... ${s_m}^{r_m}$.Then by above proposition we can write,

$$\begin{aligned}\phi(g_j) &= \phi(eg_j)\\ &= \phi(\mathrm{e}{s_1}^{r_1}\ {s_2}^{r_2}...\ {s_m}^{r_m}.)\\ &= \phi(\mathrm{e}){s_1}^{r_1}\ {s_2}^{r_2}...\ {s_m}^{r_m}.\\ &= g_i g_j\\ &= \phi_i(g_j).\end{aligned}$$

Therefore $\phi = \phi_i$ and α is surjective.

Finally we must show that α preserve the group operation. Since,

$$\begin{aligned}\phi_{ij}(g) &= (g_i g_j)g\\ &= g_i(g_j g)\\ &= \phi_i\ (g_j g)\\ &= \phi_i(\phi_j(g))\end{aligned}$$

We can deduce that,

$$\begin{aligned} \alpha(g_i g_j) &= \phi_{ij} \\ &= \phi_i \circ \phi_j \\ &= \alpha(g_i) \circ \alpha(g_j). \end{aligned}$$

Thus we have shown that given any group G, we can construct a colored digraph representation of this group whose color preserving automorphism group is isomorphic to G itself. We will now turn our attention to a more generalized version of Cayley graphs and examine their properties

3) Simple Cayley Graphs: Simple Cayley graphs make up a large amount of graphs in an important family of graphs called Vertex-Transitive graphs. They are defined nearly identically as Cayley digraphs, however they contain an edge set instead of an arc set,and the connection set S must be closed under inverses. More specifically, if s $\in$ S, then s^{-1} $\in$ S.

To further understand these graphs, let us first consider the motivation behind this restriction on S. Given a Cayley digraph, we know that there exists an arc u,v if and only if u = sv. However, if s^{-1} $\in$ S then v = s^{-1} u and therefore v,u is also an arc. Thus, we can replace these two arcs by a simple edge. This restriction on S thus replaces every pair of arcs with an edge, turning our digraph D (G, S) into a simple Cayley graph. We will denote the simple Cayley graph of a group G on the connection set S by Cay (G, S). To study these graphs further let us first review some definitions and theorems. Recall that given a group G acting on a set V , x, y $\in$ V are G - equivalent if gx = y for some g $\in$ G. This G - equivalence is an equivalence relation on V . Each partition of V into such an equivalence class is called an orbit of V under G. Also recall that the stabilisersubgroup of G for an element x $\in$ V , denoted G_x, is the set of all group elements in G that fix x.

That is,

$$G_x = \{\text{g} \in \text{G} : \text{gx} = \text{x}\}$$

Definition 2.2.5.

A graph Γ is vertex transitive if there exists a single orbit of V (Γ) under Aut(Γ). That is, given any v, u $\in$ V (Γ) there exists a $\phi \in$ Aut(Γ) such that ϕ (v) = u.

Note that here Γ denote simply a graph not a zero divisor graph.

Theorem 2.2.6.

Every Simple Cayley Graph is Vertex-Transitive.

Proof.

Let $\rho_g : v \longrightarrow vg$ for all $v \in$ V (Cay (G, S)). Clearly, ρ_g permutes the elements of V (Cay (G, S)). To show that $\rho_g \in$ Aut(Cay (G, S)) we must show that $\{v, u\} \in$

E (Cay (G, S)) if and only if $\{vg, ug\} \in$ E (Cay (G, S)).

Suppose $\{v, u\} \in$ E (Cay (G, S)). Then we know that $v = su$ for some $s \in$ S or equivalently, $vu^{-1} = s$. But

$$(vg)(ug^{-1}) = vgg^{-1}u^{-1}$$

$$= vu^{-1}.$$

Therefore, $\{v, u\} \in$ E (Cay (G, S)) if and only if $\{vg, ug\} \in$ E (Cay (G, S)) and ρ_g is the group of Cay(G, S).

Now given any vertices v, u the mapping $\rho_{v^{-1}u}$ maps v to u, since $vv^{-1}u = u$. Thus any Cayley graph is vertex-transitive.

Just as a subgroup of an automorphism group for a Cayley color graph of G contained a subgroup isomorphic to G, simple Cayley graph possess a similar property.

Definition 2.2.7.

A group G acting on a set V is semi-regular if $G_v =$ e for all $v \in$ V. If a group is semi-regular and transitive, then we say it is regular.

Theorem 2.2.8.

Let G be a group and S be an inverse closed subset of G. Then Aut(Cay(G, S)) contains a regular subgroup isomorphic to G.

Proof.

Let G be a group with connection set S and Cay(G, S) be the Cayley graph for G defined on S. Now consider the mapping $\rho_g : v \longrightarrow vg$. We know that $\rho_g \in$ Aut(Cay(G, S)),and it can be easily shown that H $= \{\rho_g : g \in$ G$\}$ is a subgroup of G. This group acts regularly on G, since it is clearly semi-regular and is also transitive by the proof of the above theorem. The map $\phi :$ H $\longrightarrow$ G defined by $\phi(\rho_g) = g$ is an isomorphism by Cayley's theorem. Thus, H $\leq$ Aut(Cay(G, S)) is isomorphic to G.

It is natural to ask whether all vertex-transitive graphs are Cayley graphs. This question was negatively answered by the counter example of the Petersen graph.

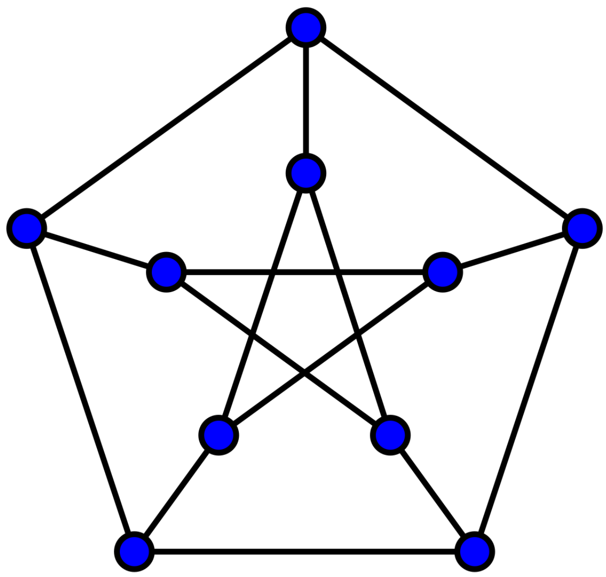

Figure 2.4

The Petersen graph on 10 vertices is the smallest example of a vertex-transitive graph which is not a Cayley graph.
The following theorem gives criteria for when a graph is indeed a Cayley graph for some group.

Theorem 2.2.9.

If a group G acts regularly on the vertices of the graph Γ, then Γ is a Cayley graph for G relative to some inverse closed connection set S.

Proof.

Let Γ be a graph with degree n and let G be a group acts normally on V (Γ) = $\{v_1, v_2, ..., v_n\}$. Since G acts normally on Γ, there exists a unique element $g_i \in$ G such that $g_i v_1 = v_i$ Now define a set S by,

$$S = \{g_i \in G : \{v_i, v_1\} \in E(\Gamma)\}$$

Now let x,y be arbitrary elements in V (Γ). Then since g_x is an automorphism of Γ, so$\{x, y\} \in E(\Gamma)$ if and only if $\{g_x^{-1}x, g_x^{-1}y\} \in E(\Gamma)$. Since $g_x u = x$, $g_x^{-1}x = u$. Furthermore, since $g_y u = y$, $g_x^{-1}g_y u = g_x^{-1}y$ .Therefore,

$g_x^{-1}g_y \in S \longleftrightarrow \{u, g_x^{-1}y\} \in E(\Gamma) \longleftrightarrow g_x^{-1}x, g_x^{-1}y \in E(\Gamma) \longleftrightarrow \{x,y\} \in E(\Gamma)$.

So, if we identify every vertex x with the group element g_x, then Γ = Cay(G, S). Since Γ is undirected, S is closed under inverses.

So given any graph Γ and a subgroup G acts regularly on V (Γ) if and only if Γ is a Cayley graph for G for some connection set S. Notice that our definition of a Simple Cayley graph does not require our connection set to be a generating set for G. Our next theorem shows the consequence of S being a generating set for G. First, let us present two definitions.

Definition 2.2.10.

A path of length r from vertex x to vertex y in a graph is a sequence of r + 1 distinct vertices starting with x and ending with y such that consecutive vertices are adjacent.

Definition 2.2.11.

A graph Γ is connected if there is a path between any two vertices of Γ.

Theorem 2.2.12.

The Cayley graph Cay(G, S) is connected if and only if S is a generating set for G.

Exercise 2.2.13.

Consider the group $G = \mathbb{Z}_5$, with connection set $S = \{2, 3\}$.Since $0 + 2^{-1} = 3$, $0 + 3^{-1} = 2$, $1 + 3^{-1} = 3$, $1 + 4^{-1} = 2$, $2 + 4^{-1} = 3$. Below is the graph ,Cay($\mathbb{Z}_5$,$\{2, 3\}$).

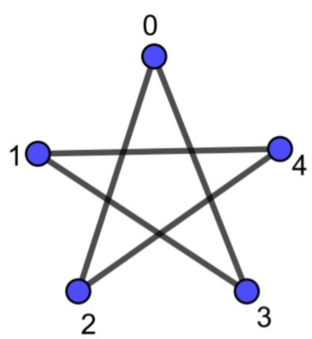

Figure 2.5

Notice that the graph is connected, since S is a generating set for G.

2.3 Graphic Regular Representation

We know that given a group G and a connection set S, there is a subgroup of Aut(Cay(G, S)) that is isomorphic to G. Often, this subgroup is a proper subgroup of Aut(Cay(G, S)), meaning the entire Aut(Cay(G, S)) is not isomorphic to G. Consider for example G = S_3. In the case of finite abelian groups Aut(Cay(G, S)) is itself isomorphic to G.

Definition 2.3.1.

A group G admits graphical regular representation if the automorphism of Cayley graph is isomorphic to G.

Let us again provide an intermediate result before presenting our main theorem for this section.

Proposition 2.3.2.

Let Cay(G, S) be a Cayley graph for G defined on the connection set S. Suppose that ϕ is an automorphism of the group G that fixes S set-wise. Then, ϕ regarded as a permutation of the vertices of Cay(G, S) fixes the vertex corresponding to the identity element of G.

Proof.

Let ϕ be a group automorphism. Then ϕ must fix the identity element. Let us show that ϕ is a graph automorphism. Suppose that v, w are adjacent vertices. Then $vw^{-1} \in$ S. Therefore, $\phi(vw^{-1}) \in$ S. But, $\phi(vw^{-1}) = \phi(v)\phi(w^{-1})$. So $\phi(v)$ and $\phi(w)$ are adjacent. Therefore, ϕ preserves adjacency and is a graph automorphism.

Theorem 2.3.3. Let Γ be a vertex-transitive graph whose automorphism group G = Aut(Γ) is abelian. Then G acts regularly on V (Γ), and G is an elementary abelian 2-group.

Proof.

Suppose $\Gamma$is a vertex-transitive graph with automorphism group G = Aut(Γ). Let $g, h \in$ G. Suppose that g fixes an arbitrary vertex $v \in$ V (Γ). Now,

$$gh(v) = hg(v) = h(v).$$

Therefore, g fixes $h(v)$ as well. However, since G acts transitively on V (Γ) any vertex, u can be written as $h(v) = u$ for some $h \in$ G. Thus g fixes every vertex and the stabilizer is the identity. Therefore, G acts regularly on V (Γ) and Γ = Cay(G, S) for some connection set S. Now consider the map $g \longrightarrow g^{-1}$. This map preserves the adjacency since Aut(Γ) is abelian and is therefore a graph automorphism. Furthermore, since S is closed under taking inverses, this mapping fixes S set-wise. Thus, by above proposition, this map must fix vertex 1. But since G acts regularly on V (Γ), this map must be the identity map.Therefore $g = g^{-1}$ and G is an abelian 2-group.

Let us think for a moment what this theorem officially states. By theorem, If a group G acts regularly on the vertices of the graph Γ, then Γ is a Cayley graph for G relative to some inverse closed connection set S, we know that if G acts regularly on a graph Γ, then Γ is a Cayley graph for G. Therefore, we know that if a graph has an abelian automorphism group, then this abelian group has a graphical regular representation. Furthermore, we know that this can only happen when G is an abelian 2-group. Thus, the only abelian groups that have a graphical regular representation are abelian 2-groups.

2.4 Basic Properties of Cayley Graphs

Cayley diagrams are one of many representations of finite groups. They provide a means of representing a group diagrammatically and various properties of groups including commutativity can be extracted from the graph. The Cayley diagram also provides sufficient information to test for isomorphism between groups, and thus is a useful tool for recognizing the type of a given group.

Let S = $\{g_1, g_2, ..., g_n\}$ be a set of distinct elements and let G is the group generated by the set S. We can define a relation $\sim$ on G such that a $\sim$ b if and only if b = $g_i a$, where $g_i \in$ S. Then the Cayley digraph is the digraph formed from the relation $\sim$, where the vertex set of the graph is the group G.

Here are various simple properties of Cayley digraphs.

1. Let a be a vertex in D(G, S) and |S| = n. Then $\deg^+$(a)

= deg$^-$(a) = n. Its proof is as follows: Let a $\in$ G. Then $g_1^{-1}a$,$g_2^{-1}a$,...$g_n^{-1}a$ n a are n distinct elements in G(by closure), since suppose that

$$g_i^{-1}a = g_j^{-1}a$$
$$\Rightarrow g_i^{-1} = g_j^{-1}$$
$$\Rightarrow g_i = g_j.$$

This is a contradiction to the fact that elements of S are distinct. Hence $g_i^{-1}a \sim a$, i = 1,2,...,n.Therefore deg$^-$(a) = n. Similarly $g_1a, g_2a, ..., g_na$ are n distinct elements in G, and a $\sim$ g_ia for all i = 1, 2, ..., n.Therefore,deg$^+(a)$ = n.

2. Cayley digraph is strongly connected(That is, there is a path from a to b and from b to a whenever a and b are vertices in the graph.

3. If G $\neq$ {e} then $\sim$ is irreflexive. That is Cayley digraphs has no loops.

4. If for all $g \in$ S, $g^{-1} \in$ S, then $\sim$ is a symmetric relation. That is, Cayley digraph is an undirected graph.

5. Let $G_1 = \langle S_1 \rangle$ and $G_2 = \langle S_2 \rangle$ be isomorphic groups, $\mid S_1 \mid \leq \mid S_2 \mid$, then D($G_1$, S_1) is isomorphic to a subgraph of

$D(G_2, S_2)$.

2.5 Applications Of Cayley Graphs

There are many meaningful applications of Cayley graphs and we conclude with a quick outline of a few of them.

1. Any problem for which graphs are used as a model and the use of edges is being optimized provides a natural setting for Cayley graphs.For example, suppose that we are to construct a network for which the number of direct links we may use is restricted, but we want to maximize the probability that the network remains connected after some links or vertices of the network are deleted. Then there is a strong tendency for the graph to be either vertex-transitive or close to it.Consequently Cayley graphs in particular the K-dimensional cube Q_K have been extensively studied by researchers working with networks.

2. It is possible to tell from a Cayley digraph whether or not the corresponding group is commutative.

3. Another application of Cayley diagrams is in that on binary representation of data, namely Gray codes. A gray code of length n is a sequence of n bit binary strings, with the property that consecutive words differ by at most one element. Gray codes are useful in mechanical encoders since a slight change in location only affects one bit. Using a typical binary code , up to

n bits could change, and slight misalignments between reading elements could cause high levels of error since flipping a bit will increase or decrease the value by a power of two. For example, an error flipping the MSB of an 8-bit word will change the value by 2^7.

Gray codes can be represented by the direct product $(\mathrm{C}_2)^n$.The difference between Gray code and normal binary code is the ordering of the elements. In gray code the greater than or equal to relation is defined as follows:

For a, b $\in (\mathrm{C}_2)^n$, a $\geq$ b if and only if a $\rightarrow$ b. The fact that $((\mathrm{C}_2)^n, \geq)$ is a totally ordered set follows from that we can always find a Hamiltonian path in $((\mathrm{C}_2)^n$, S)(since every two elements in the path are comparable). Thus $((\mathrm{C}_2)^n , \geq)$ is a well ordered set by choosing the starting vertex(element) of the Hamiltonian path as the least element.

Chapter 3

AUTOMORPHISM GROUP OF GRAPHS

Definition 3.0.1.

A graph automorphism is an isomorphism from a graph to itself. In other words, an automorphism of a graph G is a bijection $\phi : V(G) \longrightarrow V(G)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(G)$.

This definition generalizes to digraphs and graphs with loops. Here we will assume that all graphs are undirected graphs with no loops or multiple edges. Let Aut(G) denote the set of all automorphisms on a graph G. Note that this forms a group under function composition. In other words,

1. Aut(G) is closed under function composition.
2. Function composition is associative on Aut(G). This follows from the fact that function composition is associative in general.
3. There is an identity element in Aut(G). This is mapping $e(v) = v$ for all $v \in V(G)$.
4. For every $\sigma \in$ Aut(G), there is an inverse element $\sigma^{-1} \in$ Aut(G).

Since σ is a bijection, it has an inverse. By definition, this is an automorphism.

Thus, Aut(G) is the automorphism group of G. Consider the graph illustrated below.

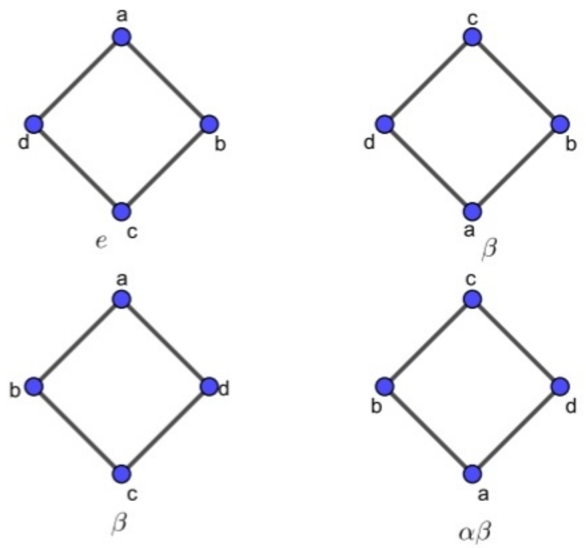

Figure 3.1

An automorphism of G can leave every vertex fixed, this is the identity automorphism e. An automorphism of G can swap vertices a and c and leave the others alone. This is the automorphism $\alpha = (a, c)$. Similarly we can swap vertices b and d while leaving a and c fixed. This results in the automorphism $\beta = (b, d)$. Finally, we can swap vertices a and c and swap vertices b and d. This results in the automorphism $\alpha\beta = (a, c)\ (b, d)$. Hence, Aut(G) is isomorphic to the Klein - 4 - group, $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$.

We can use the automorphism group to define a relation between two vertices in G. Let $u, v \in V\ (G)$, vertex u relates to v if there exists $\phi \in \text{Aut(G)}$ such that $\phi\ (u) = v$. We claim that this is an equivalence relation.

1. Reflexive: Note that $e(u) = u$for all $u \in$ V (G), where $e$is the identity automorphism.

2. Symmetric: If $\phi\ (u) = v$, then $\phi^{-1}(v) = u$.

3. Transitive: If $\phi\ (u) = v$ and $\sigma\ (v) = w$, then $\sigma(\phi(u)) = w$.

Thus the relation is an equivalence relation. Like all equivalence relations, this induces a partition of the vertex set into equivalence classes. These classes are usually called automorphism classes. If all the vertices of the graph are in the same automorphism class, then we say that the graph is vertex transitive.
Some facts about the automorphism of a graph.

Proposition 3.0.2.

Let G be a graph.

1. (Degree preserving)For all $u \in$ V (G) and for all $\phi \in$ Aut(G), deg(u) = deg($\phi(u)$).

2. (Distance preserving)For all $u, v \in$ V (G) and for all $\phi \in$ Aut(G), d(u, v) = d ($\phi(u)$, $\phi(v)$).

3. The automorphism group of G is equal to the automorphism group of the complement $\overline{G}$.

Proof.

1. Let $u \in$ V (G) with neighbors $u_1, u_2, ..., u_k$. Let $\phi \in$ Aut(G). Since ϕ preserves adjacency, it follows that $\phi(u_1)$, $\phi(u_2)$, ...,$\phi(u_k)$ are neighbors of $\phi(u)$. Therefore, deg($\phi(u)$) $\geq$ k. If v $\notin \{u_1, u_2, ..., u_k\}$ is a neighbor of $\phi(u)$, then $\phi^{-1}(v)$ is a neighbor of u. Therefore, the neighbors of $\phi(u)$ are precisely $\phi(u_1)$, $\phi(u_2)$, ..., $\phi(u_k)$.Hence deg(u) = deg($\phi(u)$).

2. Let $u, v \in$ V (G) and let $\phi \in$ Aut(G). Suppose that the distance from u to v is $d(u, v) =$ d. Further, let $u = u_0, u_1, ..., u_{d-1}, u_d = v$ be a shortest path from u to v. Since ϕ preserves adjacency, $\phi(u)$ $= \phi(u_0)$, $\phi(u_1)$, ..., $\phi(u_{d-1})$, $\phi(u_d) = \phi(v)$ is a path from $\phi(u)$ to $\phi(v)$. Thus, d($\phi(u)$, $\phi(v)$) $\leq d = d(u, v)$. Suppose that $\phi(u)$, $v_1, ..., v_{m-1}$,$\phi(v)$ is a shortest path from $\phi(u)$ to $\phi(v)$.It follows that u, $\phi^{-1}(v_1)$,...,$\phi^{-1}(v_{m-1})$, v is a shortest path from u to v. It follows that $d(u, v) \leq$ d($\phi(u)$,$\phi(v)$). Hence, have equality.

3. We will proceed by showing set inclusion in both directions. First suppose that we have some permutation $\sigma \in$ Aut(G), and an edge e $\notin E_G$. By the definition of the complement of a graph, it follows that e $\in E_{\overline{G}}$. Similarly, we know from the definition of a graph automorphism that σ(e) $\notin E_G$, and hence we find that σ(e) $\in E_{\overline{G}}$. Thus we have shown that σ is an automorphism of $\overline{G}$, and thus $\sigma \in$ Aut($\overline{G}$).Note that G is isomorphic to the complement of $\overline{G}$, so we can simply interchange G and $\overline{G}$ and find that if $\tau \in$ Aut($\overline{G}$), then $\tau \in$ Aut(G). Thus we have shown

that Aut(G) = Aut($\overline{G}$).

3.1 The Automorphism Group of Specific Graphs

In this section, we give the automorphism group for several families of graphs.Let the vertices of the path, cycle, and complete graph on n vertices be labelled $v_0, v_1, ..., v_{n-1}$ in the obvious way.

Theorem 3.1.1.

1. For all n $\geq$ 2, Aut(P_n) $\cong \mathbb{Z}_2$, the second cyclic group.
2. For all n $\geq$ 3, Aut(C_n) $\cong D_n$, the n^{th} dihedral group.
3. For all n, Aut(K_n) $\cong S_n$, the n^{th} symmetric group.

Proof.

1. As in the proof of the above proposition, any automorphism ϕ $\in$ Aut(P_n) must either map a vertex of degree one to a vertex of degree one. Thus either $\phi(v_0) = v_0$ and $\phi(v_{n-1}) = v_0$ or $\phi(v_0) = v_{n-1}$. In either case, the orbit of the remaining vertices is precisely determined by their distance from v_0. In the first case, $\phi(v_i) = v_i$ for all i. This results in the identity automorphism. In the second case, $\phi(v_i) = v_n$ - 1 - i for all i. Thus, Aut(P_n) $\cong \mathbb{Z}_2$.
2. Consider the mapping $\rho(v_i) = v_{i+1}$, where the computation on the indices is computed modulo n. Since $v_i v_{i+1}$ is an edge in the graph, ρ is an automorphism. If n is even, then consider the

mapping $\tau(v_i) = v_{n-1-i}$ and $\tau(v_{n-1-i}) = v_i$ for i = 0, 1, ...,$\frac{n}{2} - 1$. If n is odd, then consider the mapping $\tau(v_0) = v_0$, $\tau(v_i) = v_n - i$, and τ $(v_n - i) = v_i$ for i = 0, 1, ...,$\frac{n-1}{2}$. In both cases, $v_i v_{i+1}$ and $v_n - 1 - iv_n - 2 - i$ are both edges in C_n. Thus, τ is an automorphism. Note that $\rho^n = \tau^2 =$ e and $\rho^\tau = \tau\rho^{n-k}$. Hence ρ and τ generate the n^{th} dihedral group, D_n. Since we can think of C_n as a regular n - gon, we have that Aut(C_n) $\cong D_n$.

3. Since S_1 is the trivial group, the result holds for n = 1. For the remainder of the proof, let n $\geq$ 2. Let x and y be distinct vertices of K_n. Consider the mapping $\phi(x) = y, \phi(y) = x$ and ϕ(v) = v for all other $v \in$ V (K_n). Since x and y are both adjacent to every vertex, ϕ is an automorphism of K_n. Thus, every transposition of two vertices is an automorphism. Since the set of all transpositions generates S_n, the result follows.

Theorem 3.1.2.

For the complete bipartite graph, $K_{n,m}$, if n > m, then

Aut($K_{n,m}$) $\cong S_n \times S_m$.

Proof.

By above theorem, Aut(K_n) $\cong S_n$. By proposition 3.0.2 Aut($\overline{K}_n$) $\cong$ S_n. Thus, any automorphism of the form $(x_i, x_j 4\ )$ or of the form (y_k, y_l) is in Aut($K_{n,m}$). Thus, $S_n \times S_m$ is a subgroup of Aut($K_{n,m}$). Now suppose that n > m. Since deg(x_i) = m and deg($y_j\ $) = n,it follows from the proof of proposition 3.0.2 that there is no automorphism ϕ such that $\phi(x_i) = y_j$. Thus, $\text{Aut}(K_{n,m}) \cong S_n \times S_m$.

Definition 3.1.3.

The double star is the tree with two adjacent non-leaf vertices x and y such that $x_1, x_2, ..., x_n$ are the leafs adjacent to x and $y_1, y_2, ..., y_m$ are the leafs adjacent to y. This graph is denoted $S_{n,m}$.

Example,

The double star $S_{5,4}$ is,

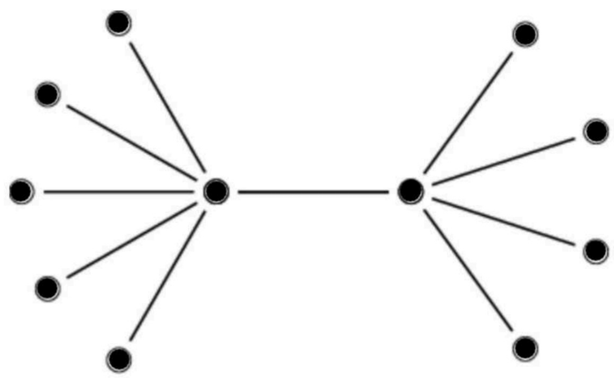

Figure 3.2

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DON BOSCO ARTS & SCIENCE COLLEGE ANGADIKADAVU

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2021-2023

Project Report on

BILINEAR FORMS

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Project Report on

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Dissertation submitted in the partial

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Kannur University

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1.

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KANNUR UNIVERSITY

BONAFIDE CERTIFICATE

Certified that this project report “BILINEAR FORMS” is the bonafide work of SWATHI SUDHAKARAN who carried out the project work under my supervision.

Signature of Head | Signature of Supervisor

DECLARATION

I, SWATHI SUDHAKARAN hereby declare that the Project work entitled BILINEAR FORMS has been prepared by me and submitted to Kannur University in partial fulfillment of requirement for the award of Master of Science is a record of original work done by me under the supervision of Mrs. RIYA BABY , Assistant Professor, Department of Mathematics, Don Bosco Arts & Science College, Angadikadavu. I, also declare that this Project work has not been submitted by me fully for the award of any Degree, Diploma, Title or recognition before any authority.

Place: Angadikadavu SWATHI SUDHAKARAN

Date: (C1PSMM1909)

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SWATHI SUDHAKARAN

CONTENTS

INTRODUCTION

Linear algebra is that branch of mathematics which treats the common properties of algebraic system which consists of a set, together with a resonable notation of a linear combination of elements in the set.This project is a brief survey of bilinear forms which is a generalization of so called inner products on real or complex spaces to any arbitrary field K.

Inner products are maps which are not completely linear, in the sense that they are linear in its first argument and conjugate linear in the second argument. So naturally a question arises is there any similar map which includes inner products as well in its collection and can be considered as a generalization of inner products.These bilinear form include certain types of inner products in its collection and these are linear in both of its argument.Bilinear form has established an important role through out many of the topics of mathematics such as Linear Algebra and Coding theory.Bilinear forms also showed importance in different fields like Engineering,Biology,Economics This made me to read and study extra in bilinear forms.

The second chapter includes the basic definitions, notations and examples which are needed to support the study of bilinear forms and the other chapters deal with types of bilinear forms.

The main aim of this project is to provide an introduction to bilinear forms and some of its basis properties and characterisation .

CHAPTER 1

PRELIMINARY RESULTS

Vector space (1.1)

Let **V** be a non empty set with two operation addition and scalar multiplication for each $\alpha, \beta \in \mathbf{V}$ and **F** be a field of scalars such that

$$\alpha + \beta \in \mathbf{V} \ \forall \ \alpha, \beta \in \mathbf{V}$$

$$\alpha \in \mathbf{V}, a \in \mathbf{F} \to a\alpha \in \mathbf{V}$$

Then **V** is called a vector space over the field **F**, if and only if the following axioms are satisfied $\forall \ \alpha, \beta \in \mathbf{V}$

1) Commutativity: $\alpha + \beta = \beta + \alpha$

2)Associativity: $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

3)Existence of identity : There exist a unique vector 0 in **V** such that

$$\alpha + 0 = 0 + \alpha = 0$$

4) Existence of inverse : There exist a unique vector -$\alpha \in \mathbf{V}$ such that

$$\alpha + (-\alpha) = 0 = (-\alpha) + \alpha$$

The element (-α) in **V** is called the additive inverse of α in **V**

5) k$(\alpha + \beta) = k\alpha + k\beta, k \in \mathbf{F}$

6) $(k_1 + k_2)\alpha = k_1\alpha + k_2\alpha$, k_1 and $k_2 \in$ F

7) For any unit scalar 1 in **F** , 1 $\alpha = \alpha$

8) $(k_1k_2)\alpha = k_1(k_2\alpha), k_1$ and $k_2 \in \mathbf{F}$

Linear combination of vector space(1.2)

Let **V(F)** be a vector space if $a_1, a_2, ...a_n \in \mathbf{V}$ then any element of the form $a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$ where $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbf{F}$ is called a linear combination of the vectors $\alpha_1, \alpha_2, ..., \alpha_n$.

Basis(1.3)

A basis for a vector space **V** is linearly independent set of vectors in **V** which spans **V**

Dimension(1.4)

The number of elements in a basis set is called the dimension of the vector space

Finite dimensional vector space(1.5)

A vector space **V(F)** is called a finite dimensional vector space , if there exist a finite subsystem of the same which generate the whole space

Linear transformation(1.6)

Let **V** and **W** be two vector space over the field **F** . A linear transformation from **V** into **W** is a function T from **V** into **W** such that , T(cx+y)= cT(x)+T(y) $\forall\, x, y \in$ **V** and c $\in$ F

Vector space isomorphism (1.7)

Let U(F) and V(F) be two vector spaces over the same field F. Then a mapping T: U $\rightarrow$ V is called an isomorphism from U to V ,if

1) T is one-one and onto
2) T$(\alpha + \beta) = T(\alpha) + T(\beta), \forall\ \alpha, \beta \in$ U
3) T(a$\alpha) = aT(\alpha)$

condition 2 and 3 united to give
T(a$\alpha + b\beta) = aT(\alpha) + bT(\beta), \forall\, \alpha, \beta \in$ U and a,b $\in$ F
Then U(F) and V(F) are said to be isomorphic vector spaces and symbolically we write U(F) $\cong$ V(F).

Transpose of a matrix(1.8)

Let A = $\left[a_{ij}\right]_{m \times n}$ then the matrix obtained by interchanging rows and columns of A is called the transpose of A and is denoted by A^t

Dual space and Dual basis(1.9)

If **V** is a vector space ,the collection of all linear functionals on **V** forms a vector space and it is denoted by V^* , call it the dual space of **V** ,

$V^* =$ L(**V**,**F**)

The basis $B^* = \{f_1, f_2, ...f_n\}$ for V^* is called the dual basis of B $= \{a_1, a_2, ..., a_n\}$ the basis for **V**.

Sum of two vector space(1.10)

Let**V** and **W** be vector spaces over a field **F** then the sum **V** + **W** is the space of all sums $\alpha + \beta$ such that $\alpha \in$ **V** and $\beta \in$ **W** That is,

$$\mathbf{V} + \mathbf{W} = \{\alpha + \beta : \alpha \in \mathbf{V}, \beta \in \mathbf{W}\}$$

Direct sum(1.11)

Let **V** be a vector space over the field **F** and W_1 and W_2 be two subspaces of **V** then **V** is said to be direct sum of W_1 and W_2 denoted by $\mathbf{V} = W_1 \oplus W_2$, if $\mathbf{V} = W_1 + W_2$ and $w_1 \cap w_2 = \{0\}$.

Group(1.12)

A group consists of the following

1) A set G ; a rule (operation) which associates with each pair of elements X,Y in G an element XY in G

2)X(YZ) = (XY)Z, $\forall$ X,Y and Z $\in$ G

3) There is an element e in G such that eX = Xe = e,$\forall$ X $\in$ G

4) To each element X in G there corresponds an element X^{-1} in G such that

$$XX^{-1} = X^{-1}X = e$$

Field(1.13)

A field is a non empty K along with function $+ : K \times K \to K$ and $\cdot : K \times K \to K$ such that

1) (K,+) is an abelian group , that is

a) $k_1 + k_2 = k_2 + k_1 \forall k_1, k_2 \in K$

b) $k_1 + (k_2 + k_3) = (k_1 + k_2) + k_3, \forall k_1, k_2, k_3 \in K$

c) There exist a unique element 0 called the zero element of K such that

$$k+0 = 0+k \ , \forall \ k \in K$$

d) To every k $\in$ K , there corresponds a unique element -k $\in$ K such that

$$k+(-k) = 0 = (-k)+k$$

2) (K-{0},$\cdot$) is an abelian group , that is

a) $k_1 \cdot k_2 = k_2 \cdot k_1, \ \forall \ k_1, k_2 \in K$

b)$k_1 \cdot (k_2 \cdot k_3) = (k_1 \cdot k_2) \cdot k_3, \forall \ k_1, k_2, k_3 \in K$

c) There exist a unique element 1 called the unit element of K such that

$$k \cdot 1 = k \text{ , for every scalar } k \in K$$

d) To every k $\in$ K , there corresponds a unique element k^{-1} $\in$ K such that

$$k^{-1}\cdot \text{ k} = \text{k } \cdot k^{-1}$$

3) $\cdot$ is distributive with respect to $+$, that is

$$k_1 \cdot (k_2 + k_3) = k_1 \cdot k_2 + k_1.k_3 \ \forall \ k_1, k_2, k_3 \in K$$

CHAPTER 2

BILINEAR FORMS

Definition(2.1)

Let **V** be a vector space of a finite dimension over the field **F**, a bilinear form on **V** is a mapping f:$\mathbf{V}\times\mathbf{V} \to F$,which satisfies,

$$(1)\text{f}(\text{a}\alpha_1 + b\alpha_2, \beta)\text{=a f}(\alpha_1, \beta)\text{+b f}(\alpha_2, \beta)$$

$$(2)\ \text{f}(\alpha, a\beta_1 + b\beta_2)\text{=a f}(\alpha, \beta_1)\text{+b f}(\alpha, \beta_2)$$

for every a,b $\in$ **F** and $\alpha, \beta, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbf{V(F)}$
We express condition 1 by saying f is linear in the first variable and condition 2 by saying f is linear in the second variable.

Results

(1) Addition of bilinear forms

Suppose f_1 and f_2 are two bilinear forms on a finite dimensional vector space $\mathbf{V(F)}$. The sum of two bilinear forms f_1 and f_2 are $f_1 + f_2$ and is such that $(f_1 + f_2)(\alpha, \beta) = f_1(\alpha, \beta)+f_2(\alpha, \beta)$
$\forall\alpha, \beta \in \mathbf{V(F)}$

For, it is easy to verify that $f_1 + f_2$ is also bilinear form on **V**
We have,

$(f_1 + f_2)(a\alpha_1 + b\alpha_2, \beta) = f_1(a\alpha_1 + b\alpha_2, \beta) + f_2(a\alpha_1 + b\alpha_2, \beta)$

$= [\mathrm{a}f_1(\alpha_1,\beta)+bf_1(\alpha_2,\beta)]+[af_2(\alpha_1,\beta)+bf_2(\alpha_2,\beta)]$

(since f_1 and f_2 are bilinear)

$= \mathrm{a}[f_1(\alpha_1,\beta)+f_2(\alpha_1,\beta)]+b[f_1(\alpha_2,\beta)+f_2(\alpha_2,\beta)]$

$= \mathrm{a}[(f_1+f_2)(\alpha_1,\beta)] + b[(f_1+f_2)(\alpha_2,\beta)]$

Also

$(f_1+f_2)(\alpha, a\beta_1+b\beta_2) = f_1(\alpha, a\beta_1+b\beta_2) + f_2(\alpha, a\beta_1+b\beta_2)$

$= [\mathrm{a}f_1(\alpha,\beta_1)+bf_1(\alpha,\beta_2)]+[af_2(\alpha,\beta_1)+bf_2(\alpha,\beta_2)]$

(since $f_1$and f_2 are bilinear)

$= \mathrm{a}[f_1(\alpha,\beta_1)+f_2(\alpha,\beta_1)]+b[f_1(\alpha,\beta_2)+f_2(\alpha,\beta_2)]$

$= \mathrm{a}[(f_1+f_2)(\alpha,\beta_1)] + b[(f_1+f_2)(\alpha,\beta_2)]$

$\Longrightarrow f_1+f_2$ is a bilinear form on **V**

(2) Scalar multiplication of bilinear form

Let f and g are bilinear forms on **V** and c is a scalar , we define cf as follows,

$$(\mathrm{cf})(\alpha,\beta) = cf(\alpha,\beta)\ \forall\ \alpha,\beta \in \mathbf{V}$$

Thus,

$$(\mathrm{cf})(\mathrm{a}\alpha_1+b\alpha_2,\beta) = cf(a\alpha_1+b\alpha_2,\beta)$$

$$= \mathrm{c}[\mathrm{af}(\alpha_1,\beta)+bf(\alpha_2,\beta)]$$

$$= \mathrm{a}\ (\mathrm{cf})(\alpha_1,\beta)+b(cf)(\alpha_2,\beta)$$

Similarly,

$$(\mathrm{cf})(\alpha, a\beta_1+b\beta_2) = a(cf)(\alpha,\beta_1)+b(cf)(\alpha,\beta_2)$$

Then cf is a bilinear form.We have g is also a bilinear form.There fore cf+g is a bilinear form.

That is any linear combination of bilinear forms on **V** is again a bilinear form.

NOTE

The set of all bilinear forms on **V** is closed under addition and scalar multiplication.[follows from the above result]

The set of all bilinear forms on **V(F)** is generally denoted by L(**V**,**V**,**F**)

(3) Zero function is a bilinear form

A function f : **V**×**V** → **F** is said to be a zero function ,

if f$(\alpha, \beta) = 0 \in \mathbf{F} \ \ \forall \ \ \alpha, \beta \in \mathbf{V}$ such a function is denoted by $\hat{0}$.

$$\hat{0}(\alpha, \beta) = 0 \in \mathbf{F} \ \forall \ \alpha, \beta \in \mathbf{V}$$

Now,

$$\begin{aligned} \hat{0}(a\alpha_1 + b\alpha_2, \beta) &= 0 = 0+0 \\ &= \text{a } 0+\text{b } 0 \\ &= \text{a } \hat{0}(\alpha_1, \beta) + \text{b } \hat{0}(\alpha_2, \beta) \end{aligned}$$

Also

$$\begin{aligned} \hat{0}(\alpha, a\beta_1 + b\beta_2) &= 0 = 0+0 \\ &= \text{a } 0+\text{b } 0 \\ &= \text{a } \hat{0}(\alpha, \beta_1)+ \text{b } \hat{0}(\alpha, \beta_2) \\ &\forall a, b \in \mathbf{F} \text{ and } \alpha, \beta, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbf{V} \end{aligned}$$

$\Longrightarrow \hat{0}$ is a bilinear form

(4) If f is a bilinear form on V then -f is also a bilinear form on V

Where ,

$$(-f)(\alpha, \beta) = -[f(\alpha, \beta)]$$

We have,

$$\begin{aligned} -f(a\alpha_1 + b\alpha_2, \beta) &= -[f(a\alpha_1 + b\alpha_2, \beta)] \\ &= -[af(\alpha_1, \beta) + bf(\alpha_2, \beta)] \\ &= -af(\alpha_1, \beta) - bf(\alpha_2, \beta) \\ &= a(-f)(\alpha_1, \beta) + b(-f)(\alpha_2, \beta) \end{aligned}$$

Similarly,

$$-f(\alpha, a\beta_1 + b\beta_2) = a(-f)(\alpha, \beta_1) + b(-f)(\alpha, \beta_2)$$

$\Longrightarrow$ -f is a bilinear form on **V**

(5)Vector space of bilinear forms

Let B(W) denote the set of all bilinear form on W .

Then it can be easily shown that

B(W)={ f : f is a bilinear form on W } is a vector space over the field **F**.

Note that , it is closed under addition and scalar multiplication.

That is ,

$$f+g(\alpha, \beta) = f(\alpha, \beta) + g(\alpha, \beta) \ \forall \ (\alpha, \beta) \in \mathbf{V} \times \mathbf{V}$$

$$(\text{cf})(\alpha, \beta) = cf(\alpha, \beta)\ \forall\ (\alpha, \beta) \in \mathbf{V}\times\ \mathbf{V} \text{ and a} \in \mathbf{F}$$

Associative and commutative property follows from the fact that the elements of the field **F** are both associative and commutative in the field **F** for addition.Similarly we can verify the laws of scalar multiplication

For example,

$$\text{a(f+g)} = \text{af + bg}$$

Now ,
$$\text{a(f+g)}(\alpha, \beta) = a[(f+g)(\alpha, \beta)]$$
$$= \text{a[f}(\alpha, \beta)\text{+g}(\alpha, \beta)]$$
$$= \text{af}(\alpha, \beta)\text{+ag}(\alpha, \beta)$$
$$= (\text{af})(\alpha, \beta) + (ag)(\alpha, \beta)$$
$$= (\text{af+ag})(\alpha, \beta)$$

Since above hold for all $\alpha, \beta \in$ W

$$\text{a(f+g)} = \text{af+ag}$$

Similarly we can prove that

$$\text{(a+b) f} = \text{a f+b f}$$

$$\text{(ab) f} = \text{a (bf)}$$

$$\text{1 f} = \text{f}$$

Examples

(1) Let **V(F)** be a vector space over the field F , T be a linear transformation on **V** and f a bilinear form on **V**. Suppose g is a function from **V** $\times$ **V** into **F** given by , $\text{g}(\alpha, \beta) = f(T_\alpha, T_\beta)$. Then g is a bilinear form on **V**.

Solution :

We have ,

$$g(a\alpha_1 + b\alpha_2, \beta) = f[T(a\alpha_1 + b\alpha_2), T_\beta]$$
$$= f(aT\alpha_1 + b\ T\alpha_2, T_\beta)$$
$$= af(T\alpha_1, T\beta) + bf(T\alpha_2, T\beta)$$
$$= ag(\alpha_1, \beta) + bg(\alpha_2, \beta)$$

Also ,

$$g(\alpha, a\beta_1 + b\beta_2) = f[T_\alpha, T(a\beta_1 + b\beta_2)]$$
$$= f(T\alpha, aT\beta_1 + bT\beta_2)$$
$$= af(T_\alpha, T_{\beta_1}) + bf(T_\alpha, T_{\beta_2})$$
$$= ag(\alpha, \beta_1) + bg(\alpha, \beta_2)$$

$\Longrightarrow$ g is a biinear form on **V**

2) Let L_1 and L_2 are linear functional on **V(F)**.Prove that the function f : $\mathbf{V}\times\mathbf{V} \to \mathbf{F}$ defined as $f(\alpha, \beta) = L_1(\alpha)L_2(\beta)$ is a bilinear form on **V**

Solution:

We have,

$$f(a\alpha_1 + b\alpha_2, \beta) = L_1(a\alpha_1 + b\alpha_2)L_2(\beta)$$
$$= [a\ L_1(\alpha_1) + bL_1(\alpha_2)]L_2(\beta)$$
$$= a\ L_1(\alpha_1)L_2(\beta) + bL_1(\alpha_2)L_2(\beta)$$
$$= a\ f(\alpha_1, \beta) + bf(\alpha_2, \beta)$$

Also ,

$$f(\alpha, a\beta_1 + b\beta_2) = L_1(\alpha)L_2(a\beta_1 + b\beta_2)$$

$= L_1(\alpha)[(aL_2(\beta_1) + bL_2(\beta_2)]$

$=$ a $L_1(\alpha)L_2(\beta_1) + bL_1(\alpha)L_2(\beta_2)$

$=$ a f$(\alpha, \beta_1) + bf(\alpha, \beta_2)$

$\Longrightarrow$ f is a bilinear form on **V**

3) Let F be a field and let m and n be two integers . Let **V** be the vector space of all m $\times$ n matrices over F , Let A be a fixed m $\times$ m matrix over F and let f_A be a function from **V** $\times$ **V** into F defined by

$$f_A(X, Y) = tr(X^tAY) \text{ where X,Y} \in \mathbf{V} .$$

Then f_A is a bilinear form on **V**

solution:

$f_A(aX + bZ, Y) = tr[(aX + bZ)^t$AY)]

$=$ tr $[aX^tAY]$ + tr $[bZ^t$AY]

$=$ a tr$[X^tAY]$ + b tr$[Z^t$AY]

$=$ a $f_A(X, Y)$+ b f_A(Z,Y)

[since trace function and transpose operation are linear]

Also,

$f_A(X, aY + bZ) = tr[X^t$, A(aY+bZ)]

$=$ tr $[aX^tAY + bX^t$AZ]

$=$ tr $[aX^tAY]$ + tr $[bX^t$AZ]

$=$ a tr$[X^tAY]$ + b tr$[X^t$AZ]

$=$ a $f_A(X, Y)$+ b f_A(X,Z)

There fore f_A is a bilinear form on **V** .

4) Let **V** be a vector space of all ordered n tuples over the field F , that is $\mathbf{V} = \mathbf{V}_n(\mathrm{F})$. If $\alpha = (a_1, a_2, ...a_n)$ and $\beta = (b_1, b_2...b_n)$ be any two elements of **V** and let f be a scalar valued function on **V**, defined by ,

$$\mathrm{f}(\alpha, \beta) = a_1b_1 + a_2b_2 + ... + a_nb_n$$

then f is a bilinear form on **V**

solution:

Let

$$\alpha = (a_1, a_2, ..., a_n)$$

$$\beta = (b_1, b_2, ..., b_n)$$

$$\gamma = (c_1, c_2, ..., c_n)$$

Then,

$$\begin{aligned} k_1\alpha + k_2\gamma &= k_1(a_1, a_2, ...a_n) + k_2(c_1, c_2...c_n) \\ &= (k_1a_1, k_1a_2, ..., k_1a_n) + (k_2c_1, k_2c_2, ..., k_2c_n\ ) \\ &= (k_1a_1 + k_2c_1, k_1a_2 + k_2c_2, ..., k_1a_n + k_2c_n) \\ k_1\beta + k_2\gamma &= k_1(b_1, b_2, ...b_n) + k_2(c_1, c_2...c_n) \\ &= (k_1b_1, k_1b_2, ..., k_1b_n) + (k_2c_1, k_2c_2, ..., k_2c_n\ ) \\ &= (k_1b_1 + k_2c_1, k_1b_2 + k_2c_2, ...k_1b_n + k_2c_n) \end{aligned}$$

Thus we have,

$$\begin{aligned} \mathrm{f}(k_1\alpha + k_2\gamma, \beta) &= [k_1a_1 + k_2c_1, k_1a_2 + k_2c_2, ...k_1a_n + k_2c_n, (b_1, b_2, ..., b_n)] \\ &= [(k_1a_1 + k_2c_1)b_1 + (k_1a_2 + k_2c_2)b_2 + ... + (k_1a_n + k_2c_n)b_n] \\ &= k_1(a_1b_1 + a_2b_2 + ... + a_nb_n) + k_2(c_1b_1 + c_2b_2 + ... + c_nb_n) \\ &= k_1f(\alpha, \beta) + k_2f(\gamma, \beta) \end{aligned}$$

$$\implies f(k_1\alpha + k_2\gamma, \beta) = k_1 f(\alpha, \beta) + k_2 f(\gamma, \beta)$$

Also,

$$\text{f}(\alpha, k_1\beta + k_2\gamma) = [(a_1, a_2, ..., a_n)(k_1b_1 + k_2c_1, k_1b_2 + k_2c_2, ..., k_1b_n + k_2c_n)]$$

$$= a_1(k_1b_1 + k_2c_1) + a_2(k_1b_2 + k_2c_2) + ... + a_n(k_1b_n + k_2c_n)$$

$$= k_1(a_1b_1 + a_2b_2 + ... + a_nb_n) + k_2(a_1c_1 + a_2c_2 + ... + a_nc_n)$$

$$= k_1 f(\alpha, \beta) + k_2 f(\alpha, \gamma)$$

$$\implies f(\alpha, k_1\beta + k_2\gamma) = k_1 f(\alpha, \beta) + k_2 f(\alpha, \gamma)$$

$\implies$ f is a bilinear form on **V**

Exercise

Which of the following functions f defined on vectors $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ in $\mathbb{R}^2$ are bilinear forms ?

1) $\text{f}(\alpha, \beta) = x_1y_2 - x_2y_1$

2) $\text{f}(\alpha, \beta) = (x_1 - y_1)^2 + x_2y_2$

Solution:

Let $\alpha = (x_1, x_2)$

$\beta = (y_1, y_2)$

$\gamma = (z_1, z_2)$

Then,

$$\text{a}\alpha + b\beta = a(x_1, x_2) + b(y_1, y_2)$$

$$= (\text{a}x_1, ax_2) + (by_1, by_2)$$

$$= (\text{a}x_1 + by_1, ax_2 + by_2)$$

1) $\text{f}(\alpha, \gamma) = f[(x_1, x_2), (z_1, z_2)]$

$$= x_1z_2 - x_2z_1$$

$$
\begin{aligned}
\mathrm{f}(\beta, \gamma) &= f[(y_1, y_2), (z_1, z_2)] \\
&= y_1 z_2 - y_2 z_1 \\
\mathrm{f}(\gamma, \alpha) &= f[(z_1, z_2), (x_1, x_2)] \\
&= z_1 x_2 - z_2 x_1 \\
\mathrm{f}(\gamma, \beta) &= f[(z_1, z_2), (y_1, y_2)] \\
&= z_1 y_2 - z_2 y_1
\end{aligned}
$$

$$
\begin{aligned}
\text{There fore, } \mathrm{f}(\mathrm{a}\alpha + b\beta, \gamma) &= f[a(x_1, x_2) + b(y_1, y_2), \gamma] \\
&= \mathrm{f}[(\mathrm{a}x_1 + by_1, ax_2 + by_2)(z_1, z_2)] \\
&= (\mathrm{a}x_1 + by_1)z_2 - (ax_2 + by_2)z_1 \\
&= \mathrm{a}(x_1 z_2 - x_2 z_1) + b(y_1 z_2 - y_2 z_1) \\
&= \mathrm{a}\ \mathrm{f}(\alpha, \gamma) + bf(\beta, \gamma)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mathrm{f}(\gamma, a\alpha + b\beta) &= f[(z_1, z_2), a(x_1, x_2) + b(y_1, y_2)] \\
&= \mathrm{f}[(z_1, z_2), (ax_1 + by_1, ax_2 + by_2)] \\
&= z_1(ax_2 + by_2) - z_2(ax_1 + by_1) \\
&= \mathrm{a}(z_1 x_2 - z_2 x_1) + b(z_1 y_2 - z_2 y_1) \\
&= \mathrm{a}\ \mathrm{f}(\gamma, \alpha) + bf(\gamma, \beta)
\end{aligned}
$$

$\Longrightarrow$ f is a bilinear form on $\mathbb{R}^2$

$$
\begin{aligned}
2)\ \mathrm{f}(\alpha, \gamma) &= f[(x_1, x_2), (z_1, z_2)] \\
&= (x_1 - z_1)^2 + x_2 z_2 \\
\mathrm{f}(\beta, \gamma) &= (y_1 - z_1)^2 + y_2 z_2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{f}(\mathrm{a}\ \alpha + \mathrm{b}\beta, \gamma) &= \mathrm{f}[(ax_1 + by_2, ax_2 + by_2), (z_1, z_2)] \\
&= (ax_1 + by_2 - z_1)^2 + (ax_2 + by_2)z_2 \text{ and}
\end{aligned}
$$

$$\text{af}(\alpha,\gamma) + bf(\beta,\gamma) = a(x_1 - z_1)^2 + ax_2z_2 + b(y_1 - z_1)^2 + by_2z_2$$
$$= \text{a}(x_1 - z_1)^2 + \text{b}\ (y_1 - z_1)^2 + (ax_2 + by_2)z_2$$

That is f(a$\alpha + b\beta, \gamma) \neq af(\alpha,\gamma) + bf(\beta,\gamma)$

$\Longrightarrow$ f is not a bilinear form on $\mathbb{R}^2$

Definition(2.2)

Let B = $\{\ \alpha_1, \alpha_2, ..., \alpha_n\ \}$ be an ordered basis for a finite dimensional vector space **V(F)** and f be a bilinear form on **V** then the matrix of f with respect to basis B that is $[f]_B$ is the matrix A = $\begin{bmatrix} a_{i,j} \end{bmatrix}_{n\times n}$ such that

A = f$(\alpha_i, \alpha_j) = a_{i,j}$ [i=1,2,3,...,n ; j=1,2,3,...,n]

$$\text{Let } \alpha = \textstyle\sum_{i=1}^{n} x_i\alpha_i$$

Therefore coordinate matrix of $\alpha = X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Therefore $X^t = \begin{bmatrix} x_1, x_2, \ldots, x_n \end{bmatrix}$

$$\beta = \textstyle\sum_{j=1}^{n} y_j\alpha_j$$

There fore coordinate matrix of $\beta = Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Now,

$$\text{f}(\alpha, \beta) = f(\textstyle\sum_{i=1}^{n} x_i \alpha_i, \sum_{j=1}^{n} y_j \alpha_j\ )$$

$$= \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j f(\alpha_i, \alpha_j)$$

$$= \begin{bmatrix} x_1, x_2, \ldots, x_n \end{bmatrix} \begin{bmatrix} a_{i,j} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= X^t A Y$$

$$= \begin{bmatrix} \alpha \end{bmatrix}_B^t A \begin{bmatrix} \beta \end{bmatrix}_B$$

Theorem(2.3)

Let **V** be a finite dimensional vector space over the field F. for each ordered basis B of **V** , the function which associates with each bilinear form on **V** it's matrix in the ordered basis B is an isomorphism of the space L(**V**,**V**,F) onto the space of n $\times$ n matrices over the field F .

Proof:

Let S be the vector space of all n $\times$ n matrices over the field F and let ϕ be a function from L(**V**,**V**,F) into S

$\phi(f) = \begin{bmatrix} f \end{bmatrix}_B$ $\forall f \in$ L(**V**,**V**,F)

Let B = { $\alpha_1, \alpha_2, ..., \alpha_n$ } be the basis of **V** .

Then for a,b $\in$ F and f,g $\in$ L(**V**,**V**,F) we have ,

$$\phi(af + bg) = \Big[af + bg\Big]_B$$

Also,

$$\text{(af+bg)} \ (\alpha_i, \alpha_j) = \text{(af)}(\alpha_i, \alpha_j) + \text{(bg)}(\alpha_i, \alpha_j)$$

$$= \text{af}(\alpha_i, \alpha_j) + \text{bg}(\alpha_i, \alpha_j)$$

$$\text{(i = 1,2, ...,n ; j = 1,2,...,n)}$$

$$\implies \Big[af + bg\Big]_B = a\Big[f\Big]_B + b\Big[g\Big]_B$$

$$\implies \phi(af + bg) = a\phi(f) + b\phi(\text{g})$$

$$\implies \phi \text{ is a linear transformation.}$$

To prove ϕ is $one - one$

Let f,g $\in$ L(**V**,**V**,F)

Then,

$$\phi(f) = \phi(g)$$

$$\implies \Big[f\Big]_B = \Big[g\Big]_B$$

$$\implies \text{f = g}$$

$$\implies \phi \text{ is one-one}$$

Next to prove that ϕ is onto

Let A = $\Big[a_{i,j}\Big]$ be any n $\times$ n matrix over F , then there exist a linear form f on **V** such that $\Big[f\Big]_B$ = A

That is ϕ(f) = A

$$\implies \phi \text{ is onto}$$

Hence ϕ is an isomorphism of L(**V**,**V**,F) onto the space of all n $\times$ n matrices over F .

Corollory(2.4)

If B = { $\alpha_1, \alpha_2, ..., \alpha_n$ } is an ordered basis for **V** and $B^* = \{L_1, L_2, ..., L_n$ } is the dual basis for $\mathbf{V}^*$, then the n^2 bilinear form $f_{i,j}(\alpha, \beta) = L_i(\alpha)L_j(\beta), 1 \leq i \leq n$; $1 \leq j \leq$ n form a basis for the space L(**V**,**V**,F). In particular the dimension of L(**V**,**V**,F) is n^2 .

Proof:

The dual basis defined by the fact that $L_i(\alpha)$ is the i^{th} coordinates of α in the ordered basis B . Also f_{ij} defined by
$f_{ij}(\alpha, \beta) = L_i(\alpha)L_j(\beta)$.
We have to prove that this is a bilinear form .

$$\begin{aligned} f_{i,j}(a\alpha_1 + b\alpha_2, \beta) &= L_i(a\alpha_1 + b\alpha_2)L_j(\beta) \\ &= [\mathrm{a}L_i(\alpha_1) + bL_i(\alpha_2)]L_j(\beta) \\ &= \mathrm{a}\ L_i(\alpha_1)L_j(\beta) + bL_i(\alpha_2)L_j(\beta) \\ &= \mathrm{a}\ f_{i,j}(\alpha_1, \beta) + bf_{i,j}(\alpha_2, \beta) \end{aligned}$$

Also

$$\begin{aligned} f_{i,j}(\alpha, a\beta_1 + b\beta_2) &= L_i(\alpha)L_j(a\beta_1 + b\beta_2) \\ &= L_i(\alpha)[aL_j(\beta_1) + bL_j(\beta_2)] \\ &= \mathrm{a}\ L_i(\alpha)L_j(\beta_1) + bL_i(\alpha)L_j(\beta_2) \\ &= \mathrm{a}\ f_{i,j}(\alpha, \beta_1) + bf_{i,j}(\alpha, \beta_2) \end{aligned}$$

$\Longrightarrow$ f is a bilinear form .

If $\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$

$$\beta = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_n\alpha_n$$

Then $f_{i,j}(\alpha, \beta) = x_i y_j$

Let f be any bilinear form **V** and let A be the matrix of f in the ordered basis B

Then,

$$\text{f}(\alpha, \beta) = \sum_{i,j} A_{i,J} x_i y_j$$

$$\implies f = \sum_{i,j} A_{i,J} f_{i,j}$$

Therefore n^2 forms f_{ij} comprise a basis for L(**V**,**V**,F)

That is dim(L(**V**,**V**,F) $= n^2$

Theorem(2.5)

Let $B_1 = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $B_2 = \{\beta_1, \beta_2, ..., \beta_n\}$ be two ordered basis of a finite dimensional vector space **V(F)** and f is a bilinear form on **V** , then there exist an invertible n $\times$ n matrix C over the field F such that

$\left[f\right]_{B_2} = C^t \left[f\right]_B$ C , where C^t denotes the transpose of matrix C

Example

Let f be a bilinear form on $\mathbb{R}^2$ defined by f$((x_1, y_1), (x_2, y_2)) = x_1x_2 + y_1y_2$. Find the matrix of f for the following basis.

$$B_1 = \{(1, 0)(0, 1)\} = \{\alpha_1, \alpha_2\}$$

$$B_2 = \{(1, -1)(1, 1)\} = \{\beta_1, \beta_2\}$$

Also find the transition matrix C from B_1 to B_2 and verify that

$\left[f\right]_{B_2} = C^t \left[f\right]_B C$

Solution:

let $\begin{bmatrix} f \end{bmatrix}_{B_1} = \begin{bmatrix} a_{i,j} \end{bmatrix}_{B_1}$

So that f(α_i, α_j) = $a_{i,j}$

$a_{1,1} = f(\alpha_1, \alpha_1)$ = f((1,0)(1,0)) = 1+0 = 1

$a_{1,2} = f(\alpha_1, \alpha_2)$ = f((1,0)(0,1)) = 0+0 = 0

$a_{2,1} = f(\alpha_2, \alpha_1)$ = f((0,1)(1,0)) = 0+0 = 0

$a_{2,2} = f(\alpha_2, \alpha_2)$ = f((0,1)(0,1)) = 0+1 = 1

Therefore

$$\begin{bmatrix} f \end{bmatrix}_{B_1} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

let $\begin{bmatrix} f \end{bmatrix}_{B_2} = \begin{bmatrix} b_{i,j} \end{bmatrix}_{B_2}$

So that f(β_i, β_j) = b_{ij}

$b_{1,1} = f(\beta_1, \beta_1)$ = f((1,-1)(1,-1)) = 1+1 = 2

$b_{1,2} = f(\beta_1, \beta_2)$ = f((1,-1)(1,1)) = 0+0 = 0

$b_{2,1} = f(\beta_2, \beta_1)$ = f((1,1)(1,-1)) = 1-1 = 0

$b_{2,2} = f(\beta_2, \beta_2)$ = f((1,1)(1,1)) = 1+1 = 2

There fore

$$\begin{bmatrix} f \end{bmatrix}_{B_2} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Let T be a linear operator on **V** such that $\mathrm{T}(\alpha_i) = \beta_i$

$i = 1, 2, \ldots$ and $\left[C_{ij}\right]$ be the matrix of T relative to basis B .

Now,

$$\mathrm{T}(\alpha_i) = \beta_i \text{ [by definition]}$$

$$\therefore \mathrm{T}(1,0) = (1,-1) = (1,0) + (-1)\ (0,1)$$

Or

$$\mathrm{T}(\alpha_1) = \alpha_1 - \alpha_2$$

$$\mathrm{T}(0,1) = (1,1) = (1,0) + (0,1)$$

Or

$$\mathrm{T}(\alpha_2) = \alpha_1 - \alpha_2$$

Then,

$$\mathrm{C} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\Longrightarrow C^t = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\Longrightarrow C^t \left[f\right]_{B_1} C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \left[f\right]_{B_2}$$

$$\Longrightarrow \left[f\right]_{B_2} = C^t \left[f\right]_{B_1} C$$

Definition(2.6)

The rank of a bilinear form f on **V** , written rank(f) is defined to be the rank of any matrix representation .

Definition(2.7)

A bilinear form f on a vector space **V** is said to be degenerate , if for all non zero α in **V** , f$(\alpha, \beta) = 0$ for every β in **V** and for all nonzero β in **V**, f$(\alpha, \beta) = 0$ for every α in **V**

A bilinear form is clled non degenerate if it is not degenerate .

That is,

if for all non zero α in **V** there is a β in **V** such that f$(\alpha, \beta) \neq 0$ and for all non zero β in **V** there is a α in **V** such that

f$(\alpha, \beta) \neq 0$

CHAPTER 3

SKEW SYMMETRIC AND SYMMETRIC BILINEAR FORMS

3.1 SYMMETRIC BILINEAR FORMS

Definition(3.1.1)

A bilinear form on the vector space **V** is said to be symmetric if, f$(\alpha, \beta) = f(\beta, \alpha)\ \forall\ \alpha, \beta \in \mathbf{V}$

Result

If **V** is a finite dimensional vector space then the bilinear form f on V is symmetric if and only if its matrix A in some ordered basis is symmetric
That is,

$$A^t = \text{A}$$

Proof : Let the matrices of α, β with respect to a certain basis B be X and Y and A $= \left[f\right]_B$

Now,

$$\text{f}(\alpha, \beta) = X^t \left[f\right]_B \text{Y}$$

$$= X^t AY$$

$$\text{f}(\beta, \alpha) = Y^t \left[f\right]_B \text{X}$$

$$= Y^t AX$$

Now, f will be a symmetric bilinear form if and only if f$(\alpha, \beta) = f(\beta, \alpha)$

$\forall\ \alpha, \beta \in \mathbf{V}$ Or $X^t AY = Y^t AX$ $\forall$ matrices X and Y .

Now $X^t AY$ is a 1×1 matrix and as such it is equal to its transpose .

$$X^t AY^t = Y^t AX$$

$$\Rightarrow Y^t A^t (X^t)^t = Y^t AX$$

$$\Rightarrow Y^t A^t X = Y^t AX$$

$$\text{(since } (X^t)^t = X)$$

$$\Rightarrow A^t = A$$

That is A is symmetric

Hence bilinear form will be symmetric if and only if its matrix is symmetric .

Theorem(3.1.2)

Let **V** be a vector space over the field F , whose characteristic is not 2, that is 1+1 $\neq$ 0 , then every symmetric bilinear form is uniquely determined by the curresponding quadratic form .

proof:

Let f be any bilinear form on **V** and let q be the corresponding quadratic form associated with f , then for all $\alpha, \beta \in \mathbf{V}$

We have ,

$$\begin{aligned}
\text{q}(\alpha+\beta) &= f(\alpha+\beta, \alpha+\beta) \\
&= \text{f}(\alpha, \alpha+\beta) + f(\beta, \alpha+\beta) \\
&= \text{f}(\alpha, \alpha) + f(\alpha, \beta) + f(\beta, \alpha) + f(\beta, \beta) \\
&= \text{q}(\alpha) + f(\alpha, \beta) + f(\alpha, \beta) + q(\beta) \\
&\text{(since f}(\alpha, \beta) = f(\beta, \alpha) \text{ , that is symmetric)} \\
&= \text{q}(\alpha) + (1+1)f(\alpha, \beta) + q(\beta)
\end{aligned}$$

$$(1+1)\ \text{f}(\alpha, \beta) = q(\alpha+\beta) - q(\alpha) - q(\beta) ....(1)$$

Since 1+1 $\neq = 0$, f(α, β) is uniquely determined by the corresponding quadractic form.

Further extension

Let F be a subfield of the complex numbers ,then ......(1) yields the polarization identity

That is,

$$\text{f}(\alpha, \beta) = \tfrac{1}{4}\ q(\alpha+\beta) \text{ - } \tfrac{1}{4}\ q(\alpha-\beta)$$

From above theorem we have ,

$$2f(\alpha, \beta) = q(\alpha + \beta) - q(\alpha) - q(\beta) ....(1)$$

Also

$$\begin{aligned} q(\alpha - \beta) &= f(\alpha - \beta, \alpha - \beta) \\ &= f(\alpha, \alpha - \beta) - f(\beta, \alpha - \beta) \\ &= f(\alpha.\alpha) - f(\alpha, \beta) - f(\beta, \alpha) + f(\beta, \beta) \\ &= q(\alpha) + q(\beta) - 2f(\alpha, \beta) \end{aligned}$$

(since f is symmetric)

$$2\ f\ (\alpha, \beta) = q(\alpha) + q(\beta) - q(\alpha - \beta)\ ....(2)$$

Adding ....(1) and ......(2) we get,

$$4\ f\ (\alpha, \beta) = q(\alpha + \beta) - q(\alpha - \beta)$$

$$\Rightarrow f(\alpha, \beta) = \frac{1}{4}\ q(\alpha + \beta) - \frac{1}{4}\ q(\alpha - \beta)$$

Theorem (3.1.3)

Let **V** be a finite dimensional vector space over a field of characteristic zero and let f be a symmetric bilinear form on **V** , then there is an ordered basis for **V** in which f is represented by a diagonal matrix

.

proof:

We want to prove that there is a basis B $= \{\alpha_1, \alpha_2, ..., \alpha_n\}$ for **V** such that f$(\alpha_i, \alpha_j) = \left[a_{ij}\right] = 0$ for i $\neq$ j

If dim **V** = n = 1 or f = 0 then the theorem is true

We may suppose f $\neq 0$ and dim **V** $= n > 1$

If q is the quadratic form associated with f and if f$(\alpha, \alpha) = 0$

for each $\alpha \in \mathbf{V}$

Then q(α) = 0

Thus by polarization identity ,

$$f(\alpha, \beta) = \frac{1}{4}q(\alpha + \beta) - \frac{1}{4}q(\alpha - \beta)$$

We get f(α, β) = 0 $\quad \forall\ \alpha, \beta \in \mathbf{V}$

Thus f = 0 which controdicts our assumption and hence there must exist a vector $\alpha \in \mathbf{V}$ such that f(α, α) = $q(\alpha) \neq 0$

Let U be the one dimensional subspace of $\mathbf{V}$ which is spanned by vectors α and W be the set of all vectors β such that f(α, β) = 0

Then W is a subspace of $\mathbf{V}$

claim: V = U $\oplus$ W

1) To show that U $\cap$ W = $\{0\}$

Let $\delta \in U\cap$ W

$\Rightarrow \delta \in$ U and $\delta \in$ W

$\delta \in$ U $\Rightarrow \delta = k\alpha$ for some scalar k

Also,

$\delta \in$ W $\Rightarrow 0 = f(\alpha, \delta) = f(\alpha, k\alpha) = kf(\alpha, \alpha)$

But f(α, α) $\neq 0$

Therefore,

k =0 and so $\delta = k\alpha$=0

$\Rightarrow U \cap W = \{0\}$

2) To show that V = U+W

Let $\gamma \in \mathbf{V}$

Put $\beta = \gamma - \frac{f(\gamma,\alpha)}{f(\alpha,\alpha)}\alpha$

We have,

$$f(\alpha, \beta) = f(\alpha, \gamma - \frac{f(\gamma,\alpha)}{f(\alpha,\alpha)}\alpha)$$

$$= f(\alpha, \gamma) - \frac{f(\gamma,\alpha)}{f(\alpha,\alpha)}f(\alpha, \alpha)$$

$$= f(\alpha, \gamma) - f(\gamma, \alpha)$$

$$= 0$$

Hence $\beta \in$ W

Thus $\gamma = \frac{f(\gamma,\alpha)}{f(\alpha,\alpha)}\alpha + \beta$

That is $\gamma \in$ U+W

Thus by (1) ans (2) we get V = U $\oplus$ W

$$\Rightarrow dimW = dimV - dimU$$

$$= \text{n -1}$$

Let h be the restriction of f from **V** to W . It is bilinear form on W and dim W = n-1 $<$ n

Thus by mathematical induction we assume there exist a basis $\{\alpha_2, \alpha_3, ...\alpha_n\}$ of W such that f(α_i, α_j) = 0 , i $\neq$ j (i$\geq$ 2,j$\geq$ 2)

Again $\{\alpha_1\}$ is the basis of U and we have V = U $\oplus$ W

Therefore it follows that $\{\alpha_1, \alpha_2, ...\alpha_n\}$ is the basis of **V** such that f(α_i, α_j) = 0 , i $\neq$ j

That is the matrix of f with respect to the basis B is a diagnal matrix.

Corollory(3.1.4)

Let F be a subfield of complex numbers and let A be an n×n symmetric matricx over F then there is an invertible n×n matrix C over F such that C^tAC is diagonal

Proof:

Let **V** be a finite dimensional vector space and let X = $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be an ordered basis of **V**

Then $\left[f\right]_X$ = A , where f is bilinear form on **V**

But as A is symmetric matrix so the bilinear form f is also symmetric.

Therefore by preceding theorem there exist an ordered basis Y of **V** such that $\left[f\right]_Y = C^t$AC

$\Rightarrow C^t$AC is also a diagonal matrix

Hence the proof

3.2 SKEW SYMMETRIC BILINEAR FORMS

Definition(3.2.1)

If f is a bilinear form on the vector space **V** then it is said to be skew symmetric if,

$$\text{f}(\alpha, \beta) = -f(\beta, \alpha) \ \forall \ \alpha, \beta \in \mathbf{V}$$

Theorem(3.2.2)

Every bilinear form on a vector space **V** over a subfield F of the complex numbers can be uniquely expressed as the sum of a symmetric and skew symmetric bilinear forms

Proof:

Let f be a bilinear form on a vector space **V**

Let

$f_1(\alpha, \beta) = 1/2[f(\alpha, \beta) + f(\beta, \alpha)]$ ......(1)

$f_2(\alpha, \beta) = 1/2[f(\alpha, \beta) - f(\beta, \alpha)]$......(2)

It is easy to prove that both f_1 and f_2 are bilinear forms on **V**

Further we also have ,

$$f_1(\beta, \alpha) = \tfrac{1}{2} \ [\ \text{f}(\beta, \alpha) + f(\alpha, \beta)]$$
$$= f_1(\alpha, \beta)$$
$$\Rightarrow f_1 \text{ is symmetric}$$

Then

$$f_2(\beta, \alpha) = \tfrac{1}{2} \ [\ \text{f}(\beta, \alpha) - f(\alpha, \beta)]$$

$$= - 1/2\ [\ \text{f}(\alpha,\beta) - f(\beta,\alpha)]$$

$$= -f_2(\alpha,\beta)$$

$\Rightarrow f_2$ is skew symmetric bilinear form

From (1) and (2) we have

$$f_1(\alpha,\beta) + f_2(\alpha,\beta) = \text{f}\ (\alpha,\beta)$$

$$\Rightarrow (f_1 + f_2)(\alpha,\beta) = f(\alpha,\beta)\ \forall\ \alpha,\beta \in \mathbf{V}$$

$$\Rightarrow f_1 + f_2 = f$$

Uniqueness:

Further suppose that f = $f_3 + f_4$ where f_3 is symmetric and f_4 is skew symmetric

We have,

$$\text{f}\ (\alpha,\beta) = (f_3 + f_4)(\alpha,\beta)$$

$$= f_3(\alpha,\beta) + f_4(\alpha,\beta) \qquad ....(3)$$

Also,

$$\text{f}\ (\beta,\alpha) = (f_3 + f_4)(\beta,\alpha)$$

$$= f_3(\alpha,\beta) - f_4(\alpha,\beta) \qquad ....(4)$$

(since f_3 is symmetric and f_4 is skew symmetric)

Hence adding (3) and (4) we get ,

$$\text{f}\ (\alpha,\beta) + f(\beta,\alpha) = 2f_3(\alpha,\beta)$$

$$\Rightarrow f_3(\alpha,\beta) = \tfrac{1}{2}[f(\alpha,\beta) + f(\beta,\alpha)]$$

$$= f_1(\alpha,\beta)$$

$$\Rightarrow f_3 = f_1$$

Substracting (4) from (3) we also get ,

$$\text{f }(\alpha,\beta) - f(\beta,\alpha) = 2f_4(\alpha,\beta)$$

$$\Rightarrow f_4(\alpha,\beta) = \tfrac{1}{2}[f(\alpha,\beta) - f(\beta,\alpha)]$$

$$= f_2(\alpha,\beta)$$

$$\Rightarrow f_4 = f_2$$

Hence the solution f = $f_1 + f_2$ is unique

Theorem (3.2.3)

Let **V** be a finite dimensional vector space then the bilinear form f on **V** is skew symmetric if and only if its matrix A in some ordered basis is skew symmetric

That is A^t = -A

Proof:

Let B be an ordered basis for **V** and let α, β be any two vectors in **V**

Then f(α,β) = X^tAY and f(β,α) = Y^tAX , where f is a bilinear form on **V** and X and Y are co-ordinate matrices of α and β

Also A is the matrix of f in the ordered basis B

Now f will skew symmetric if and only if X^tAY =-Y^tAX for all column matrix X and Y

But X^tAY is 1×1 matrix

Therefore,

$$X^t\text{AY} = (X^tAY)^t$$

$$= Y^tA^t(X^t)^t$$

$$= Y^t A^t X$$

The bilinear form f will skew symmetric if and only if $Y^t A^t X$ = -$Y^t A^t X$ forall column matrices X and Y

That is A^t = -A

That is A is skew symmetric

CHAPTER 4

GROUP PRESERVING BILINEAR FORMS

Definition(4.1)

Let f be a bilinear form on **V** and let T be a bilinear operator on the same vector space **V** over the field **F** then we say that T preserves f if f($T_\alpha, T_\beta) = f(\alpha, \beta)$ $\forall\alpha, \beta \in$ **V**

Remarks

1)The identity operator I preserves every bilinear form

For , f($I\alpha, I\beta) = f(\alpha, \beta), \forall\alpha, \beta \in$ **V**, f being any bilinear form

2)Let S and T be linear operators which preserve f , then the product ST also preserve f

For f($ST_\alpha, ST_\beta) = f(T_\alpha, T_\beta)$ (since S preserves f)

= f(α, β) (since T preserves f)

$\Rightarrow$ ST preserves f

3) Let G be the set of all linear operators on **V** which preserve f then we say that G is closed with respect to the operation of product of two bilinear operators .

Theorem(4.2)

Let f be a non-degenerating bilinear form on a finite dimensional vector space **V** then the set G of all bilinear operators on **V** which preserve f is a group under the operation of composition.

Proof:

Let G = { T : T preserves f }

Let T and S be two linear operators in G which preserve bilinear form

Therefore

$$\text{f}(T_\alpha, T_\beta) = f(\alpha, \beta)$$
$$\text{f}(S_\alpha, S_\beta) = f(\alpha, \beta)$$

Closure property:

$$\text{f}(TS_\alpha, TS_\beta) = f(S_\alpha, S_\beta) \text{ (since T preserves f)}$$
$$= \text{f}(\alpha, \beta) \text{ (since S preserves f)}$$

Therefore TS also preserves f.So TS $\in$ G

Therefore G is closed for the given composition.

Associative:

For each T,S,P $\in$ G we have,

$$(\text{T(SP)})(\alpha) = T(SP)(\alpha)$$
$$= \text{T[S(P}(\alpha))]$$
$$= \text{(TS)(P}(\alpha))$$
$$= \text{(TS)P}(\alpha)$$
$$\Rightarrow \text{T(SP)} = \text{(TS)P}$$

Hence it is associative.

Identity :

f$(I\alpha, I\beta) = f(\alpha, \beta)$ (since I$\alpha = \alpha$ and I$\beta = \beta$)

Therefore the identity operator I also preserves f and as such belong to G .

Hence G has an identity element.

Inverse:

Let T $\in$ G so that,

f$(T\alpha, T\beta) =$ f$(\alpha, \beta), \forall \alpha, \beta \in \mathbf{V}$

We have to show that T is invertible and $T^{-1} \in$ G

Since $\mathbf{V}$ is finite dimensional it will be sufficient to show that T is nonsingular

That is

$$T\alpha = 0$$
$$\Rightarrow \alpha = 0$$

Choose a vector α in the nullspace of T , so that $T\alpha = 0$

Then for any vector β in $\mathbf{V}$

We have,

$$\text{f}(\alpha, \beta) = f(T\alpha, T\beta) \text{ [since T preserves f]}$$

$$= \text{f}(0, T\beta)$$

$$= 0$$

Now f$(\alpha, \beta) = 0, \forall \beta \in \mathbf{V}$ and f is non degenerate .That is for all non zero α there exist $\beta \in \mathbf{V}$ such that f$(\alpha, \beta) \neq 0$

Therefore

$$\alpha = 0$$

Thus

$$\text{T}\alpha = 0$$

$$\Rightarrow \alpha = 0$$

$$\Rightarrow \text{T is non singular}$$

$$\Rightarrow \text{T is invertible and } T^{-1} \text{ be inverse of T}$$

So T $T^{-1} = T^{-1}$T = I Now,we have shown the existence of T^{-1} and we have to show that T^{-1} is in G

That is , it preserves f

f $(T^{-1}(\alpha), T^{-1}(\beta)) = f(TT^{-1}\alpha, TT^{-1}\beta)$ (since T preserves f)

$$= \text{f(I } \alpha, I\beta)$$

$$= \text{f}(\alpha, \beta)$$

Which shows that T^{-1} also preserves f.

That is $T^{-1} \in$ G

Thus inverse of each element of G exists

Hence G is a group.

CONCLUSION

By doing this project we learned about bilinear forms on finite dimensional vector spaces. In the first section we treats the space of bilinear forms on a vector space of dimension n .The matrix of bilinear form in an ordered basis is introduced, and the isomorphism between the space of forms and the space of n $\times$ n matrices is established.The rank of bilinear form is defined,and non- degenerate bilinear forms are introduced. The second section discusses symmetric bilinear forms and skew-symmetric bilinear forms. The third section discusses the group preserving a bilinear forms.

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Project Report on

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Dissertation submitted in the partial

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Kannur University

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BONAFIDE CERTIFICATE

Certified that this project report “FUZZY ALGEBRA” is the bonafide work of ANJU JAYARAJ who carried out the project work under my supervision.

Signature of Head

Signature of Supervisor

DECLARATION

I, ANJU JAYARAJ hereby declare that the Project work entitled FUZZY ALGEBRA has been prepared by me and submitted to Kannur University in partial fulfillment of requirement for the award of Master of Science is a record of original work done by me under the supervision of Mrs. RIYA BABY Assistant Professor, Department of Mathematics, Don Bosco Arts & Science College, Angadikadavu. I, also declare that this Project work has not been submitted by me fully for the award of any Degree, Diploma, Title or recognition before any authority.

Place: Angadikadavu ANJU JAYARAJ

Date: (C1PSMM1904)

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ANJU JAYARAJ

CONTENTS

INTRODUCTION

The concept of sets and functions to represent problem were defined in the 20th century. This way of representing problems is more rigid. In many circumstances the solutions using this concepts are meaningless. This difficulty was overcome by the introduction of fuzzy concept.

In 1965, L.A Zadeh mathematically formulated the fuzzy subset concept. He introduced the notion of a fuzzy subset of a set as a method for representing uncertainity. He defined the fuzzy subset of a non-empty set as a collection of objects with grade of membership in a continuum, with each object being assigned a value between 0 and 1 by a membership function. Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate or useful notion in describing the real life problems, because every object encountered in this real physical world carries some degree of fuzziness. Further the concept of grade of membership is not a probabilistic concept.

As know to us, a relation is a subset of the Cartesian product of two sets. A relation is fuzzified while a subset is fuzzified. Infact whether two objects have a relation is not always easy to determine. For example, the relation "greater than" on the set of real numbers is a crisp one because we can determine the order relation of any two real numbers without vagueness. However the relation "much greater than" is a fuzzy one because it is impossible for us to figure out the exact minimum difference of two numbers satisfying this relation. In real world problems, there exists a lot of such relations.

For example, "being friend of" and "being confident in" between some people. These relations will be termed as fuzzy relations.

After the concept of fuzzy sets first introduced by Zadeh, the theory of fuzzy mathematics has been widely used in Mathematics and many more areas.

In 1971, Rosenfeld introduced the concept of fuzzy subgroup, which spreads the area of fuzzy algebra. Fuzzy algebra plays an important role in the field of computer, such as fuzzy codes, fuzzy finite state machines, regular fuzzy languages and codes and so on, so it is necessary for us to do further research on fuzzy algebra theory.

In 1981, W.J Liu introduced the concept of fuzzy ideals of a ring. It was followed by the studies of Mukherjee and Sen who defined and examined fuzzy prime ideals of a ring. Fuzzy ideals were further investigated by Malik and Mordeson and they gave complete characterization of fuzzy prime ideals of an arbitrary ring.

PRELIMINARIES

Definition 1.0.1

Let X be a nonempty set. A fuzzy set A in X is characterised by membership function $\mu_A : X \to [0,1]$ and $\mu_A(x)$ is interpreted as a degree of membership of element x in fuzzy set A, for each $x \in$ X. It is clear that A is completely determined by the set of tuple

$$A = \{(x, \mu_A(x) : x \in X)\}$$

i.e., A mapping $A : X \to [0,1]$ is called a fuzzy set on X. The value $A(x)$ of A at $x \in X$ stands for the degree of membership of x in A.

- The set of all fuzzy sets on X will be denoted by $F(X).A(x) = 1$ means full membership, $A(x) = 0$ means non-membership and intermediate values between 0 and 1 means partial membership.

$A(x)$ is referred to as a membership function as x varies in X.

Proposition 1.0.2

Let X be a non empty set. Then there exists an isomorphism between $(P(X), \cap, \cup^c)$ and $(Ch(X), \vee, \wedge, ^c)$, where $P(X)$ is the powerset of X and $Ch(X)$ is the set of two valued characteristic function on X. - It follows from the above proposition that every subset of X may be regarded as a mapping from X to $\{0,1\}$

In this sense an ordinary set is also a fuzzy set whose membership function is just its characteristic function. Accordingly we shall identify the membership degree $A(x)$ with the value $\chi_A(x)$ of the characteristic function χ_A at x when A is an ordinary set.

For the two extreme cases $\emptyset$ and X, the membership functions are defined by

$$\forall x \in X, \emptyset(x) = 0 \text{ and } X(x) = 1$$

- In contrast with fuzzy sets, ordinary sets are sometimes termed as crisp sets.

Example 1.0.3

A realter wants to classify the house he offer to his client. One indicator of comfort of these house is the number of bedrooms in it. Let $X = \{1, 2, \ldots, 10\}$ be the set of available types of houses described by $x =$ the number of bedrooms in house. Then the fuzzy set comfortable type of house for a four person family may be described as $A = \{(1, 0.2), (2, 0.5)(3, 0.8)(4, 1)(5, 0.7)(6, 0.3)\}$.

In the set of ordered pair, the first element denote element and second element denote the degree of membership.

- If $\forall x \in X, A(x) \leq B(x)$, then we call that A is a subset of B or A is contained in B, denoted by $A \subseteq B$.

If $\forall x \in X, A(x) = B(x)$, then A and B are called equal, denoted by $A = B$. Obviously, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

If $A \neq \emptyset, A \subseteq B$ and $\exists x \in X$ such that $A(x) < B(x)$, then we say that A is properly contained in B, denoted by $A \subset B$, where $A, B \in F(X)$.

Proposition 1.0.4

$\forall A, B, C, D \in F(X)$

1. $A \cap B \subseteq A$ and $A \subseteq A \cup B$ 2. $A \subseteq B \Leftrightarrow A \cup B = B \Leftrightarrow A \cap B = A$

2. $A \subseteq B$ and $C \subseteq D \Rightarrow A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$

3. $A \subseteq B \Rightarrow B^c \subseteq A^c$

- $(F(X), \cap, \cup, {}^c)$ is a soft algebra, i.e., $F(X)$ satisfies $\forall A, B, C, D \in F(X)$

1. idempotency : $A \cup A = A, A \cap A = A$

2. commutativity : $A \cup B = B \cup A, A \cap B = B \cap A$

3. associativity : $(A \cup B) \cup C = A \cup (B \cup C), (A \cap B) \cap C = A \cap (B \cap C)$

4. absorption law : $A \cup (A \cap B) = A, A \cap (A \cup B) = A$

5. distributivity : $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

6. the existence of the greatest and least elements : $\emptyset \subseteq A \subseteq X$

7. involution : $(A^c)^c = A$

8. De- Morgan's law : $(A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c$

From the above property $(F(X), \cup, \cap, ^c)$ are largely dependent on properties of $([0,1], \max, \min^{c\,c}) = ([0,1], \vee, \wedge, ^c)$

- The set $\{x \mid A(x) = 1\}$ is said to be the kernel of $A, \operatorname{Ker} A$.
- The set $\{x \mid A(x) > 0\}$ is called the support of A, denoted by $\operatorname{supp}(A)$.
- The number $\bigvee_{x \in X} A(x)$ is called the height of $A, \operatorname{hgt}(A)$.
- The number $\bigwedge_{x \in X} A(x)$ is referred to as the plinth of A, denoted by $\operatorname{plt}(A)$.
- If $\operatorname{Ker}(A) = \emptyset$, then A is called a fuzzy set.

Definition 1.0.5

Let A be a fuzzy set on X. For $\alpha \in [0,1]$, the α cut A_α of A is defined as $A_\alpha = \{x \mid A(x) \geq \alpha\}$, and the strong α cut $\hat{A}_\alpha$ of A is defined as $\hat{A}_\alpha = \{x \mid A(x) > \alpha\}$

1. $A \cap B \subseteq A$ and $A \subseteq A \cup B$

Chapter 1

FUZZY RELATION

Definition 2.0.1

Let X and Y be two non-empty sets. A mapping $R : X\times\ Y \rightarrow [0,1]]$ is called a fuzzy(binary) relation from X to Y. For $(x, y) \in X \times Y, R(x, y) \in [0, 1]$ is referred to as the degree of relationship between x and y.

Particularly, a fuzzy relation from X to X is called a fuzzy(binary) relation on X

By definition, a fuzzy relation R is a fuzzy set on $X \times Y$, i.e., $R \in F(X \times Y)$

Definition 2.0.2

Let R be a fuzzy relation from X to Y. The R-afterset xR of $x(x \in X)$ is a fuzzy set on Y defined by, $\forall y \in Y, (xR)(y) = R(x, y)$.

The R-foreset Ry of $y(y \in Y)$ is a fuzzy set on X defined by, $\forall x \in X, (Ry)(x) = R(x, y)$

Since fuzzy relations are fuzzy sets, they have the same set-theoretic operations as fuzzy sets.

Let R and S be fuzzy relations from X to $Y.R$ is contained in S, denoted $R \subseteq S$ if and only if $\forall (x, y) \in X \times Y, R(x, y) \leq S(x, y)$; R is equal to S, denoted $R = S$, if and only if $\forall (x, y) \in X \times Y, R(x, y) =$

$S(x, y)$. Clearly, $R = S$ if and only if $R \subseteq S$ and $S \subseteq R$ The union $R \cup S \in F(X \times Y)$ of R and S is defined by, $\forall (x, y) \in X \times Y$,

$$(R \cup S)(x, y) = R(x, y) \vee S(x, y)$$

The intersection $R \cap S \in F(X \times Y)$ of R and S is defined by $\forall (x, y) \in X \times Y$,

$$(R \cap S)(x, y) = R(x, y) \wedge S(x, y)$$

The complement $R^c \in F(X \times Y)$ of R is defined by $\forall (x, y) \in X \times Y$,

$$(R^c)(x, y) = 1 - R(x, y)$$

The inverse $R^{-1} \in F(X \times Y)$ of R is defined by $\forall (x, y) \in X \times Y$,

$$R^{-1}(y, x) = R(x, y)$$

In addition, if $R_i \in F(X \times Y)$ for $i \in I$ indexing set, then $\bigcup_{i \in I} R_i$ is defined by, $\forall (x, y) \in X \times Y$

$$\left(\bigcup_{i \in I} R_i \right)(x, y) = \bigvee_{i \in I} R_i(x, y)$$

and $\bigcap_{i \in I} R_i$ is defined by $\forall (x, y) \in X \times Y, \bigcap_{i \in I} (\mathrm{R_i})(x, y) = \wedge_{i \in i} \mathrm{R_i}(x, y)$

Proposition 2.0.8 $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Proposition 2.0.9 $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

Proposition 2.0.10 $(R^c)^{-1} = \left(R^{-1}\right)^c$

A fuzzy relation also has the concept of (strong) α cut. The crisp relation, $R_\alpha = \{(x, y) \mid R(x, y) \geq \alpha\}$ for $\alpha \in [0, 1]$ will be called the α-cut relation of R. $R_\alpha = \{(x, y) \mid R(x, y) > \alpha\}$ for $\alpha \in [0, 1]$ will be called the strong α-cut relation of R.

Fuzzy Equivalence

Definition 2.1.1

If $R(x, x) = 1, \forall x \in X$, then R is called a reflexive(fuzzy) relation.

Definition 2.1.2

R is reflexive if and only if $\forall \alpha \in [0, 1], R_\alpha$ is reflexive.

Proof. If R is reflexive, then $\forall \alpha \in [0, 1], R(x, x) = 1 \geq \alpha$. Hence $(x, x) \in R_\alpha$. Thus R_α is reflexive.

Conversely, assume that $\forall \alpha \in [0, 1], R_\alpha$ is reflexive.

Particularly, R_1 is reflexive.

Hence $\forall x \in X, (x, x) \in R_1$ or $R(x, x) = 1$

It follows from that R is reflexive if and only if R_1(1-cut relation of R)is reflexive.

Definition 2.1.3

If $\forall x, y \in X, R(x, y) = R(y, x)$, then R is called a symmetric(fuzzy) relation.

Obviously, R is symmetric if and only if $R = R^{-1}$

Definition 2.1.3

R is symmetric if and only if $\forall \alpha \in [0, 1], R_\alpha$ is a symmetric relation.

Definition 2.1.4

Proof. If R is symmetric and $(x, y) \in R_\alpha$, then $R(y, x) = R(x, y) \geq \alpha$.

Hence $(y, x) \in R_\alpha$, which proves the symmetry of R_α

Conversely, assume that $\forall \alpha \in [0, 1], R_\alpha$ is symmetric.

For any $x, y \in X$, take $\alpha = R(x, y)$. Then $(x, y) \in R_\alpha$ and hence $(y, x) \in R_\alpha$ due to the symmetry of R_α

Therefore, $R(y, x) \geq \alpha = R(x, y)$

Next, $x, y \in R_\alpha$ and hence $(y, x) \in R_\alpha$ implies $R(x, y) \geq \alpha$ and $R(y, x) \geq \alpha$. We can take $R(y, x) = \alpha$ so that $R(x, y) \geq R(y, x)$. Combining the two inequalities yields, $R(x, y) = R(y, x)$

Definition 2.1.5

If $R \supseteq R^2$, then R is said to be a transitive(fuzzy)relation

Definition 2.1.6

R is transitive if and only if $\forall x, y, z \in X, R(x, z) \geq R(x, y) \wedge R(y, z)$

Proof.

$$
\begin{aligned}
R \text{ is transitive } &\Leftrightarrow R \supseteq R^2 \\
&\Leftrightarrow \forall x, z \in X, R(x,z) \geq R^2(x,y) \\
&\Leftrightarrow \forall x, y, z \in X, R(x,z) \geq R(x,y) \wedge R(y,z)
\end{aligned}
$$

Definition 2.1.7

R is transitive if and only if $\forall \alpha \in [0,1], R_\alpha$ is transitive.

Proof. Suppose that R is transitive. Let $(x,y),(y,z) \in R_\alpha$, for any fixed $\alpha \in [0,1]$. It follows that $R(x,y) \geq \alpha$ and $R(y,z) \geq \alpha$. Then,

$$R(x,y) \wedge R(y,z) = \alpha \geq R(x,z) \Rightarrow (x,z) \in R_\alpha$$

.Conversely, let $\forall \alpha \in [0,1], R_\alpha$ is transitive.

We prove $\forall x, y, z \in X, R(x,z) \geq R(x,y) \wedge R(y,z)$

By letting, $R(y,x) \wedge R(y,z) = \alpha$, we have $R(x,y) \geq \alpha$ and $R(y,z) \geq \alpha$,

so that $(x,y) \in R_\alpha$ and $(y,z) \in R_\alpha$.

Hence $R(x,z) \geq R(y,x) \wedge R(y,z) = \alpha$.

Since R_α is transitive.

Definition 2.1.8

If R is reflexive, symmetric and transitive, then R is called a fuzzy equivalence relation. R is a fuzzy equivalence relation if and only if $\forall \alpha \in [0,1], R_\alpha$ is an equivalence relation.

Definition 2.1.9

Let R be a fuzzy equivalence relation on X. A fuzzy set $[a]_R$ for $a \in X$ defined by;

$$\forall x \in X, [a]_R(x) = R(a, x)$$

is called the fuzzy equivalence class of a by R. The set $X/R = \{[a]_R \mid a \in X\}$ of all fuzzy equivalence classes is called the fuzzy quotient set of X by R.

Definition 2.1.10

If R is a fuzzy equivalence relation, then $[a]_R = [b]_R$ if and only if $R(a, b) = 1$

Proof. Suppose that $[a]_R = [b]_R$. $R(a, b) = [a]_R(b) = [b]_R(b) = R(b, b) = 1$ Conversely let $R(a, b) = 1$. Then $\forall x \in X$,

$$[a]_R(x) = R(a, x) \geq R(a, b) \wedge R(b, x) = R(b, x) = [b]_R(x)$$

.Since R is transitive.

Similarly, $[b]_R(x) = R(b, x) \geq R(b, a) \wedge R(a, x) = R(a, x) = [a]_R(x)$.

We have $[b]_R(x) \geq [a]_R(x)$

Consequently $[a]_R = [b]_R$.

Chapter 2
FUZZY GROUPS

Definition 3.0.11

Definition 4.0.11. Let G be a group. A fuzzy subset A on G is called a fuzzy subgroup of G if it satisfies the following conditions:

1. $A(xy) \geq A(x) \wedge A(y)$ for any $x, y \in G$
2. $A\left(x^{-1}\right) \geq A(x)$ for any $x \in G$.

Definition 3.0.12

Let A be a fuzzy subset of a set S. For $t \in [0,1]$, the set

$$A_t = \{x \in S \mid A(x) \geq t\}$$

is called a level subset of the fuzzy subset A.

Note 3.0.13

A_t is a subset of S in the ordinary sense. The terminology 'level set' was introduced by Zadeh.

Proposition 3.0.14

Let A be a fuzzy subgroup of G. For any $x \in G$,

1. $A(x) \leq A(e)$
2. $A\left(x^{-1}\right) = A(x)$ 3. $A\left(x^n\right) \geq A(x)$, where n is an arbitrary integer.

Proof. 1. $A(e) = A\left(xx^{-1}\right) \geq A(x) \wedge A\left(x^{-1}\right) \geq A(x) \wedge A(x) = A(x)$

2. $A(x)=A\left(\left(x^{-1}\right)^{-1}\right) \geq A\left(x^{-1}\right)$

We have from the definition of fuzzy subgroup that, $A\left(x^{-1}\right) \geq A(x)$

Therefore, $A\left(x^{-1}\right)=A(x)$.

3. The proof is by mathematical induction on n.

Let $n=2$. Then we have from the definition of fuzzy subgroup

$$A\left(x^{2}\right) \geq A(x) \wedge A(x)=A(x)$$

i.e. the result holds for $n=2$.

Assume it holds for k.

Now,

$$\begin{aligned} A\left(x^{k+1}\right) & \geq A(x) \wedge A\left(x^{k}\right) \\ & \geq A(x) \wedge A(x) \quad [\text{ by induction hypothesis }] \\ & =A(x) \end{aligned}$$

Hence the result is true for any arbitrary integer n.

Proposition 3.0.15

Let $A \in F(G)$. Then A is a fuzzy subgroup of G if and only if

$$A\left(x y^{-1}\right) \geq A(x) \wedge A(y)$$

holds for any $x, y \in G$. Proof. Assume that A is a fuzzy subgroup of G. Then we have,

$$
\begin{aligned}
A\left(xy^{-1}\right) &\geq A(x) \wedge A\left(y^{-1}\right) \\
&= A(x) \wedge A(y)
\end{aligned}
$$

Conversely, suppose that $A\left(xy^{-1}\right) \geq A(x) \wedge A(y)$, holds for any $x, y \in G$. Then for any $x \in G$,

$$A(e) = A\left(xx^{-1}\right) \geq A(x) \wedge A(x) = A(x)$$

i.e.,

$$A(x) \leq A(e)$$

Thus for any $x \in G$,

$$A\left(x^{-1}\right) = A\left(ex^{-1}\right) \geq A(e) \wedge A(x) = A(x)$$

i.e.,

$$A\left(x^{-1}\right) \geq A(x)$$

Meanwhile for any $x, y \in G$

$$A(xy) = A\left(x\left(y^{-1}\right)^{-1}\right) \geq A(x) \wedge A\left(y^{-1}\right) \geq A(x) \wedge A(y)$$

Therefore,

A is a fuzzy subgroup of G.

Proposition 3.0.16

Let G be a group and A be a fuzzy subgroup of G. Then the level subset A_t, for $t \in [0,1], t \leq A(e)$, is a subgroup of G, where e is the identity of G.

Proof. We have,

$$A_t = \{x \in G \mid A(x) \geq t\}$$

Since $A(e) \geq t, e \in A_t$

Clearly, $A_t \neq \emptyset$

Let $x, y \in A_t$. Then $A(x) \geq t$ and $A(y) \geq t$

Since A is a fuzzy subgroup of G,

$$A(xy) \geq A(x) \wedge A(y) \geq t$$

i.e.,

$$A(xy) \geq t$$

Hence $xy \in A_t$

Again $x \in A_t \Rightarrow A(x) \geq t$

Since A is a fuzzy subgroup,

$$A\left(x^{-1}\right) \geq A(x)$$

and hence $A\left(x^{-1}\right) \geq t$ i.e.

$$x^{-1} \in A_t$$

Therefore, A_t is a subgroup of G

Proposition 3.0.17

Proposition 4.0.17. Let G be a group and A be a fuzzy subset of G such that A_t is a subgroup of G for all $t \in [0,1], t \leq A(e)$, then A is a fuzzy subgroup of G.

Proof. Assume A_t is a subgroup of G for all $t \in [0,1], t \leq A(e)$. Let $x, y \in G$ and let $A(x) = t_1$ and $A(y) = t_2$. Then, $x \in A_{t_1}$ and $y \in A_{t_2}$

Let us assume that $t_1 < t_2$. Then it follows $A_{t_2} \subseteq A_{t_1}$.

So $y \in A_{t_1}$.

Thus $x, y \in A_{t_1}$ and since A_{t_1} is a subgroup of G, by hypothesis, $xy \in A_{t_1}$ Therefore,

$$A(xy) \geq t_1 = A(x) \wedge A(y)$$

Next, let $x \in G$ and let $A(x) = t$. Then $x \in A_t$

Since A_t is a subgroup, $x^{-1} \in A_t$

Therefore,

$$A\left(x^{-1}\right) \geq t$$

And hence,

$$A\left(x^{-1}\right) \geq A(x)$$

Thus, A is a fuzzy subgroup of G.

Definition 3.0.18

(Level Subgroup) Let G be a group and A be a fuzzy subgroup of G. The subgroups $A_t, t \in [0,1]$ and $t \leq A(e)$, are called level subgroups of A.

Particularly,

$$A_{A(e)} = \{x \in G \mid A(x) = A(e)\}$$

is a subgroup of G if A is a fuzzy subgroup of G. We shall denote this subgroup by A^*.

The binary multiplicative operation in G can be extended to $F(G)$ using the Zadeh's extension principle.

Definition 3.0.19

Let $A, B \in F(G)$. Then $A \circ B$ is defined by:

for any $z \in G$,

$$(A \circ B)(z) = \bigvee_{z=xy} (A(x) \wedge B(y))$$

In addition, for every $A \in F(G)$, we shall define $A^{-1} \in F(G)$ by: for any $x \in G, A^{-1} = A\left(x^{-1}\right)$. With these notions, we present an equivalent statement of a fuzzy subgroup.

Propostion 3.0.20

Let $A \in F(G)$. Then A is a fuzzy subgroup of G if and only if

$$A \circ A^{-1} = A$$

Definition 3.0.21

Let A be a fuzzy subgroup of G, and let f be an epimorphism of G onto a group G'. Then $f(A)$ is a fuzzy subgroup of G'.

Definition 3.0.22

Let f be a homomorphism from a group G to a group G', and let B be a fuzzy subgroup of G'. Then $f^{-1}(B)$ is a fuzzy subgroup of G.

Proof. For any $x, y \in G$,

$$\begin{aligned} f^{-1}(B)\left(xy^{-1}\right) &= B\left(f\left(xy^{-1}\right)\right) \\ &= B\left(f(x)f\left(y^{-1}\right)\right) \\ &\geq B(f(x)) \wedge B\left(f(y)^{-1}\right) \\ &= B(f(x)) \wedge B\left(f\left(y^{-1}\right)\right) \\ &\geq B(f(x)) \wedge B(f(y)) \\ &= f^{-1}(B)(x) \wedge f^{-1}(B)(y) \end{aligned}$$

Definition 3.0.23

Let $A_1, A_2, \ldots, A_n$ be fuzzy subgroups of $G_1, G_2, \ldots, G_n$ respectively. Then the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$ is a fuzzy subgroup of $G_1 \times G_2 \times \ldots \times G_n$

Proof. Form a tuple for $x_i, y_i \in G_i$. Since we have,

$$A_i\left(x_i y_i^{-1}\right) \geq A_i\left(x_i\right) \wedge A_i\left(y_i\right)$$

Then,

$$\begin{aligned}
\left(A_1 \times A_2 \times \ldots \times A_n\right)\left(x_1 y_1^{-1}, x_2 y_2^{-1}, \ldots, x_n y_n^{-1}\right) &= \bigwedge_{i=1}^{n} A_i\left(x_i y_i^{-1}\right) \\
&\geq \bigwedge_{i=1}^{n} A_i\left(x_i\right) \wedge A_i\left(y_i\right) \\
&= \left(A_1 \times A_2 \times \ldots \times A_n\right)\left(x_1 y_1, x_2 y_2,\right.
\end{aligned}$$

Suppose G is a finite group, then the number of subgroups of G is finite whereas the number of level subgroups of a fuzzy subgroup A appears to be infinite. But since every level subgroups is indeed a subgroup of G, not all these level subgroups are distinct.

The next proposition characterises this aspect:

Proposition 3.0.24

Let G be a group and A be a fuzzy subgroup of G. Two level subgroups A_{t_1}, A_{t_2} (with $t_1 < t_2$)of A are equal if and only if there is no $x \in G$ such that $t_1 < A(x) < t_2$.

Proof. Let us consider $A_{t_1} - A_{t_2}$. Suppose there exists $x \in G$ such that $t_1 < A(x) < t_2$, then $A_{t_2} \subset A_{t_1}$, since x belongs to A_{t_1} but not in A_{t_2}, which contradicts the hypothesis.

Conversely, let there be no $x \in G$ such that $t_1 < A(x) < t_2$. Since $t_1 < t_2$, we have $A_{t_2} \subseteq A_{t_1}$

Let $x \in A_{t_1}$, then $A(x) \geq t_1$, and hence $A(x) \geq t_2$, since $A(x)$ does not lie between t_1 and t_2.

Therefore, $x \in A_{t_2}$

So, $A_{t_1} \subseteq A_{t_2}$

Thus, $A_{t_1} = A_{t_2}$

Lemma 3.0.25

Let G be a finite group of order n and A be a fuzzy subgroup of G. Let

$$\text{Im}(A) = \{t_i \mid A(x) = t_i, \text{ for some } x \in G\}$$

Then $\{A_{t_i}\}$ are the only level subgroups of A.

Proposition 3.0.26

Any subgroup H of a group G can be realized as a level subgroup of some fuzzy subgroup of G.

Proposition 3.0.27

Let $\bar{A}$ be the collection of all fuzzy subgroups of group G and $\bar{B}$ be the collection of all level subgroups of members of $\bar{A}$. Then there is a one to one correspondence between the subgroups of G and the equivalence classes of level subgroups (under a suitable equivalence relation on $\bar{B}$)

Definition 3.0.28

Let G be a group of prime power order. Then G is cyclic if and only if there exists a fuzzy subgroup A of G such that for $x, y \in G$,

1. if $A(x) = A(y)$, then $< x >=< y >$ 2. if $A(x) > A(y)$, then $< x >\subset< y >$

Proposition 3.0.29

Let G be a cyclic p-group of order p^n, where p is a prime. Let A be a fuzzy subgroup of G then for $x, y \in G$

1. if $O(x) > O(y)$, then $A(y) \geq A(x)$
2. if $O(x) = O(y)$, then $A(x) = A(y)$

Definition 3.0.30

Let A and B be two fuzzy subgroups of a group G. Then A and B are said to be conjugate fuzzy subgroups of G if for some $x \in G$,

$$A(y) = B\left(x^{-1}yx\right), \forall y \in G$$

Proposition 3.0.31

If A and B are conjugate fuzzy subgroups of the group G, then $O(A) = O(B)$

Normal fuzzy subgroups and Cosets

Now we introduce the notion of fuzzy middle coset of a group G.

Definition 3.1.1

Let A be a fuzzy subgroup of a group G. Then for any $x, y \in G$, a fuzzy middle coset xAy of the group G is defined by,

$$(xAy)(a) = A\left(x^{-1}ay^{-1}\right), \forall a \in G$$

Proposition 3.1.2

If A is a fuzzy subgroup of a group G, then for any $x \in G$ the fuzzy middle coset xAx^{-1} of the group G is also a fuzzy subgroup of the group G

Proposition 3.1.3

Let A be any fuzzy subgroup of a group G and xAx^{-1} be a fuzzy middle coset of the group G. Then

$$O\left(xAx^{-1}\right) = O(A)$$

for any $x \in G$.

Proof. Let A be a fuzzy subgroup of a group G and $x \in G$. Then by the above proposition,the fuzzy middle coset xAx^{-1} is a fuzzy subgroup of the group G. By the definition of a fuzzy middle coset of the group G,

$$\left(xAx^{-1}\right)(a) = A\left(x^{-1}ax\right)$$

$\forall a \in G$

Hence for any $x \in G, A$ and xAx^{-1} are conjugate fuzzy subgroups of the group G as there exists $x \in G$ such that

$$\left(xAx^{-1}\right)(a) = A\left(x^{-1}ax\right), \forall a \in G.$$

Thus $O\left(xAx^{-1}\right) = O(A)$, for any $x \in G$.

Proposition 3.1.4

Let A be a fuzzy subgroup of a finite group G. Then $O(A)$ divides $O(G)$.

Proof. Let A be a fuzzy subgroup of a finite group G with e as its identity element. Clearly,

$$H = \{x \in G \mid A(x) = A(e)\}$$

is a subgroup of the group G for H is a t-level subset of the group G, where $t = A(e)$

By Lagrange's theorem,

$$O(H) \mid O(G)$$

Thus by the definition of order of a fuzzy subgroup of the group G,

$$O(A) \mid O(G)$$

Proposition 3.1.5

Let A and B be any two improper fuzzy subgroups of a group G. Then A and B are conjugate fuzzy subgroups of the group G if and only if $A = B$

Proposition 3.1.6

Let A and B be two fuzzy subsets of an abelian group G. Then A and B are conjugate fuzzy subsets of the group G if and only if $A = B$.

Proof. Let A and B be conjugate fuzzy subsets of the group G. Then for some $x \in G$, we have

$$\begin{aligned} A(a) &= B\left(x^{-1} a x\right), \forall a \in G \\ &= B\left(x^{-1} x a\right), \forall a \in G \\ &= B(a), \forall a \in G \end{aligned}$$

Hence $A = B$.

Conversely if $A = B$, then for the identity element e of the group G, we have

$$A(a) = B\left(e^{-1}ae\right), \forall a \in G$$

Hence A and B are conjugate fuzzy subsets of the group G.

Proposition 3.1.7

A fuzzy subgroup A of G is called normal if

$$A(xy) = A(yx)$$

holds for any $x, y \in G$.

Proposition 3.1.8

A fuzzy subgroup A of G is normal if and only if

$$A\left(xyx^{-1}\right) = A(y)$$

holds for any $x, y \in G$. Proof. Let A be a fuzzy subgroup of G.

Suppose A is normal. By definition, for any $x, y \in G$,

$$
\begin{aligned}
A\left(xyx^{-1}\right) &= A\left((xy)x^{-1}\right) \\
&= A\left(x^{-1}(xy)\right) \\
&= A\left(x^{-1}xy\right) \\
&= A(y)
\end{aligned}
$$

Conversely, suppose that $A\left(xyx^{-1}\right) = A(y)$ holds for any $x, y \in G$.Then,

$$
\begin{aligned}
xy &= x(yx)x^{-1} \\
A(xy) &= A\left(x(yx)x^{-1}\right) \\
&= A(yx)
\end{aligned}
$$

i.e., A is normal.

Proposition 3.1.9

$A \in F(G)$ is a normal fuzzy subgroup of G if and only if

$$
A \circ A^{-1} = A \text{ and } A \circ B = B \circ A
$$

holds for all $B \in F(G)$. Proof. For any fuzzy subgroup, $A \circ A^{-1} = A$. Take

$$
\begin{aligned}
(A \circ B)(z) &= \bigvee_{z=xy} A(x) \wedge B(y) \\
&= \bigvee_{y \in G} A\left(zy^{-1}\right) \wedge B(y) \\
&= \bigvee_{y \in G} A\left(y^{-1}z\right) \wedge B(y) \\
&= \bigvee_{z=yx} A(x) \wedge B(y) \\
&= \bigvee_{z=yx} B(y) \wedge A(x) \\
&= (B \circ A)(z)
\end{aligned}
$$

Conversely, $A \circ A^{-1} = A$ implies A is a fuzzy subgroup. To show that A is normal, take $B = \left\{x^{-1}\right\}$. Then

$$
\begin{aligned}
A(xy) &= \left(\left\{x^{-1}\right)\right\} \circ A\right)(y) \\
&= \left(A \circ \left\{x^{-1}\right\}\right)(y) \\
&= \bigvee_{y=st} A(s) \wedge \left\{x^{-1}\right\}(t) \\
&= A(yx)
\end{aligned}
$$

Proposition 3.1.10

$A \in F(G)$ is a normal fuzzy subgroup of G if and only if A_t is a normal subgroup of G for any $t \in [0,1], t \leq A(e)$

Proof. A is a subgroup if and only if A_t is one. For normality, take $x \in G$ and $y \in A_t$. It follows that $A\left(xyx^{-1}\right) = A(y) \geq t$. Hence, $xyx^{-1} \in A_t$, and thus A_t is normal.

Conversely, take $x, y \in G$ and $t = A(y)$. Then, $t \in \{A(x) \mid x \in G\}$ and $y \in$

A_t. Hence $xyx^{-1} \in A_t$. Consequently

$$A\left(xy^{-1}x\right) \geq t = A(y)$$

. As a result, A is a normal fuzzy subgroup of G.

Particularly, A^* is a normal subgroup of G if A is a normal fuzzy subgroup of G.

Definition 3.1.11

Let A be a fuzzy subgroup of G. For every $x \in G$, define $xA, Ax \in F(G)$ by; $\forall y \in G, (xA)(y) = A\left(x^{-1}y\right)$ and $(Ax)(y) = A\left(yx^{-1}\right)$.

Then xA and Ax are called the left coset and right coset of A with respect to x respectively. Clearly, $xA = Ax$ holds for any $x \in G$, if A is a normal fuzzy subgroup of G. In this case, we simply call $xA(= Ax)$ a coset. Write $G/A = \{xA \mid x \in G\}$

Lemma 3.1.12

Let A be two normal fuzzy subgroups of G. Then $xA \circ yA = (xy)A$ holds for any two cosets $xA, yA \in G/A$.

Proof. On the one hand, for any $z \in G$

$$
\begin{aligned}
(xA \circ yA)(z) &= \bigvee_{z=z_1z_2} \left((xA)\left(z_1\right) \wedge (yA)\left(z_2\right)\right) \\
&\geq (xA)(x) \wedge (yA)\left(x^{-1}z\right) \\
&= A\left(x^{-1}x\right) \wedge A\left(y^{-1}x^{-1}z\right) \\
&= A(e) \wedge A\left(y^{-1}x^{-1}z\right) \\
&= A\left((xy)^{-1}z\right) \\
&= ((xy)A)(z)
\end{aligned}
$$

We have the following result concerning $(G/A, \circ)$.

Proposition 3.1.13

Let A be a normal fuzzy subgroup of G. Then

1. $(G/A, \circ)$ is a group
2. G/A is isomorphic to G/A^*

Proof. 1

Clearly, the operation $\circ$ is associative, A is the identity of G/A and the inverse of xA is $x^{-1}A$.Hence, $(G/A, \circ)$ is a group.

2. For any $x \in G$, let $f : xA \to xA^*$. Then, for any $x, y \in G$,

$$
\begin{aligned}
f(xA \circ yA) &= f(xyA) \\
&= xyA^* \\
&= xA^*yA^* \\
&= f(xA)f(yA)
\end{aligned}
$$

Hence, f is a homomorphism. In order to prove that f is injective, suppose that $xA = yA$. Then $A\left(x^{-1}z\right) = A\left(y^{-1}z\right)$ for all $z \in G$.

Particularly, $A\left(x^{-1}y\right) = A(e)$ when $z = y$. Thus $x^{-1}y \in A^*$. As a result, $xA^* = yA^*$. Hence f is injective. It is clear that f is surjective.

In summary, f is an isomorphism between G/A and G/A^*.

G/A will be called the quotient group of G by a normal fuzzy subgroup A of G.

Definition 3.1.14

Let A be a fuzzy subgroup of a group G and $x \in G$. Then the pseudo fuzzy coset $(xA)^p$ is defined by,

$$(xA)^p(y) = p(x)A(y), \forall y \in G$$

and for some $p \in P$.

Proposition 3.1.15

Let A be a fuzzy subgroup of a group G. Then the pseudo fuzzy coset $(xA)^p$ is a fuzzy subgroup of $G, \forall x \in G$.

Proof. Let A be a fuzzy subgroup of a group $G . \forall x, y \in G$,

$$\begin{aligned}
(aA)^p\left(xy^{-1}\right) &= p(a)A\left(xy^{-1}\right) \\
&\geq p(a) \wedge \{A(x), A(y)\} \\
&= \bigwedge\{p(a)A(x), p(a)A(y)\} \\
&= \bigwedge\{(aA)^p(x), (aA)^p(y)\}
\end{aligned}$$

i.e.

$$(aA)^p \left(xy^{-1}\right) \geq \bigwedge \left\{(aA)^p(x), (aA)^p(y)\right\}, \forall x, y \in G$$

This proves that $(aA)^p$ is a fuzzy subgroup of the group G.

Proposition 3.1.16

Let A be a normal fuzzy subgroup of G. Define $\bar{A} : G/A \to [0,1]$ by

$$\forall xA \in G/A, \bar{A}(xA) = A(x)$$

Then $\bar{A}$ is a normal fuzzy subgroup of G/A. Proof. Firstly, for any $xA \in G/A$,

$$\begin{aligned} \bar{A}\left((xA)^{-1}\right) &= \bar{A}\left(x^{-1}A\right) \\ &= A\left(x^{-1}\right) \\ &= A(x) \\ &= \bar{A}(xA) \end{aligned}$$

and for any $xA, yA \in G/A$,

$$\begin{aligned} \bar{A}(xAyA) &= \bar{A}(xyA) \\ &= A(xy) \\ &\geq A(x) \wedge A(y) \\ &= \bar{A}(xA) \wedge \bar{A}(yA) \end{aligned}$$

Hence, $\bar{A}$ is a fuzzy subgroup of G/A. Next, for any $xA, yA \in G/A$,

$$\begin{aligned}\bar{A}(xA \circ yA) &= \bar{A}(xyA) \\ &= A(xy) \\ &= A(yx) \\ &= \bar{A}(yxA) \\ &= \bar{A}(yA \circ xA).\end{aligned}$$

Hence $\bar{A}$ is a normal fuzzy subgroup of G/A.

Proposition 3.1.17

Let A be a normal fuzzy subgroup of G and let f be an epimorphism of G onto a group G. Then $f(A)$ is a normal fuzzy subgroup of G.

Proof. We have, A is a normal fuzzy subgroup of G and f be an epimorphism of G onto G. Then $f(A)$ is a fuzzy subgroup of G. Let $u, v \in G$. Then there exists $x \in G$ such that $f(x) = u$ since f is surjective. Hence, we obtain successively

$$
\begin{aligned}
f(A)\left(uvu^{-1}\right) = & \bigvee_{f(z)=uvu^{-1}} A(z) \\
= & \bigvee_{f(z)=f(x)v(f(x))^{-1}} A(z) \\
= & \bigvee_{f(x^{-1}zx)=v} A(z) \quad (f \text{ is a homomorphism }) \\
= & \bigvee_{f(y)=v} A\left(xyx^{-1}\right) \\
= & \bigvee_{f(y)=v} A(y) \\
= & f(A)(v)
\end{aligned}
$$

Hence $f($ A) is a normal fuzzy subgroup of G.

Proposition 3.1.18

Let f be a homomorphism from G to a group G and let B be a normal fuzzy subgroup of G. Then $f^{-1}(B)$ is a normal fuzzy subgroup of G.

Proof. We have, if f is a homomorphism from G to G and B is a fuzzy subgroup of G, then $f^{-1}(B)$ is a fuzzy subgroup of G. Now, let $x, y \in G$. Then

$$\begin{aligned} f^{-1}(B)(xy) &= B(f(xy)) \\ &= B(f(x)f(y)) \\ &= B(f(y)f(x)) \\ &= B(f(yx)) \\ &= f^{-1}(B)(yx) \end{aligned}$$

Hence, $f^{-1}(B)$ is a normal fuzzy subgroup of G.

Proposition 3.1.19

Let $A_1, A_2, \ldots, A_n$ be normal fuzzy subgroups of $G_1, G_2, \ldots, G_n$ respectively. Then the Cartesian product $\prod_{i=1}^{n} A_i$ is a normal fuzzy subgroup of $G_1 \times G_2 \times \ldots \times G_n$ Proof. We have, if $A_1, A_2, \ldots, A_n$ be fuzzy subgroups of $G_1, G_2, \ldots, G_n$ respectively. Then the Cartesian product $\prod_{i=1}^{n} A_i$ is a fuzzy subgroup of $G_1 \times G_2 \times \ldots \times G_n$. Furthermore, $\forall \left(x_1, x_2, \ldots, x_n\right), \left(y_1, y_2, \ldots, y_n\right) \in G_1 \times G_2 \times \ldots \times G_n$

$$\begin{aligned} \left(\prod_{i=1}^{n} A_i\right)\left(x_1, x_2, \ldots, x_n\right)\left(y_1, y_2, \ldots, y_n\right) &= \left(\prod_{i=1}^{n} A_i\right)\left(x_1y_1, x_2y_2, \ldots, x_ny_n\right) \\ &= \bigwedge_{i=1}^{n} A_i\left(x_iy_i\right) \\ &= \bigwedge_{i=1}^{n} A_i\left(y_ix_i\right) \\ &= \left(\prod_{i=1}^{n} A_i\right)\left(y_1x_1, y_2x_2, \ldots, y_nx_n\right) \end{aligned}$$

Chapter 3
FUZZY SUBRINGS

In this and next subsection, we assume $(R, +, \circ)$ is a ring. For convenience, we write xy instead of $x \circ y$ for $x, y \in R$.

Definition 4.0.1

$A \in F(R)$ is called a fuzzy sub ring of R if A satisfies that

1. $\forall x, y \in R, A(x - y) \geq A(x) \wedge A(y)$
2. $\forall x, y \in R, A(xy) \geq A(x) \wedge A(y)$

Proposition 4.0.2

$A \in F(R)$ is a fuzzy sub ring of R if and only if A_t is a sub ring of R for every $t \in [0, 1], t \leq A(e)$

By proposition,

$$A^* = \{x \mid A(x) = A(0)\}$$

is a sub ring of R. The operations on R can be extended to $F(R)$ as follows: $\forall A, B \in F(R), \forall z \in R$

$$(A + B)(z) = \bigvee_{x+y=z} (A(x) \wedge B(y)$$

$$(A - B)(z) = \bigvee_{x-y=z} (A(x) \wedge B(y)$$

$$(A \circ B)(z) = \bigvee_{xy=z} (A(x) \wedge B(y)$$

Remark 4.0.3

$A \in F(R)$ is a fuzzy sub ring of R if and only if $A - A \subseteq A$ and $A \circ A \subseteq A$

Proof. Let A be a fuzzy sub ring of R. Since A is a fuzzy group under addition, $A - A \subseteq A$. Moreover,

$$\forall z \in R, A \circ A(z) = \bigvee_{xy=z} (A(x) \wedge A(y) \leq A(xy) = A(z)$$

i.e.,

$$A \circ A \subseteq A$$

Conversely, suppose that $A - A \subseteq A$ and $A \circ A \subseteq A$. Then $\forall x, y \in R$,

$$\begin{aligned} A(x-y) &\geq (A-A)(x-y) \\ &= \bigvee_{s-t=x-y} (A(s) \wedge A(t)) \\ &\geq A(x) \wedge A(y) \end{aligned}$$

Similarly,

$$\begin{aligned} A(xy) &\geq (A \circ A)(xy) \\ &= \bigvee_{xy=st} (A(s) \wedge A(t)) \\ &\geq A(x) \wedge A(y). \end{aligned}$$

Consequently, A is a fuzzy sub ring of R.

Proposition 4.0.4

Let A be a fuzzy sub ring of R and let f be an epimorphism of R onto a ring R. Then $f(A)$ is a fuzzy sub ring of R.

Proof. Let $u, v \in R$. Then there exists $x, y \in R$ such that $f(x) = u$ and $f(y) = v$ since f is surjective. Hence, we obtain successively

$$\begin{aligned}
f(A)(u) \wedge f(A)(v) &= \bigvee_{f(x)=u} A(x)\Bigg) \wedge \Bigg(\bigvee_{f(y)=v} A(y)\Bigg) \\
&= \bigvee_{f(x)=u, f(y)=v} A(x) \wedge A(y) \\
&\leq \bigvee_{f(x)=u, f(y)=v} A(x-y) \quad (A \text{ is a fuzzy subring of } R) \\
&\leq \bigvee_{f(x)-f(y)=u-v} A(x-y) \quad (f \text{ is a homomorphism }) \\
&= \bigvee_{f(z)=u-v} A(z) \\
&= f(A)(u-v)
\end{aligned}$$

Similarly, $f(A)(uv) \geq f(A)(u) \wedge f(A)(v)$

Hence $f(A)$ is a sub ring of R.

Proposition 4.0.5

Let f be a homomorphism from R to a ring R and let B be a fuzzy sub ring of R. Then $f^{-1}(B)$ is a sub ring of R. Proof. For any $x, y \in R$,

$$
\begin{aligned}
f^{-1}(B)(xy) &= B(f(xy)) \\
&= B(f(x)f(y)) \\
&\geq B(f(x)) \wedge B(f(y)) \\
&= f^{-1}(B)(x) \wedge f^{-1}(B)(y)
\end{aligned}
$$

Similarly,

$$f^{-1}(B)(x-y) \geq f^{-1}(B)(x) \wedge f^{-1}(B)(y)$$

Thus $f^{-1}(B)$ is a fuzzy sub ring of R.

For a fixed element $k \in [0,1]$, the fuzzy set $\hat{k}$ on R defined by $\hat{k}(x) = k, \forall x \in R$ is called a constant fuzzy set on R.

Obviously all constant fuzzy sets on R satisfy both the axioms of a fuzzy ring. Hence all constant fuzzy sets on R are fuzzy rings on R.

Now turning to non-constant fuzzy sets on R, we get the following proposition.

Proposition 4.0.6

A non-constant fuzzy set A on R is a fuzzy ring on R if and only if A_t is a sub ring of $R, \forall t \in [0,1]$

- If R_1, R_2 are rings, $f : R_1 \to R_2$ is a function, A is a fuzzy set on R_1 and B is a fuzzy set on R_2, then the image of A under f is the fuzzy set $f(A)$ on R_2 defined by,

$$f(A)(y) = \bigvee \left\{ A(x) \mid x \in f^{-1}(y) \right\}$$
$$= 0, \text{ iff}$$

The pre image of B under f is the fuzzy set $f^{-1}(B)$ on R_1 defined by,

$$f^{-1}(B)(x) = B[f(x)], \forall x \in R_1$$

Proposition 4.0.7

Let R_1, R_2 be rings, $f : R_1 \to R_2$ be a ring homomorphism. A be a fuzzy ring on R_1 and B be a fuzzy ring on R_2. Then $f(A)$ is a fuzzy ring on R_2 and $f^{-1}(B)$ is a fuzzy ring on R_1.

Fuzzy Ideals

Definition 4.1.1

A fuzzy ring A on a ring R is said to be a fuzzy left ideal if

$$A(xy) \geq A(y), \forall x, y \in R$$

and fuzzy right ideal if

$$A(xy) \geq A(x), \forall x, y \in R$$

A is called a fuzzy ideal if it is both a fuzzy left ideal and a fuzzy right ideal. In other words, a fuzzy set A on R is a fuzzy ideal if $\forall x, y \in R$.

1. $A(x-y) \geq \bigwedge\{A(x), A(y)\}$
2. $A(xy) \geq \bigvee\{A(x), A(y)\}$

Proposition 4.1.2

A fuzzy ring on R is a fuzzy ideal if and only if A_t is an ideal of $R, \forall t \in [0,1]$ Proof. Suppose that A is a fuzzy ideal of R. Then A_t, for $t \in [0,1]$ is a sub ring of R.

Let $x, y \in A_t$ and $z \in R$. Then

$$A(x-y) \geq A(x) \wedge A(y) \geq t$$

and

$$A(zx) \geq A(z) \vee A(x) \geq A(x) \geq t$$

Hence $x - y \in A_t$ and $zx \in A_t$. Thus A_t is an ideal of R.

Conversely suppose that A_t is an ideal of R for every $t \in [0,1]$. Then A is a fuzzy subring of R. Let $x, y \in R$ and $t = A(x)$. Then $t \in [0,1]$ and $x \in A_t$. Since A_t is an ideal $xy \in A_t$.

Hence $A(xy) \geq t = A(x)$.

Similarly, $A(xy) \geq A(y)$

Therefore $A(xy) \geq A(x) \vee A(y)$

Thus A is a fuzzy ideal of R.

Particularly,

$$A^* = \{x \mid A(x) = A(0)\}$$

is an ideal of R if A is a fuzzy ideal of R.

Proposition 4.1.3

Let A be a fuzzy ideal of R and let f be an epimorphism of R onto a ring R. Then $f(A)$ is a fuzzy ideal of R

If A is a fuzzy subring on R, we shall use the notation A_0 for $\{x \in R \mid A(x) = A(0)\}$

CONCLUSION

We conclude this project by knowing the different aspect of fuzzy algebra. This project is all about fuzzy set., fuzzy relation, fuzzy subring. From this we get most of the ideas of fuzzy algebra and relation between fuzzy sets. Fuzzy sets can be applied in the following fields: Engineering, Psychology, medicine, Artificial intelligence, Ecology, Decision making theory, Sociology, Computer science, Manufacturing and others.

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BONAFIDE CERTIFICATE

Certified that this project report "FUZZY MATRICES " is the bonafide work of AMRUTHA P M who carried out the project work under my supervision.

Signature of Head   Signature of Supervisor

DECLARATION

I, AMRUTHA P M hereby declare that the Project work entitled FUZZY MATRICES has been prepared by me and submitted to Kannur University in partial fulfillment of requirement for the award of Master of Science is a record of original work done by me under the supervision of Mrs. PRIJA V, Assistant Professor, Department of Mathematics, Don Bosco Arts & Science College, Angadikadavu. I, also declare that this Project work has not been submitted by me fully for the award of any Degree, Diploma, Title or recognition before any authority.

Place: Angadikadavu AMRUTHA P M

Date: (C1PSMM1902)

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AMRUTHA P M

CONTENTS

INTRODUCTION

We deal with fuzzy matrices, that is matrices over the fuzzy algebra $\mathbf{F}$= [0,1] under max-min operations (+,·) defined as a+b = max{a,b} and a · b =min{a,b} for all a,b $\in \mathbf{F}$ and the standard order '$\leq$' of real numbers. Let $\mathbf{F}_{mn}$ be the set of all m×n matrices over $\mathbf{F}$. In short $\mathbf{F}_n$ denotes $\mathbf{F}_{nn}$. For A $\in \mathbf{F}_n$, A^T, R(A), C(A), ρ_r(A), ρ_c(A) and ρ(A) denotes the transpose, row space, column space, row rank, column rank and rank of A. Algebraic operations on matrices are max-min operations, which are different from that of the standard operations on real matrices. In practice, fuzzy matrices have been proposed to represent fuzzy relations in a system based on fuzzy sets theory, the behaviour of the dynamic fuzzy systems depends heavily on the products of fuzzy matrices in the matrix representations of the system. A $\in \mathbf{F}_{mn}$ is regular if there exists X such that AXA = A; X is called a generalized (g^-) inverse of A and is denoted by A^-. $A^-\{1\}$ denotes the set of all g-inverses of a regular matrix A. A regular matrix as one that has a generalized inverse lays the foundation in the study on fuzzy relational equations. Regular fuzzy matrices play an important role in estimation and inverse problem in fuzzy relational equations and in fuzzy optimization problems. This motivates us to develop the study on generalized regular fuzzy matrices. The power of a fuzzy matrix are either convergent to a fuzzy matrix (or) oscillating with finite period. For a fuzzy matrix A, $A^{k+d} = A^k$ for some integers k,d $\geq$ 0. Therefore, all fuzzy matrices have an index and a period. On the other hand, most matrices over the non negative real numbers will not have an index and a period. Spectral inverses, such as group inverse and Drazin inverse are defined for fuzzy matrices, analogous to that for complex matrices. For A $\in \mathbf{F}_n$, The Drazin inverse of A is a solution of the equations : A^k XA = A^k, XAX = X, AX = XA, for some positive integer k. Group inverse is the solution of the equations : AXA = A, XAX = X, AX = XA. Hence Drazin inverse and group inverse are identical when k = 1. We define the regularity index of a matrix A $\in \mathbf{F}_n$ as a generalization of the index of A. A characterization of a matrix whose regularity index coincides with the index of A is established. It is shown that for a matrix, regularity index is less than (or) equal to the index of a matrix. Numerical examples are provided to illustrate the relation between the regularity index and index of a fuzzy matrix.

CHAPTER 1

PRELIMINARIES

Matrices are one of the most important tools in mathematics. Also the Matrices are not only used as a representation of the coefficients in system of linear equations, but utility of matrices far exceeds that use.

Definition 1.0.1

A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.
We denote matrices by capital letters. The following are some examples of matrices:

$$\text{A} = \begin{pmatrix} -2 & 5 \\ 0 & \sqrt{5} \\ 3 & 6 \end{pmatrix} \qquad \text{B} = \begin{pmatrix} 2+i & 3 & -1/2 \\ 3.5 & -1 & 2 \\ \sqrt{3} & 5 & 5/7 \end{pmatrix}$$

$$\text{C} = \begin{pmatrix} 1+x & x^3 & 3 \\ cosx & sinx+2 & tanx \end{pmatrix}$$

In the above examples, the horizontal lines of elements are said to constitute, rows of the matrix and the vertical lines of elements are said to constitute, columns of the matrix. Thus A has 3 rows and 2 columns, B has 3 rows and 3 columns while C has 2 rows and 3 columns.

Definition 1.0.2

Order of a matrix-A matrix having m rows and n columns is called a matrix of order mn.

Note 1.0.3

1. We shall follow the notation, namely A = $[a_{ij}]_{mn}$ to indicate that A is a matrix of order m $\times$ n.

2. We shall consider only those matrices whose elements are real numbers or functions taking real values.

Definition 1.0.4

Identity matrix-A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an identity matrix. In other words, the square matrix A = $[a_{ij}]_{n\times n}$ is an identity matrix, if

$$a_{ij} = 1 \text{ if } i = j$$

$$0 \text{ if } i \neq j$$

We denote the identity matrix of order n by In. When order is clear from the context, we simply write it as I. For example

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1.1 Operation on Matrices

we shall introduce certain operations on matrices, namely, addition of matrices, multiplication of a matrix by a scalar, difference and multiplication of matrices.

Definition 1.1.1

Addition of matrices - The operation of addition of matrices are defined only if the matrices which are being added are of the same order. If A and B are two (m × n) matrices with elements a_{ik} and b_{ik} respectively, then their sum A + B is the (m × n) matrix C whose elements c_{ik} are given by $c_{ik} = a_{ik} + b_{ik}$

$$\mathrm{A} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix} \text{ and } \quad \mathrm{B} = \begin{pmatrix} 3 & 4 & 1 \\ 0 & 2 & -1 \end{pmatrix}$$

then $\qquad \mathrm{A+B} = \begin{pmatrix} 4 & 6 & 4 \\ -1 & 3 & 1 \end{pmatrix}$

Two matrices are said to be conformable to addition if they are of the same order.

Definition 1.1.2

Multiplication of a matrix - Two matrices A and B can be multiplied together to form their product BA (in that order) only when the number of columns of B is equal to the number of rows of A. A and B are then said to be conformable to the product BA. We shall see shortly, however, that A and B need not be conformable to the product AB, and that, even when they are, the product AB does not necessarily equal the product BA. That is, matrix multiplication is in general non-commutative. Suppose now A is a matrix of order (m × p) with elements a_{ik} and B is a matrix of order (p × n) with elements b_{ik}. Then A and B are conformable to the product AB which is a matrix C, say, of order (m × n) with elements c_{ik} defined by,

$$c_{ik} = \sum_{s=1}^{p} a_{is} b_{sk}$$

For example, if A and B are the matrices

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{pmatrix}$$

Then, $$AB = \begin{pmatrix} 10 & 13 \\ 11 & 14 \end{pmatrix}$$

Now the product BA is also defined in this case since the number of columns of B is equal to the number of rows of A. However, it is readily found that

$$\begin{pmatrix} 7 & 3 & 8 \\ 11 & 4 & 9 \\ 12 & 5 & 13 \end{pmatrix}$$

Properties of matrix addition

1. Commutative Law If A = $[a_{ij}]$, B = $[b_{ij}]$ are matrices of the same

order, say m × n, then A + B = B + A.

2. Associative Law For any three matrices A = $[a_{ij}]$, B = $[b_{ij}]$, C = $[c_{ij}]$ of the same order, say m × n, (A + B) + C = A + (B + C).

3. Existence of additive identity Let A = $[a_{ij}]$ be an m × n matrix and O be an m × n zero matrix, then A + O = O + A = A. In other words, O is the additive identity for matrix addition.

4. The existence of additive inverse Let A = be any matrix, then we have another matrix as A = $[a_{ij}]_{m\times n}$ such that A+(A) = (A)+A = O. So A is the additive inverse of A or negative of A.

Properties of multiplication of matrices

1. The associative law For any three matrices A, B and C. We have (AB)C = A(BC), whenever both sides of the equality are defined.

2. The distributive law For three matrices A, B and C.

(i) A(B+C) = $AB + AC$ (ii) (A + B)C = $AC + BC$, whenever both sides of equality are defined.

3. The existence of multiplicative identity For every square matrix A, there exist an identity matrix of same order such that IA = AI = A.

Properties of scalar multiplication of a matrix

1. k(A + B) = $kA + kB$

2. (k + l)A = $kA + lA$

Transpose of a matrix

If A = $[a_{ij}]$ be an m × n matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. Transpose of the matrix A is denoted by A' or (A^T). In other words, if $A = [a_{ij}]_{m\times n}$, then
$A' = [a_{ij}]_{n\times m}$. For example,

$$\text{If A} = \begin{pmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & -1/3 \end{pmatrix}_{3\times 2} \quad \text{Then } A' = \begin{pmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & -1/3 \end{pmatrix}_{2\times 3}$$

Definition 1.1.3 Rank of a matrix

The rank of a matrix is the maximum number of independent rows (or, the maximum number of independent columns). A square matrix An×n is non-singular only if its rank is equal to n. For an m × n matrix,

- If m is less than n, then the maximum rank of the matrix is m.
- If m is greater than n, then the maximum rank of the matrix is n.

The rank of a matrix would be zero only if the matrix had no elements. If a matrix had even one element, its minimum rank would be one.

Definition 1.1.4

The number of non-zero rows in the row reduced form of a matrix

A is called the rank of A, denoted rank(A).

Method of find the rank of a matrix

Reduce the given matrix A into the row Echolon form or row reduced echolon form by performing elementary operations. Then rank of A is equal to the number of non zero rows of the row echolon form is equal to the number of non-zero rows of row reduced form.

Solution of a system of linear equations

Consider the following system

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots \; a_{2n}x_n = b_2$$

$$\begin{array}{ccc} . & . & . \\ . & . & . \\ . & . & . \end{array}$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m$$

The matrix form the system is $AX =$ b where

$$\begin{pmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots a_{mn} \end{pmatrix}, \quad \text{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Here number of variables = n.

1. Consistency of the system:

If Rank of (A, b) = rank of A = r ,Then the system has solution. In this case we say that the system is consistent.

Case-1: If n = r, then the system has a unique solution.

Case-2: If r < n then system has infinite number of solutions.

2. In-consistency of the system:

If Rank of (A, b) $\neq$ rank of A. Then the system has no solution. In this case we say that the system is in-consistent.

CHAPTER 2

FUNDAMENTAL CONCEPTS

The fundamental concepts of fuzzy matrices; A fuzzy algebra is a mathematical system (F,+,.) with two binary operations +,. defined on a set **F** satisfying the following properties:

(P1) Idempotence $\quad a+a = a$

$a.a = a$

(P2) Commutativity $\quad a+b = b+a$

$a.b = b.a$

(P3) Associativity $\quad a+(b+c) = (a+b)+c$

$a.(b.c) = (a.b).c$

(P4) Distributivity $\quad a.(b+c) = (a.b)+(a.c)$

$a+(b.c) = (a+b).(a+c)$

(P5) Universal bounds $\quad a+0 = a \; ; \; a+1 = 1$

$a.0 = 0 \; ; \; a.1 = a$

A Boolean algebra with complementarity, that is, for each element a, there exists an element $\overline{a}$, called the complement of a such that $a+\overline{a} = 1 \; ; \; a.\overline{a} = 0$.

Thus we have introduced the concepts of fuzzy algebra as a generalization of a Boolean algebra. By a fuzzy matrix, we mean a matrix over a fuzzy algebra. A Boolean matrix is a special case of a fuzzy matrix with entries from the set {0,1}.

2.1 Fuzzy vector

A fuzzy vector is an n-tuple of elements of the fuzzy algebra **F** = [0, 1].

Definition 2.1.1

Let V_n denote the set of all n-tuples $(x_1, x_2, ..., x_n)$ over **F**. An element of V_n is called a fuzzy vector of dimension n. The operations (+,.) are defined on V_n as follows:

For $x = (x_1 , x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ in V_n,

$x{+}y = (x_1{+}y_1 , x_2{+}y_2 , ..., x_n{+}y_n)$

ax = (ax_1, ax_2..., ax_n), for a ∈ **F**.

The system V_n together with these operations of component wise addition and fuzzy multiplication is called fuzzy vector (or) a vector space over **F**, as the scalars are restricted to **F**. This will be sufficient for our purpose. Let O denote the zero vector (0, 0,...,0).

Definition 2.1.2

Let $V^n = \{x^T : x \in V_n\}$ where x^T is the transpose of the vector x. For u, v ∈ V^n, a ∈ **F**, define av = $(a^T V^T)$; u+v = $(u^T + V^T)^T$. Then V^n is a fuzzy vector space. If we write an element of V_n as a 1 × n matrix, it is called a row vector. The elements of V^n are column vectors.

For any result about V_n there exist a corresponding result about V^n. Thus V^n is isomorphic to V_n as a fuzzy algebra.

Definition 2.1.3

A subspace of V_n is a subset W of V_n such that $O \in W$ and for x, $y \in$ W, $x+y \in$ W .

Example

The set B_n of all n-tuples $(a_1,\ a_2,\ ...,a_n)$ over the two elements Boolean algebra B = $\{0,\ 1\}$ is a Boolean vectorspace of dimension n. B_n is a subspace of V_n.

Example

W =$\{$(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1), (1,0,1) (1,1,1)$\}$ is a subspace of V_3.

Definition 2.1.4

A linear combination of elements of set S is a finite sum $\sum a_i x_i$ where $x_i \in$ S and $a_i \in \mathbf{F}$. The set of all linear combinations of elements of S is called the span of S, denoted as $<S>$.

Definition 2.1.5

For subsets S, W of V_n if $<S>$ = W, then S is called a spanning set (or) set of generators for W. If W is a subspace of V_n, then $<W>$ = W

Definition 2.1.6

A basis for a subspace W of V_n is a minimal spanning set for W.

Example

W = $\{(x,\ x) :\ x \in \mathbf{F}\}$ is a subspace of $\mathbf{F} \times \mathbf{F}$. The singleton set

S = {(1,1)} is the minimal spanning set for W, since every element $(x, x) = x(1,1)$ with $x \in \mathbf{F}$ is a linear combination of (1,1) in S, S is a basis for W.

Definition 2.1.7

A set S of vectors over a fuzzy algebra **F** is independent if and only if each element of S is not a linear combination of other elements of S, that is, no element v ∈ S is a linear combination of $S \setminus \{v\}$.(where $\setminus$ denotes the set theoretic difference)

Definition 2.1.8

A set S of vectors over **F** is dependent if it is not a linearly independent set.

Proposition 2.1.1

1. The set consisting of zero vector along is linearly dependent.
2. If X ⊂ Y and if X is linearly dependent then so is Y.
3. If X ⊂ Y and if Y is linearly dependent then so is X.

Example

The set of vectors {(0.5, 1), (1, 0.6), (0.7, 0.9)} is a dependent set, since (0.7, 0.9) = 0.9(0.5, 1) + 0.7(1, 0.6).

Example

Let B = $\{b_1, b_2\}$ be a subset of V_3 , where b_1 = (b, 0.3, 0.7) with b ≥ 0.8 and b_2 = (0.4, 0.6, 0.5). We claim that the span of B, that is, $<B>$ contains the set S = $\{a_1, a_2, a_3, a_4\}$ be a subset of V_3 given by a_1 = (0.6, 0.3, 0.6), a_2 = (0.4, 0.5, 0.5),

a_3 = (0.5, 0.6, 0.5) and a_4 = (0.8, 0.6, 0.7).

a_1 = (0.6, 0.3, 0.6) = 0.6(b, 0.3, 0.7) + 0.3(0.4, 0.6, 0.5),

that is, a_1 =(0.6)b_1 + (0.3)b_2

a_2 = (0.4, 0.5, 0.5) = 0.4(b, 0.3, 0.7) + 0.5(0.4, 0.6, 0.5),

that is, a_2 = (0.4)b_1 + (0.5)b_2

a_3 = (0.5, 0.6, 0.5) = 0.5(b, 0.3, 0.7) + 0.6(0.4, 0.6, 0.5),

that is, a_3 = (0.6)b_1 + (0.3)b_2

a_4 = (0.8, 0.6, 0.7) = 0.8(b, 0.3, 0.7) + 0.6(0.4, 0.6, 0.5),

that is, a_4 = (0.8)b_1 + (0.6)b_2

Thus each a_i(i = 1 to 4) is a linear combination of b_1 and b_1.

Hence, $<B> \supseteq$ S

Definition 2.1.9

Let A be a subset of V_n and $\mathcal{A}$ be the set of all subsets of V_n whose span contains A.The Schein rank of the set A, denoted as $\rho_s(A)$ is the minimum of the cardinality of the set in $\mathcal{A}$. That is, $\rho_s(A)$ = min{ |C| : C $\in \mathcal{A}$}

2.2 Standard Basis

The boolean algebra {0,1} any finitely generated subspace of V_n has a unique basis. We define a standard basis and prove that any finitely generated subspace of V_n over the fuzzy algebra **F** = [0,1] has a unique standard basis.

Definition 2.2.1

A basis C over the fuzzy algebra **F** is a standard basis if and only if whenever $c_i = \sum a_{ij}c_j$ for c_i, $c_j \in$ C and $a_{ij} \in$ **F** then $a_{ii}c_i = c_i$.

Example

The basis {(0.5,1,0.5), (0,1,0.5), (0,0.5,1)} is not a standard basis for (0.5,1,0.5) = (0.5)(0.5,1,0.5)+1(0,1,0.5)+(0.3)(0,0.5,1)
but (0.5,1,0.5) $\neq$ (0.5)(0.5,1,0.5). However the basis {(0.5,0.5,0.5), (0,1,0.5), (0,0.5,1)} is a standard basis for the same subspace.

Theorem 2.2.1

Over the fuzzy algebra **F** =[0,1], any two bases for a finitely generated subspace have the same cardinality. Any finitely generated subspace over **F** has a unique standard basis.

Proof

We first show that for any finite basis C, there exists a standard basis having the same cardinality. Let S be the set of all fuzzy vectors each of whose entries equals some entry of a vector C. Then S is a finite set.

Suppose C is not a standard basis, then $c_i = \sum a_{ij}c_j$ for some $c_i \in$ C and $a_{ij} \in$ [0,1] with $c_i \neq a_{ii}c_i$, that is $c_i \neq \min\{a_{ii}, c_i\}$, therefore $a_{ii}c_i < c_i$.

Let C_1 be the set obtained from C by replacing c_i by $a_{ii}c_i$. Then |C| = $|C_1|$ and $<C> = <C_1>$ and it can be verified that C_1 is independent set and all the vectors of C_1 are all in S.

Let us define an order relation on finite subsets of S as follows; Let the weight of a finite subset be the sum of all entries of members of the subset, regarded as real numbers.

We define that $F_1 \leq F_2$, for finite subsets F_1 and F_2 of S if weight of $F_1 \leq$ weight of F_2. Clearly this is a partial order relation on finite subsets of S, Since $a_{ii}c_i < c_i$, $C_1 \leq C$ and $|C_1| = |\text{C}|$ is finite.

If C_1 is a standard basis, then C_1 is the required standard basis with the same cardinality as C. If not, then repeat the process of replacing C_1 by a basis C_2 and proceed.

Therefore after replacing bases of the form C by bases of the form C_1 , the process must terminate after a finite number of steps.

This can happen only if we have obtained a standard basis with the same cardinality as C. This proves that for any finite basis, there exist a standard basis with the same cardinality.

Next we show that there is only one standard basis. Let C be a standard basis.

Suppose for $c_k \in$ C, we have $c_k = \sum_j a_j$, where $a_j \in <C>$, then a_j can be expressed as a linear combination of basis vector in C, that is $a_j = \sum b_{ji}c_i$ with $b_{ji} \in \mathbf{F}$ and $c_i \in$ C. Therefore,

$$\begin{aligned} c_k &= \sum_j a_j \\ &= \sum_{i,j} b_{ji}c_i \\ &= \sum_i (\sum_j b_{ji}c_i) \end{aligned}$$

Since C is a standard basis. We have, $(\sum_j b_{jk})c_k = c_k$;

by using the fact that the fuzzy sum is the maximum, $b_{jk}c_k = c_k$ for some j and from $a_j = \sum_{ij} b_{ji} c_i$ we get $a_j \geq c_k$.

From $c_k = \sum a_j$, we have $c_k \geq a_j$ .Therefore, it follows that $c_k = a_j$ for some j.

Thus we conclude that whenever $c_k = \sum_j a_j$, then c_k equals some summand a_j.

Next to prove the uniqueness, if possible, let us assume that C and $C^{'}$ are two standard basis with $\mid C \mid = \mid C^{'} \mid$. Since $C^{'}$ is a basis , each element of C can be expressed as a linear combination of elements of $C^{'}$. By proceeding argument, each element a_i of C must be a multiple of some element b_j of $C^{'}$. Since fuzzy multiplication is minimum , it follows that $a_i \leq b_j$.

In the same manner, by using that each element of $C^{'}$ is a multiple of some element of C, and |C| = $|C^{'}|$ it follows that $a_i = b_j$ and therefore C = $C^{'}$.

This proves the uniqueness of the standard basis. Hence the theorem.

Definition 2.2.2

The dimension of the finitely generated subspace S of a vector space V_n over the fuzzy algebra **F** denoted by dim(S) is defined to be the cardinality of the standard basis of S.

Example

The set {(1,0,0), (0,1,0), (0,0,1)} forms the standard basis for V_3. dim(V_3) = 3

Example

Let S = {(0.5, 0.5, 0.5), (0, 1, 0.5), (0, 0.5, 1)} be a standard basis of V_3. The subspace of V_3 generated by S is W = $<S>$ where W = $\{(x, y, z) : 0 \leq x \leq 0.5 \leq y, z \leq 1\} \cup \{(x, y, z) : 0 \leq x \leq y = z \leq 0.5\}$. Here dim(W) = |S| = 3

Remark 2.2.3

For vector spaces over a field, dim(S) = dim(W) if and only if S = W. This fails for vector spaces over **F**. For spaces W = $\{(x, y, z) : 0 \leq x \leq 0.5 \leq y, z \leq 1\} \cup \{(x, y, z) : 0 \leq x \leq y = z \leq 0.5\}$ and the set {(1,0,0), (0,1,0), (0,0,1)} forms the standard basis for V_3, dim(W) = dim(V_3) = 3, but W $\neq V_3$.

Example

Let W = $\{(x, y, z) : 0 \leq x \leq 0.5 \leq y, z \leq 1\} \cup \{(x, y, z) : 0 \leq x \leq y = z \leq 0.5\}$ be the space. The vector (0.4, 0.5, 0.6) $\in$ W can be expressed as a linear combination of the standard basis vectors (0.5, 0.5, 0.5), (0, 1, 0.5), (0, 0.5, 1) as

(0.4, 0.5, 0.6)=α(0.5, 0.5, 0.5) + β(0, 1, 0.5) + γ(0, 0.5, 1) for α = 0.4, $\beta \leq 0.5$ and $\gamma = 0.6 \in$ **F**. Thus expression for the vector space x is not unique. However, the unique standard basis of a finitely generated subspace over **F** admits a unique representation.

Theorem 2.2.2

Let S be a finitely generated subspace of V_n and $\{c_1, c_2, ..., c_n\}$ be the standard basis for S. Then any vector $x \in$ S can be expressed uniquely as a linear combination of the standard basis vectors.

Proof

Since $\{c_1, c_2,..., c_n\}$ is the standard subspace for S, x is a linear combination of the standard basis vectors. Let

$$x = \sum_{j=1}^{n} \beta_j c_j \text{ where } \beta_j \in \mathbf{F}$$

In this expression, the coefficients β_j's are not unique. If we write this in the matrix form as $x = (\beta_1, \beta_2, \cdots \beta_n)\cdot$C, where C is the matrix whose rows are the basis vectors, then $x =$ p$\cdot$C has a solution $(\beta_1, \beta_2, \cdots, \beta_n)$. It follows that this equation has a unique maximal solution $(p_1, p_2, \cdots, p_n)$.Then

$x = \sum_{j=1}^{n} p_j c_j$ with $p_j \in \mathbf{F}$ is the unique representation of the vector x.

Theorem 2.2.3

Let S be a vector space over **F** and be the linear span of the vectors $x_1, x_2..., x_m$. If some x_i is a linear combination of x_1 , $x_2..., x_{i-1}$, $x_{i+1}...,x_m$, then the vectors $x_1, x_2 ,....,x_{i-1}, x_{i+1}..., x_m$ also spans S.

Proof

Let W = $\{x_1, x_2,..., x_m\}$ such that S = $<W>$, since x_i is a linear combination of $x_1, x_2,...,x_{i-1}, x_{i+1},....,x_m$, there exist β_j's for j = 1 to m and j $\neq$ i $\in \mathbf{F}$, such that

$$x_i = \sum_{j=1:j\neq i}^{m} \beta_j x_j$$

Since S = $<W>$, any vector $y \in$ S can be expressed as

$$y = \alpha_1 x_1 + \alpha_2 x_2 + ..., \alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1} + ... + \alpha_m x_m$$

$= \sum_{j=1:j \neq i}^{m} \alpha_j x_j + \alpha_i x_i$

$= \sum_{j=1:j \neq i}^{m} \alpha_j x_j + \alpha_i \left(\sum_{j=1:j \neq i}^{m} \beta_j x_j\right)$

$= \sum_{j=1:j \neq i}^{m} \gamma_j x_j$

where, $\gamma_j = \alpha_j + \alpha_i \beta_j$ for j = 1 to m and j $\neq$ i are elements in **F**.

Since y is arbitrary vector in S, we have S = $<W/\{x_i\}>$.
Thus the vectors x_1 , x_2..., x_{i-1}, x_{i+1}..., x_m spans S. Hence the theorem.

Theorem 2.2.4

Let S be a vector space over **F** of dimension n and let x_1 , x_2..., x_m be linearly independent vectors in S.Then there exists a basis for S containing x_1 , x_2..., x_m.

Proof

Let y_1 , y_2..., y_n be the unique standard basis for S. Then the set W = $\{x_1$, x_2..., x_m , y_1 , y_2..., $y_n\}$ is a linearly dependent subset of S.Therefore y_i for some i is a linear combination of the vectors in W/$\{y_i\}$. Since S is a subspace, $<S>$ = S, and W = $<S>$ .Thus S is a linear span of W.
By the above theorem,W/$\{y_i\}$ also spans S. If the set is linearly independent set, then we have a basis for S as required. Otherwise we continue the process until we get a basis containing x_1 , x_2..., x_m .

Theorem 2.2.5

Any set of (n + 1) vectors in V_n is linearly dependent.

Proof

If the set of (n + 1) vectors linearly independent,then by the above theorem, we can find a basis for V_n containing the set. This is a contradiction, since every basis for V_n must contain precisely n vectors.

CHAPTER 3

FUZZY MATRICES

By a fuzzy matrix, we mean a matrix over a fuzzy algebra. Here we confine with matrices over the fuzzy algebra $\mathbf{F} = [0,1]$ under the max-min operations and with the usual ordering on real numbers. Fuzzy matrices have quite different properties from matrices over a field, due to the fact that addition in a fuzzy algebra does not form a group. Here we will develop that every fuzzy linear transformation on V_n can be represented by a unique fuzzy matrix. One of the most important ways to study a fuzzy matrix is to consider its row space, that is, the subspace of V_n spanned by its rows.

Definition 3.0.4

Let $\mathbf{F}_{mn}$ denote the set of all $m \times n$ matrices over $\mathbf{F}$. If m = n, in short, we write $\mathbf{F}_n$. Elements of $\mathbf{F}_{mn}$ are called as membership value matrices, binary fuzzy relation matrices (or) in short, fuzzy matrices. Boolean algebra $\{0, 1\}$ are special types of fuzzy matrices.

Definition 3.0.5

Let A = $(a_{ij}) \in \mathbf{F}_{mn}$. Then the element a_{ij} is called the (i, j) entry of A. Let A_{i*} (or A_{*j}) denote the i^{th} row (column) of A . The row space R(A) of A is the subspace of V_n generated by the rows $\{A_{i*}\}$ of A. The column space C(A) of A is the subspace of V_m generated by the columns $\{A_{*j}\}$ of A. The null space (or) kernel of A is the set

$\{x : xA = 0\}$. Note that row (column) vector is just an element of V_n (V^n).

Definition 3.0.6

The $n \times m$ zero matrix **O** is the matrix all of whose entries are zero. The $n \times n$ identity matrix **I** is the matrix (δ_{ij}) such that

$$\delta_{ij} = \begin{cases} 1 & \text{if i = j} \\ 0 & \text{if i} \neq \text{j} \end{cases}$$

The $n \times m$ universal matrix J is the matrix all of whose entries are 1.

3.1 Addition of Matrices

Definition 3.1.1

Let A = $(a_{ij}) \in \mathbf{F}_{mn}$ and B = $(b_{ij}) \in \mathbf{F}_{mn}$. Then the matrix A+B = (sup$\{a_{ij}, b_{ij}\} \in \mathbf{F}_{mn}$ is called the sum of A and B.

Example

If $\quad A = \begin{pmatrix} 0.5 & 0 & 1 \\ 0.8 & 0.2 & 0.3 \\ 0 & 0.6 & 0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 & 0.4 & 0.6 \\ 0.5 & 0.3 & 0.3 \\ 0.7 & 0.8 & 0 \end{pmatrix}$

Then, $\quad A+B = \begin{pmatrix} 0.5 & 0.4 & 1 \\ 0.8 & 0.3 & 0.3 \\ 0.7 & 0.8 & 0.1 \end{pmatrix}$

Theorem 3.1.1

The set $\mathbf{F}_{mn}$ is a fuzzy algebra under the component wise addition and multiplication operation ($+$, $\odot$) defined as follows ;

For A = (a_{ij}) and B = (b_{ij}) in $\mathbf{F}_{mn}$,

A + B = $(\sup\{a_{ij}, b_{ij}\})$

A $\odot$ B = $(\inf\{a_{ij}, b_{ij}\})$

Proof

The properties (P1) to (P4) of fuzzy algebra are automatically held. A+0 = A and A $\odot$ J = A, for all A$\in$ $\mathbf{F}_{mn}$, Hence the zero matrix **O** is the additive identity and the universal matrix **J** is the multiplicative identity.

Thus identity element relative to the operations + and $\odot$ exist. Further A+J = J and A.0 = 0. Therefore (P6) holds.

For A = (a_{ij}), B = (b_{ij}) and C = $(c_{ij}) \in \mathbf{F}_{mn}$, Since (a_{ij}) , (b_{ij}) and (c_{ij}) are all real numbers in [0, 1] they are comparable.

If $a_{ij} \leq b_{ij}$ (or) c_{ij} then in both cases,

$\inf\{a_{ij}$, $\sup\{b_{ij}, c_{ij}\}\} = a_{ij}$ and $\sup\{\inf\{a_{ij}$, $b_{ij}\}$, $\inf\{a_{ij}$, $c_{ij}\} = a_{ij}$.

Therefore ij^{th} entry of A $\odot$(B+C) = ij^{th} entry of (A $\odot$ B) + (A $\odot$ C).

If $a_{ij} \geq b_{ij}$ and c_{ij} then we have two cases ,

$a_{ij} \geq b_{ij} \geq c_{ij}$ (or) $a_{ij} \geq c_{ij} \geq b_{ij}$ then,

inf { a_{ij} , $\sup\{b_{ij}$,$c_{ij}\}\} = b_{ij} = \sup\{\inf\{a_{ij}$,$b_{ij}\}$, $\inf\{a_{ij}$, $c_{ij}\}\}$ (or) $\inf\{a_{ij}$, $\sup\{b_{ij}$, $c_{ij}\}\} = c_{ij} = \sup\{\inf\{a_{ij}, b_{ij}\}, \inf\{a_{ij}, c_{ij}\}\}$. Therefore

ij^{th} entry of A $\odot$(B+C) = ij^{th} entry of (A $\odot$ B) + (A $\odot$ C).

Hence A $\odot$(B+C) = (A $\odot$ B) + (A $\odot$ C).

Similarly, if $a_{ij} \geq b_{ij}$ (or) c_{ij}, then both cases $\sup\{a_{ij}, \inf\{b_{ij}, c_{ij}\}\} = a_{ij}$ and $\inf\{\sup\{a_{ij}, b_{ij}\}, \sup\{a_{ij}, c_{ij}\}\} = a_{ij}$. Therefore ij^{th} entry of A + (B ⊙ C) = ij^{th} entry of (A + B) ⊙ (A + C).

If $a_{ij} \leq b_{ij}$ and c_{ij}, then we have two cases $a_{ij} \leq b_{ij} \leq c_{ij}$ or $a_{ij} \leq c_{ij} \leq b_{ij}$ then,

$\sup\{a_{ij}, \inf\{b_{ij}, c_{ij}\}\} = b_{ij} = \inf\{\sup\{a_{ij}, b_{ij}\}, \sup\{a_{ij}, c_{ij}\}\}$ (or)

$\sup\{a_{ij}, \inf\{b_{ij}, c_{ij}\}\} = c_{ij} = \inf\{\sup\{a_{ij}, b_{ij}\}, \sup\{a_{ij}, c_{ij}\}\}$. Therefore ij^{th} entry of A + (B ⊙ C) = ij^{th} entry of (A + B) ⊙ (A + C).

Hence, A + (B ⊙ C) = (A + B) ⊙ (A + C) can be proved.

Thus the property (P5) of distributivity holds. Thus $\mathbf{F}_{mn}$ is a fuzzy algebra with the operation +, ⊙ .

Definition 3.1.2

Let $A = a_{ij} \in \mathbf{F}_{mn}$ and c $\in \mathbf{F}$ then the fuzzy multiplication, that is, scalar multiplication with scalar restriction to **F** is defined as

$$\text{c}A = (\inf\{\text{c}, a_{ij}\}) \in \mathbf{F}_{mn} \tag{3.1}$$

For the universal matrix **J**, by the definition ,

c**J** = (inf{c, 1}) is the constant matrix all of whose entries are c. Further under componentwise multiplication,

$$\text{c}\mathbf{J} \odot A = (\inf\{\text{c}, a_{ij}\}) = \text{c}A \tag{3.2}$$

Proposition 3.1.2

The set $\mathbf{F}_{mn}$ is a fuzzy vector space under the operations defined as

$A + B = (\sup\{a_{ij}, b_{ij}\})$ and

$\text{c}A = (\inf\{\text{c}, a_{ij}\})$ for $A = (a_{ij})$, $B = (b_{ij}) \in \mathbf{F}_{mn}$ and c $\in \mathbf{F}$.

Proof

For A, B, $C \in \mathbf{F}_{mn}$,

$$A + B = B + A \in \mathbf{F}_{mn} \qquad \text{(commutativity)}$$

$$A + (B + C) = (A + B) + C \quad \text{(Associativity)}$$

For all $A \in \mathbf{F}_{mn}$, there exist an element O $\in \mathbf{F}_{mn}$ such that $A+$ O $= A$.

For c $\in \mathbf{F}$,

$$\begin{aligned} \mathrm{c}(A{+}B) &= \mathrm{cJ} \odot (A{+}B) \\ &= (\mathrm{cJ} \odot A){+}(\mathrm{cJ} \odot B) \quad \text{(By Theorem 3.1.1)} \\ &= \mathrm{c}A{+}\mathrm{c}B \quad \text{(By (3.2))} \end{aligned}$$

For c_1, $c_2 \in \mathbf{F}$,

$$\begin{aligned} (c_1 + c_2)A &= (\ c_1 + c_2)\mathrm{J} \odot A \quad \text{(by(3.2))} \\ &= (c_1\mathrm{J} + c_2\mathrm{J}) \odot A \\ &= (c_1\mathrm{J}) \odot \mathrm{A} + (c_2\mathrm{J}) \odot A \\ &= c_1\mathrm{A} + c_2\mathrm{A} \end{aligned}$$

Hence $\mathbf{F}_{mn}$ is a vector space over $\mathbf{F}$. In particular for m = 1, it reduces to definition.

3.2 Max-Min Composition of Matrices

Definition 3.2.1

For $A = (a_{ij}\ ) \in \mathbf{F}_{mp}$ and $B = (b_{ij}) \in \mathbf{F}_{pn}$, the max-min product

$AB = (sup_k\{\inf\{a_{ik},\, b_{kj}\}\}) \in \mathbf{F}_{mn}$

The product AB is defined if and only if the number of columns of A is same as the number of rows of B ; A and B are said to be conformable for multiplication.

Example

$$A = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 1 \end{pmatrix} , \quad B = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.3 \end{pmatrix}$$

Then,

$$AB = \begin{pmatrix} (0.8\ 0.1)\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} & (0.8\ 0.1)\begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} \\ (0.2\ \ 1)\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} & (0.2\ \ 1)\begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} sup\{inf\{0.8, 0.6\}, inf\{0.1, 0.4\} & sup\{inf\{0.8, 0.5\}, inf\{0.1, 0.3\} \\ sup\{inf\{0.2, 0.6\}, inf\{1, 0.4\} & sup\{inf\{0.2, 0.5\}, inf\{1, 0.3\} \end{pmatrix}$$

$$= \begin{pmatrix} sup\{0.6, 0.1\} & sup\{0.5, 0.1\} \\ sup\{0.2, 0.4\} & sup\{0.2, 0.3\} \end{pmatrix}$$

$$= \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.3 \end{pmatrix}$$

Remark 3.2.2

For elements $a_1,\, a_2 \in \mathbf{F}$, the sum $\sum_{i=1}^{2} a_i = a_1 + a_2 \ = \sup\{a_1,\, a_2\}$ and product ab = inf{a, b}, hence the matrix product AB can be expressed as $AB = (\sum_{k=1}^{p} a_{ik} b_{kj})$.

In vector notation AB = $(A_{i*} B_{*j})$ where A_{i*} is the i^{th} row of A and B_{*j} is the j^{th} column of B.

Proposition 3.2.1

For any three matrices A, B, C over **F** of order $m \times n$, $n \times p$, $p \times q$ respectively, (AB)C = A(BC)

Proof

With the given type of matrices (AB)C and A(BC) are both defined and are of type $m \times q$. Let A = (a_{ij}) , B = (b_{jk}), C = (c_{kl}) such that the ranges of the suffixes i, j, k, l are 1 to m, 1 to n, 1 to p and 1 to q respectively. Now (ik^{th}) element of the product $AB = \sum_{j=1}^{n} a_{ij}b_{jk}$. The (il^{th}) element in the product (AB)C is the sum of products of the corresponding elements in the (i^{th}) row of AB and (l^{th}) column of C with k common. Thus

$$(il^{th}) \text{ element of } (AB)C = \sum_{k=1}^{p} \{(\sum_{j=1}^{n} a_{ij}b_{jk})c_{kl}\}$$
$$= \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij}\ b_{jk}\ c_{kl} \qquad (3.3)$$

Now (jl^{th}) element of the product BC = $\sum_{k=1}^{p} b_{jk}\ c_{kl}$. Again the il^{th} element in the product $A(BC)$ is the sum of products of the corresponding elements in the i^{th} row of A and l^{th} column of BC.

$$(il^{th}) \text{ element of } A(BC) = \sum_{j=1}^{n} a_{ij}(\sum_{k=1}^{p} b_{jk}c_{kl})$$
$$= \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ij}\ b_{jk}\ c_{kl} \qquad (3.4)$$

From (3.3) and (3.4), we see that $(AB)C = A(BC)$.

Proposition 3.2.2

For any three matrices A, B, C over **F** of order $m \times n$, $n \times p$, $p \times q$ respectively, $A(B + C) = AB + AC$

Proof

Let $A = (a_{ij})$, $B = (b_{jk})$, $C = (c_{Jk})$ such that the ranges of the suffixes i, j, k are 1 to m, 1 to n, 1 to p respectively. Now

(jk^{th}) element of B + C = $b_{jk} + c_{jk} = \sup\{b_{jk}, c_{jk}\}$

(ik^{th}) element in the product of A and (B + C), that is, of A(B + C) is the sum of the products of the corresponding elements in i^{th} row of A and k^{th} column of (B + C).

$= \sum_{j=1}^{n} a_{ij}(b_{jk} + c_{jk})$

$= \sum_{j=1}^{n} a_{ij}\ b_{jk} + \sum_{j=1}^{n} a_{ij}\ c_{jk}$

$= (ik^{th})$ element of AB + (ik^{th}) element of AC

$= (ik^{th})$ element of $(AB + AC)$

Thus $A(B + C) = AB + AC$

Definition 3.2.3

We use the notation A^2 to designate the product AA, $A^3 = AA^2 = A^2A$; and in general $A^k = A.A^{k-1} = A^{k-1}.A$, for any positive integer k. The matrix A^k is called the k^{th} power of A. The notations $a_{ij}^{(k)}$ and $A_{ij}^{(k)}$ denote the $(ij)^{th}$ entry of A^k and $(ij)^{th}$ block of A^k. The notation $(A_{ij})^k$ means the k^{th} power of the $(ij)^{th}$ block of A.

Remark 3.2.4

We note that the matrix multiplication is not in general commutative, that is, AB $\neq$ BA,Further AB = 0, need not imply A = 0 (or) B = 0 as in the case of real matrices.

Example

Let A $= \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 1 \end{pmatrix}$, B $= \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.3 \end{pmatrix}$, C $= \begin{pmatrix} 0.6 & 0.2 \\ 0.7 & 0.3 \end{pmatrix}$

$$AB = \begin{pmatrix} (0.8\ 0.1)\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} & (0.8\ 0.1)\begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} \\ (0.2\ 1)\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} & (0.2\ 1)\begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} sup\{0.6, 0.1\} & sup\{0.5, 0.1\} \\ sup\{0.2, 0.4\} & sup\{0.2, 0.3\} \end{pmatrix}$$

$$= \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.3 \end{pmatrix}$$

$$BA = \begin{pmatrix} (0.6\ 0.5)\begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} & (0.6\ 0.5)\begin{pmatrix} 0.1 \\ 1 \end{pmatrix} \\ (0.4\ 0.3)\begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} & (0.4\ 0.3)\begin{pmatrix} 0.1 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} sup\{0.6, 0.2\} & sup\{0.1, 0.5\} \\ sup\{0.4, 0.2\} & sup\{0.1, 0.3\} \end{pmatrix}$$

$$= \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.3 \end{pmatrix}$$

Therefore AB = BA

$$BC = \begin{pmatrix} (0.6\ 0.5)\begin{pmatrix}0.6\\0.7\end{pmatrix} & (0.6\ 0.5)\begin{pmatrix}0.2\\0.3\end{pmatrix} \\ (0.4\ 0.3)\begin{pmatrix}0.6\\0.7\end{pmatrix} & (0.4\ 0.3)\begin{pmatrix}0.2\\0.3\end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} sup\{0.6, 0.5\} & sup\{0.2, 0.3\} \\ sup\{0.4, 0.3\} & sup\{0.2, 0.3\} \end{pmatrix}$$

$$= \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.3 \end{pmatrix}$$

$$CB = \begin{pmatrix} (0.6\ 0.2)\begin{pmatrix}0.6\\0.4\end{pmatrix} & (0.6\ 0.2)\begin{pmatrix}0.5\\0.3\end{pmatrix} \\ (0.7\ 0.3)\begin{pmatrix}0.6\\0.4\end{pmatrix} & (0.7\ 0.3)\begin{pmatrix}0.5\\0.3\end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} sup\{0.6, 0.2\} & sup\{0.5, 0.3\} \\ sup\{0.6, 0.3\} & sup\{0.5, 0.3\} \end{pmatrix}$$

$$= \begin{pmatrix} 0.6 & 0.5 \\ 0.6 & 0.5 \end{pmatrix}$$

Therefore BC $\neq$ CB

Example

Let $A = \begin{pmatrix} 0.5 & 0 \\ 0.3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0.5 \end{pmatrix}$

But, AB = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

AB = 0

BA = $\begin{pmatrix} 0 & 0 \\ 0.5 & 0 \end{pmatrix}$

BA $\neq$ 0

Definition 3.2.5

A square matrix is called a permutation matrix if every row and every column contains exactly one 1 and all the other entries are 0. Let P_n denotes the set of all $n \times n$ permutation matrices.

Definition 3.2.6

For A $\in \mathbf{F}_{mn}$, the transpose is obtained by interchanging its rows and columns and is denoted by A^T.

Note that if A $\in P_n$, then $AA^T = A^TA = I_n$, the identity matrix of order n.

3.3 Comparable Matrices

Definition 3.3.1

Let A = (a_{ij}) and B = (b_{ij}) $\in \mathbf{F}_{mn}$. We write A $\leq$ B if $a_{ij} \leq b_{ij}$ for all i, j and we say that A is dominated by B (or) B dominates A. A and B are said to be comparable, if either A $\leq$ B (or) B $\leq$ A.

Example

0 $\leq$ A $\leq$ J for all A $\in F_{mn}$.

Proposition 3.3.1

Let A, B $\in F_{mn}$. Then A $\leq$ B $\iff$ A + B = B

Proof

If A $\leq$ B, then

$$
\begin{aligned}
A + B &= (\sup\{a_{ij}, b_{ij}\}) \\
&= (b_{ij}) \\
&= B
\end{aligned}
$$

Conversely, if A + B = B , then $a_{ij} \leq b_{ij}$, that is, A $\leq$ B

Thus A $\leq$ B $\iff$ A + B = B

Proposition 3.3.2

Let A, B $\in F_{mn}$. If A $\leq$ B then for any C $\in F_{np}$, AC $\leq$ BC and for any D $\in F_{pm}$, DA $\leq$ DB.

Proof

A $\leq$ B, $a_{ik} \leq b_{ik}$ for all i = 1 to m and k = 1 to n. By fuzzy multiplication, $a_{ik}c_{kj} \leq b_{ik}c_{kj}$ for j = 1 to p.

By fuzzy addition we get, $\sum_k a_{ik}c_{kj} \leq \sum_k b_{ik}c_{kj}$

Thus AC $\leq$ BC.

Similarly, Since A $\leq$ B, $a_{ik} \leq b_{ik}$, for all i = 1 to n and k = 1 to p.

By fuzzy multiplication, $a_{ik}c_{kj} \leq b_{ik}c_{kj}$ for j = 1 to m.

By fuzzy addition we get, $\sum_k d_{ik}a_{kj} \leq \sum_k d_{ik}b_{kj}$

Thus DA $\leq$ DB.

Proposition 3.3.3

Let A_1 and $A_2 \in \mathbf{F}_{mn}$; B_1 and $B_2 \in \mathbf{F}_{np}$. If $A_1 \leq A_2$ and $B_1 \leq B_2$, then $A_1 B_1 \leq A_2 B_2$.

Proof

Let $A_1 = (a_{ij})$, $A_2 = (a'_{ij})$, $B_1 = (b_{jk})$, and $B_2 = (b'_{jk})$. Since $A_1 \leq A_2$ and $B_1 \leq B_2$, we have $a_{ij} \leq a'_{ij}$ and $b_{jk} \leq b'_{jk}$ for all i = 1 to m , j = 1 to n , k = 1 to p. Therefore, $a_{ij}\ b_{jk} \leq a'_{ij}\ b'_{jk}$. Hence

$$\sum_j a_{ij} b_{jk} \leq \sum_j a'_{ij} b'_{jk}.$$

Thus $A_1 B_1 \leq A_2 B_2$

Example

$$\text{Let } A_1 = \begin{pmatrix} 0.8 & 0.5 \\ 0.4 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A_1 \leq A_2$$

$$B_1 = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.3 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0.6 & 0.2 \\ 0.7 & 0.3 \end{pmatrix}$$

$$B_1 \nleq B_2$$

$$A_1 B_1 = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.4 \end{pmatrix} \nleq \begin{pmatrix} 0.6 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} = A_2 B_2$$

Proposition 3.3.4

For A $\in \mathbf{F}_{mn}$, $AA^T\ A \geq A$.

Proof

Let A = (a_{ij}) for i = 1 to m and j = 1 to n. Then (il^{th}) element of

$AA^T\ A = \sum_j (\sum_k\ a_{ik} a_{jk}) a_{jl}$

Since sum in **F** is the maximum, and multiplication is minimum, this expression is greater than or equal to each term in the summation.

Therefore , for k = l, j = i, we have (il^{th}) element of

$AA^T\ A \geq a_{ik} a_{ik} a_{il} = a_{il}^3 = a_i$

Thus (il^{th}) element of $AA^T\ A \geq (il^{th})$ element of A.

Hence $AA^T\ A \geq A$.

CHAPTER 4

RANKS

Rank is one of the fundamental concepts for the development of fuzzy matrix theory as it is for field based matrices. However the row rank and column rank of a fuzzy matrix need not be equal. There is also a third rank concept, called fuzzy rank(or) Schein rank of great importance in fuzzy matrix theory.

Definition 4.0.2

The row space R(A) of an m×n matrix A is the subspace of V_n generated by the rows of A. The row rank $\rho_r(A)$ is the smallest possible size of a spanning set of R(A). The column space C(A) and column rank ρ_c(A) are defined in dual fashion. Thus C(A) = R(A^T) and ρ_c(A) = $\rho_r(A^T)$.

Definition 4.0.3

Let $A \in \mathbf{F}_{mn}$. The fuzzy rank $\rho_f(A)$ is the smallest integer t such that A = BC where $B \in \mathbf{F}_{mt}$ and $C \in \mathbf{F}_{tn}$. This decomposition is called the fuzzy rank factorization.

Remark 4.0.4

The row rank, column rank and the fuzzy rank of a zero matrix is 0. For a finite matrix A, $\rho_r(A)$ is the maximum number linearly independent rows of A.

Example

Let $\quad A = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 1 \end{pmatrix}$

Any element (x, y) in R(A) is of the form

$(x, y) = \alpha(0, 0.5) + \beta(0.5, 1) \quad$ for $\alpha, \beta \in \mathbf{F}$

$= (0, \alpha \wedge 0.5) + (\beta \wedge 0.5, \beta)$. where, $\wedge$ denotes the minimum.

$x = \beta \wedge 0.5$ and $y = \max\{\alpha \wedge 0.5, \beta\}$

Therefore $0 \leq x \leq 0.5$ and $y \geq \alpha \wedge 0.5$ and

$\beta \implies y \geq \beta \geq \beta \wedge 0.5 \geq x$. Hence

$R(A) = \{(x\ , y)\colon 0 \leq x \leq y \leq 0.5\} \cup \{(x\ , y)\colon 0.5 = x \leq y \leq 1\}$.
Here $A = A^T$. Hence C(A) = R(A) and $\rho_f(A) = \rho_r(A) = \rho_c(A) = 2$

Example

Let $A = \begin{pmatrix} 1 & 0.8 & 0 \\ 0.8 & 0.7 & 0 \\ 0.7 & 0.6 & 0 \end{pmatrix}$

$(0.8, 0.7, 0.6)^T$ and $(1, 0.8, 0.7)^T$ are linearly independent. Hence ρ_c(A) = 2. Each row of A is not a linear combination of the other two rows, that is, all the three rows are linearly independent. Hence ρ_r(A) = 3

$$A = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 0.7 \\ 0.7 & 0.6 \end{pmatrix} \begin{pmatrix} 1 & 0.6 & 0 \\ 0.7 & 0.8 & 0 \end{pmatrix} = BC$$

where, $B \in \mathbf{F}_{32}$ and $C \in \mathbf{F}_{23}$; A = BC is the fuzzy rank factorization. Hence $\rho_f(A) = 2$.

Proposition 4.0.5

For A, $B \in \mathbf{F}_{mn}$, we have the following:

1.R(B) $\subseteq$ R(A) if and only if B = XA for some $X \in \mathbf{F}_m$

2.C(B) $\subseteq$ C(A) if and only if B = AY for some $Y \in \mathbf{F}_n$

Proof

1. If B = XA, then by definition For A = $(a_{ij}) \in \mathbf{F}_{mp}$ and B = (b_{ij}) $\in \mathbf{F}_{pn}$, the max-min product

AB = (sup $\{$inf $\{a_{ik}, b_{kj}\}\}) \in \mathbf{F}_{mn}$

The product AB is defined if and only if the number of columns of A is the same as the number of rows of B; A and B are said to be conformable for multiplication, the i^{th} row of XA, that is

(XA)$_{i*} = \sum_J x_{ij} A_{i*} \in$ R(A)

Hence R(B) $\subseteq$ R(A).

Conversely, suppose R(B) $\subseteq$ R(A), then each row of B is a linear combination of the rows of A.

Hence $B_i * = \sum x_{ij} A_{j*}$ and from which it follows that B = XA.

2. By using the facts C(A) = R(A^T) and

(XA)^T = A^T X^T. Then by 1 we get the result.

Proposition 4.0.6

For a pair of matrices A and B if the product AB is defined, then

R(AB) = R(A)B $\subseteq$ R(B) and C(AB) $\subseteq$ C(A).

Proof

Any vector $y \in \mathrm{R}(AB)$ is of the form $y = uAB = xB$ where $x = uA \in \mathrm{R}(A)$. Hence $y \in \mathrm{R}(B)$.

Thus,

$$\mathrm{R}(AB) = \mathrm{R}(A)B \subseteq \mathrm{R}(B)$$

$$\mathrm{C}(AB) = \mathrm{R}(\mathrm{AB})^T = \mathrm{R}(B^T\ A^T) \subseteq \mathrm{R}(A^T) = \mathrm{C}(A).$$

Remark 4.0.6

For matrices over a field, $\mathrm{R}(AB) \subseteq \mathrm{R}(B)$ implies $\rho_r(AB) \leq \rho_r(B)$.

However, this fails for fuzzy matrices.

Proposition 4.0.7

Let $A \in \mathbf{F}_{mn}$ with $\rho_r(A) =$ r. Then there exist matrices $B \in \mathbf{F}_{mr}$ and $C \in \mathbf{F}_{rn}$ such that $\rho_r(A) = \rho_r(C) =$ r and $A = BC$. This decomposition is called a row rank factorization of A.

Proof

Since row rank of A is r, if $\mathrm{R}(A)$ is generated by the rows of an r $\times$ n matrix C, then $\rho_r(C) =$ r and there exist an m $\times$ r matrix B such that $A = BC$. Thus $A = BC$ as required with $\rho_r(A) = \rho_r(C) =$ r.

Proposition 4.0.8

Let $A \in \mathbf{F}_{mn}$ with $\rho_c(A) =$ s. Then there exist matrices $B \in \mathbf{F}_{ms}$ and $C \in \mathbf{F}_{sn}$ such that $\rho_c(A) = \rho_c(B) =$ s and $A = BC$. This decomposition is called a column rank factorization of A.

Proof

Since column rank of A is s, if C(A) is generated by the rows of an s $\times$ n matrix C, then $\rho_c(C)$ = s and there exist an m $\times$ s matrix B such that $A = BC$. Thus $A = BC$ as required with $\rho_c(A) = \rho_c(C)$ = s.

Proposition 4.0.9

Let $A \in \mathbf{F}_{mn}$ with $\rho(A) = \rho_r(A) = \rho_c(A) =$ r, then there exist matrices $B \in \mathbf{F}_{mr}$ and $C \in \mathbf{F}_{rn}$ such that $A = BC$ with $\rho(A) = \rho_c(B) = \rho_r(C) =$ r. This is called a rank factorization of A.

Example

For the matrix, A = $\begin{pmatrix} 1 & 0.8 & 0 \\ 0.8 & 0.7 & 0 \\ 0.7 & 0.6 & 0 \end{pmatrix}$, $\rho_f(A) = \rho_c(A) = 2$ and $\rho_r(A)$ = 3.

The decomposition $A = BC$ with B $\in \mathbf{F}_{32}$, C $\in \mathbf{F}_{23}$ is a column rank factorization and also the fuzzy rank factorization. However A = BC is not a row rank factorization.

Remark 4.0.7

For field based matrices A, B if R(A) $\subseteq$ R(B) and ranks are equal then R(A) = R(B). However this fails for fuzzy matrices.

Example

Let A $= \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.3 & 0.5 & 1 \end{pmatrix}$ and B $= \begin{pmatrix} 0 & 0.5 & 1 \\ 1 & 0.5 & 0 \end{pmatrix}$;

A $= \begin{pmatrix} 0.5 & 0 \\ 1 & 0.3 \end{pmatrix} \begin{pmatrix} 0 & 0.5 & 1 \\ 1 & 0.5 & 0 \end{pmatrix} = XB.$

Hence by result, R$(A) \subseteq$ R(B). Here , (1, 0.5, 0) $\neq \alpha$(0, 0.5, 0.5) + β(0.3, 0.5, 1) for α, $\beta \in \mathbf{F}$.There fore 2^{nd} row of B is not a linear combination of the rows of A.Thus R(B) $\nsubseteq$ R(A) but ρ_r(A) = ρ_r(B) = 2 and ρ_c(A) = ρ_c(X) = 2. Hence, $A = XB$ is a rank factorization of A.

Definition 4.0.8

In a row (column) rank factorization $A = BC$ if R$(A) =$ R(C) (C(A) = C(B)) then it is said to be a full row (column) rank factorization of A. Further,if $A = BC$ with R$(A) =$ R(C) and C$(A) =$ C(B) then it is said to be a full rank factorization.

Remark 4.0.9

For matrices A , B and X in previous example

$A = \begin{pmatrix} 0.5 & 0 \\ 1 & 0.3 \end{pmatrix} \begin{pmatrix} 0 & 0.5 & 1 \\ 1 & 0.5 & 0 \end{pmatrix} = XB$ is a rank factorization of A. In $A = XB$, since C$(A) =$ C(X), it is full column rank factorization, but not a full rank factorization. However, for field based matrices, rank factorization is the same as full rank factorization.

Definition 4.0.10

For vectors v $\in V_m$ and w $\in V_n$ the cross-vector c(v, w) is the matrix $A = (a_{ij}) \in \mathbf{F}_{mn}$ such that $a_{ij} = v_i w_j$, that is, A $= v^T$ w.

Proposition 4.0.10

A non-zero matrix A is a cross-vector if and only if ρ_r(A) $= 1$ if and only if ρ_c(A) $= 1$ if and only if ρ_f(A) $= 1$

Proof

If A is a cross-vector, then we have $A = v^T$w, since A is non-zero, v and w are non-zero vectors. Hence ρ_r(w) $= 1$, $\rho_c(v^T) = \rho_r$(w) $=$ 1. Therefore, ρ_r(A) $= 1$ and ρ_c(A) $= 1$.

Conversely, if ρ_r(A) $= 1$, then by the result we get, A has a row rank factorization of the form $A = BC$ where B^T and C are row vectors. Hence A is a cross-vector. Similarly, when ρ_c(A) $= 1$, then by the result, we can see that A is a cross-vector. Again if A is a cross-vector, then A $= v^T$w is a fuzzy rank factorization of A and $\rho_f(A) = 1$. Conversely, if $\rho_f(A) = 1$, then A has a fuzzy rank factorization $A = BC$ with BT and C as row vectors. Thus A is a cross-vector.

Proposition 4.0.11

Let A $\in \mathbf{F}_{mn}$. The fuzzy rank ρ_f(A) satisfy the following properties.

1. ρ_f(A) $\leq \inf\{\rho_r$(A), ρ_c(A)$\}$
2. $\rho_f(PAQ) \leq \rho_f$(A), if the matrix PAQ is defined.

3. ρ_f(A) is the smallest size of a set S of vectors such that

R(A) $\subseteq <S>$.

4. ρ_f(A) is the least number of rank 1 matrices whose sum is A.

Proof

1. Let ρ_r(A) = r, then A has a row rank factorization $A = BC$ with ρ_r(A) = ρ_r(C) = r ; B $\in$ C $\in$ $\mathbf{F}_{rn}$.
Then by definition we get, $\rho_f(\mathrm{A}) \leq \rho_r(\mathrm{A})$. Similarly by proposition, using column rank factorization we get $\rho_f(\mathrm{A}) \leq \rho_c(\mathrm{A})$. Therefore $\rho_f(\mathrm{A}) \leq \inf\{\rho_r(\mathrm{A}),\ \rho_c(\mathrm{A})\}$. Thus 1 holds.

2. Let ρ_f(A) = t, then by definition, t is the least integer such that $A = BC$ with $B \in \mathbf{F}_{mt}$, C $\in \mathbf{F}_{tn}$ is the fuzzy rank factorization of A.
Then for P $\in \mathbf{F}_{pm}$ Q $\in \mathbf{F}_{nq}$,P$AQ = (PB)(CQ) = VW$, where V $\in \mathbf{F}_{pt}$ and W $\in \mathbf{F}_{tq}$. Thus we have a decomposition for PAQ. Therefore by definition, $\rho_f(\mathrm{P}AQ) \leq$ t = ρ_f(A). Thus $\rho_f(\mathrm{P}\ AQ) \leq \rho_f(\mathrm{A})$ and 2 holds.

3. By definition of fuzzy rank, A = BC is a decomposition with B $\in \mathbf{F}_{mt}$ and C $\in \mathbf{F}_{tn}$ where t is the least integer.
By the result, R(A) = R(BC) $\subseteq$ R(C) = $<S>$ where S is the smallest spanning set of R(C). Thus 3 holds.

4. Let s be the least number of rank 1 matrices A_i's such that A = $A_1 + A_2 + \ . \ . \ . \ A_s$. Since each A_i is of rank 1, By the above proposition each A_i is a cross-vector, that is, there exist vectors $v_i \in V_m$ and $w_i \in V_n$ such that $A_i = v_i^T w_i$ for each i = 1 to s.

Let V $\in \mathbf{F}_{mt}$ and W $\in \mathbf{F}_{tn}$ be defined such that the i^{th} column of V is v_i^T and i^{th} row of W is w_i.

Then A $= VW$ is a fuzzy rank decomposition, that is ρ_f(A) = s. Thus 4 holds.

Corollary 4.0.12

Let $A \in \mathbf{F}_{mn}$ and $B \in \mathbf{F}_{np}$. Then the following hold:

1. $\rho_f(AB) \leq \min\{\rho_r(A), \rho_r(B)\}$
2. $\rho_f(AB) \leq \min\{\rho_c(A), \rho_c(B)\}$
3. $\rho_f(AB) \leq \min\{\rho_r(A), \rho_r(B), \rho_c(A), \rho_c(B)\}$

Proof

1. $\rho_f(AB) \leq \rho_f(A)$ By above proposition condition 2

$\leq \rho_r(A)$ By above proposition condition 1

$\rho_f(AB) \leq \rho_f(B)$ By above proposition condition 2

$\leq \rho_r(B)$ By above proposition condition 1

Hence $\rho_f(AB) \leq \min\{\rho_r(A), \rho_r(B)\}$. Thus 1 holds.

2. $\rho_f(AB) \leq \rho_f(A)$ By above proposition condition 2

$\leq \rho_c(A)$ By above proposition condition 1

$\rho_f(AB) \leq \rho_f(B)$ By above proposition condition 2

$\leq \rho_c(B)$ By above proposition condition 1

Hence $\rho_f(AB) \leq \min\{\rho_c(A), \rho_c(B)\}$. Thus 2 holds.

3. From 1 and 2, we get $\rho_f(AB) \leq \min\{\rho_r(A), \rho_r(B), \rho_c(A), \rho_c(B)\}$

Definition 4.0.11

The Schein rank $\rho_s(A)$ of a matrix A is defined as the least number of rank 1 matrices whose sum is M.

Remark 4.0.12

For a Boolean matrix, Schein rank is defined as the least number of cross vectors whose sum is A.

CHAPTER 5

GREEN'S RELATION

The fuzzy matrix under max-min composition, that is fuzzy multiplication form a semigroup, that is associative law holds. A fuzzy matrix is invertible if and only if it is a permutation matrix. We introduce Green's equivalence classes for fuzzy matrices. Green defined five equivalence relations **R**, **LH**, **D**, **J** in any semigroup. For fuzzy matrices these five equivalence relations can be characterized in terms of row and column spaces.

Definition 5.0.13

For any two elements x, y of a semigroup **S**, we define the following relations: 1. $x\mathbf{R}y$ if $x = y$ or if there exist a, b $\in$ **S** such that $x\text{a} = y$ and $y\text{b} = x$

2. $x\mathbf{L}y$ if $x = y$ or if there exist a, b $\in$ **S** such that

$\text{a}x = y$ and $\text{b}y = x$

3. $x\mathbf{H}y$ if $x\mathbf{R}y$ and $x\mathbf{L}y$

4. $x\mathbf{D}y$ if there exist z such that $x\mathbf{R}z$ and $z\mathbf{L}y$

5. $x\mathbf{J}y$ if there exist a, b, c, d such that $\text{a}x\text{b} = y$, $\text{c}y\text{d} = x$ (or) if $x\mathbf{R}y$ (or) $x\mathbf{L}y$. It is well known that the above relations are all equivalence relations.

Proposition 5.0.15

For n $\times$ n matrices X, Y over a commutative semiring **R** with identities 0 and 1, we have the following:

1. $A\mathbf{R}B$ if and only if $\mathrm{C}(A) = \mathrm{C}(B)$

2. $A\mathbf{L}B$ if and only if R(A) = R(B)

3. $A\mathbf{H}B$ if and only if C(A) = C(B) and R(A) = R(B)

4. $A\mathbf{D}B$ if and only if there exist matrices X, Y such that $\mathrm{R}(A)X$ $= \mathrm{R}(AX) = \mathrm{R}(B)$ and $AXY = A$

5. $A\mathbf{J}B$ if and only if there exist matrices X, Y such that $\mathrm{R}(A)X \supset \mathrm{R}(B)$ and $\mathrm{R}(B)Y \supset \mathrm{R}(A)$.

Proof

1. By proposition, $\mathrm{C}(B) \subseteq \mathrm{C}(A) \implies B = Ay$ for some $y \in \mathbf{F}_n$ By definition, $A\mathbf{R}B$ if $A = B$ or if there exist A, $B \in \mathbf{S}$ such that Ay $= B$ or $Bx = A$

That is,

$$A\mathbf{R}B \text{ if } Ay = B \iff \mathrm{C}(B) \subseteq \mathrm{C}(A)$$

$$A\mathbf{R}B \text{ if } Bx = A \iff \mathrm{C}(A) \subseteq \mathrm{C}(B)$$

$$\implies A\mathbf{R}B \iff \mathrm{C}(B) = \mathrm{C}(A)$$

2. By proposition, $\mathrm{R}(B) \subseteq \mathrm{R}(A) \iff B = xA$ for some $x \in \mathbf{F}_m$

By definition, $A\mathbf{L}B$ if $A = B$ or if there exist A, B $\in \mathbf{S}$ such that $Ax = B$ or $By = A$

That is,

$$A\mathbf{L}B \text{ if } Ax = B \iff \mathrm{R}(B) \subseteq \mathrm{R}(A)$$

$$A\mathbf{L}B \text{ if } By = A \iff \mathrm{R}(A) \subseteq \mathrm{R}(B)$$

$$\implies A\mathbf{L}B \iff \mathrm{R}(B) = \mathrm{R}(A)$$

By definition we have, $A\mathbf{H}B$ if $A\mathbf{R}B$ and $A\mathbf{L}B$

Then from 1 and 2,

$A\mathbf{R}B \iff$ C(A) = C(B)

$A\mathbf{L}B \iff$ R(A) = R(B)

$\implies A\mathbf{H}B \iff$ C(A) = C(B) and R(A) = R(B)

4. Suppose $A\mathbf{D}B$, then $A\mathbf{R}C$ and $C\mathbf{R}B$ for some matrix C.

Therefore, R(B) = R(C). Further by proposition, R(B) $\subseteq$ R(A) $\iff$ B $= XA$ for some $X \in \mathbf{F}_m$,

$AX =$ C, $CY =$ A for some matrices X and Y

$\mathrm{R}(A)X = \mathrm{R}(AX) = \mathrm{R}(C) = \mathrm{R}(B)$

$AXY = CY = A$

Conversely, let $\mathrm{R}(AX) = \mathrm{R}(B)$ and $AXY = A$

Then, $\mathrm{R}(AX) = \mathrm{R}(B)$ and $(AX)Y = A$

By 2 it follows that $AX\mathbf{L}B$, by the result - $\mathrm{C}(B) \subseteq \mathrm{C}(A)$ if and only if $B = AY$ for some $Y \in \mathbf{F}_n$ we get,

$\mathrm{C}(A) = \mathrm{C}(AXY) \subseteq \mathrm{C}(AX) \subseteq \mathrm{C}(A)$

Therefore, $\mathrm{C}(A) = \mathrm{C}(AX)$ by 1 It follows that, $A\mathbf{R}(AX)$.

Hence, $A\mathbf{D}B$.Thus 4 holds.

5. Suppose, $A\mathbf{J}B$ then there exist Z, W, X, Y such that $ZAX = B, W\ BY = A$.

$\mathrm{R}(B) = \mathrm{R}(ZAX) \subseteq \mathrm{R}(AX) \subseteq \mathrm{R}(A)X$ and

$\mathrm{R}(A) = \mathrm{R}(W\ BY) \subseteq \mathrm{R}(BY) \subseteq \mathrm{R}(B)Y$

Conversely, if $\mathrm{R}(B) \subseteq \mathrm{R}(A)X$ and $\mathrm{R}(A) \subseteq \mathrm{R}(B)Y$, then R(B) $\subseteq$ R(AX) and R(A) $\subseteq$ R(BY) again by the result - R(B) $\subseteq$ R(A) if and only if B = XA for some $Y \in \mathbf{F}_m$, we get there exist matrices W, Z such that $ZAX = B$ and W $BY = A$. Hence,$A\mathbf{J}B$.

Definition 5.0.14

Two subspace V and W of V_n are isotopic if and only if there exist matrices X, Y such that $VX \subseteq$ W and $WY \subseteq$ V ; both XY and YX are identity mappings restricted to V and W respectively.

Example

The subspaces $\{(0, x) : x$ is a real number$\}$ and $\{(x, 0) : x$ is a real number$\}$ are isotopic by the permutation matrices

$$X = Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proposition 5.0.16

Two matrices are D-equivalent if and only if their row spaces are isotopic if and only if their column spaces are isotopic.

Proposition 5.0.17

Two D-equivalent matrices have the same row and column rank.Two J-equivalent matrices have the same fuzzy rank.

Example

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0.6 & 0 \\ 0.6 & 0 & 0 \end{pmatrix} ; B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0 \end{pmatrix}$$

$\rho_c(A) = 2$ and $\rho_c(B) = 3$. Hence A and B are not D-equivalent. However R(A) and R(B) are isomorphic.

Proposition 5.0.18

For fuzzy matrices, X and Y , $X\mathbf{D}Y$ if and only if $X\mathbf{J}Y$

Proof

Suppose $X\mathbf{J}Y$,then $X = AYB$, Y $= CXD$ for some matrices A, B, C, D.

Let K be the subset of the fuzzy algebra F = [0, 1] consisting of all entries of X, Y, A, B, C, D.

Let **S** be the semigroup of all fuzzy matrices whose entries lie in K. Then **S** is finite, since under max-min composition, sum of entries of matrices AB, that is s$(AB) \subseteq$ s$(A) \cup$ s(B) and $X\mathbf{J}Y$ in **S**. Therefore $X\mathbf{D}Y$ in **S** and $X\mathbf{D}Y$ is in the semigroup of all fuzzy matrices.

CONCLUSION

Fuzzy matrices is one of the deepest and fascinating topic in mathematics. Through this project we covered the basic informations about fuzzy matrices. We were understand that one of the most important ways to study a fuzzy matrix is to consider its raw space. The applications of fuzzy matrix are in retrieval system, medical diagnosis, database management system, decision making theory and dynamical system. Also fuzzy matrices is one of the main research area in mathematics.

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DON BOSCO ARTS & SCIENCE COLLEGE ANGADIKADAVU

DEPARTMENT OF MATHEMATICS

2021-2023

Project Report on

GRAPH DOMINATION

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Project Report on

GRAPH DOMINATION

Dissertation submitted in the partial

Fulfillment of the requirement for the award of

MSc Degree in Mathematics of

Kannur University

Name : SURYA CHACKO

Roll No. : C1PSMM1908

Examiners:

1.

2.

KANNUR UNIVERSITY

BONAFIDE CERTIFICATE

Certified that this project report “GRAPH DOMINATION ” is the bonafide work of SURYA CHACKO who carried out the project work under my supervision.

Signature of Head   Signature of Supervisor

DECLARATION

I, SURYA CHACKO hereby declare that the Project work entitled GRAPH DOMINATION has been prepared by me and submitted to Kannur University in partial fulfillment of requirement for the award of Master of Science is a record of original work done by me under the supervision of Mr.ANIL M V, Assistant Professor, Department of Mathematics, Don Bosco Arts & Science College, Angadikadavu. I, also declare that this Project work has not been submitted by me fully for the award of any Degree, Diploma, Title or recognition before any authority.

Place: Angadikadavu SURYA CHACKO

Date: (C1PSMM1908)

ACKNOWLEDGEMENT

Introduction is the proper place to begin. But first I bow my head before the Almighty who is always with me. Also I must express my deepest gratitude to people along the way. No words can adequately express the sense of gratitude, still I try to express my heartfelt thanks through words. The outset, I am deeply indebted to my project supervisor Mr. ANIL M V Assistant Professor, Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu, for the invaluable guidance, loving encouragement and meticulous care towards me throughout my career. I express my deep sense of gratitude to all the faculty members of the Department of Mathematics, Don Bosco Arts and Science College, Angadikadavu. I can never forget the support and encouragement rendered by the Principal and the Staff of Don Bosco Arts & Science College, Angadikadavu. I could not name many who sincerely supported and helped for the successful completion of this Project. It is my pleasure and duty to thank each and everyone of them who walked with me.

SURYA CHACKO

CONTENTS

INTRODUCTION

Graph theory is a branch of Mathematics which has become quite rich and interesting for several reasons. In last several decades, hundreds of research articles have been published in Graph Theory. There are several areas of Graph Theory which have received good attention from mathematicians. Some of these areas are Coloring of Graphs, Matching Theory, Domination Theory, Labeling of Graph and areas related to Algebraic Graph Theory.

I found that the Theory of "Domination in Graph" deserves further attention.Thus, I choose this area for my project work. In this project I introduce the basic concepts and variance of domination in graphs. And it is reasonable to believe that "Domination in graphs" has its origin in "chessboard domination".

Technical systems like communication networks, power grids, traffic management systems and enterprise data networks have a net like structure.Let us use a computer network for an example. Here the computers are the components of the graph and the links between them represents the edges.Now imagine, we want to monitor the functions of each of the computers by one or a small number of computers in such a way that every one of these computers can control its neighbors. In Graph Theory we call these controllers a dominating set. A variety of new problems appears as soon as we impose additional properties on the dominating set.

PRELIMINARIES

Definition 1

A graph G is an ordered triplet (V(G),E(G),I_G), where V(G) is a nonempty set, E(G) is a set disjoint from V(G) and I_G is an incidence map that associates each elements of E(G) to an unordered pair of element of V(G).

• Elements of V (G) are called vertices.

• Elements of E (G) are called edges.

• If for the edge e of G, I_G(e) = {u,v} then we write e = u v.

• If I_G(e) = {u,v} then u and v are called end vertices of e.

• If e is an edge with end vertices u and v then we say that e is incident with u and v.

• A set of two or more edges of a graph G is called multiple edges or parallel edges, if their points are the same.

• An edge with two ends are the same is called a loop.

Definition 2

A vertex u is a neighbor of v in G if uv is an edge in G an u≠v.

• The set of all neighbors of v is the open neighborhood of v or neighbor set of v and it is denoted by N(v).

• The set N[v] = N(v) ∪ {v} is called the closed neighborhood of v in G. If the graph is explicit then open neighborhood and closed neighborhood of v are denoted by NG (v), NG[v] respectively.

Definition 3

A sub graph H of G is said to be a spanning sub graph of G if V(H) = V(G).

Definition 4

A subset S of the vertex set V of a graph G is called independent if no two of its vertices are adjacent in G.

Definition 5

• Vertices u and v are adjacent to each other in G if and only if there is an edge of G with u and v as its end.

• Two edges e and f are adjacent to each other in G if and only if they have common end vertices.

Definition 6

• We denote number of vertices of a graph G by n (G) and it is called order of the graph.

• The number of edges in a graph G is denoted by m(G) and it is called the size of the graph.

Definition 7

A sub graph H of G is said to be induced sub graph of G if each edge of G having its ends in V(H) is also an edge of H. The induced sub graph G with vertex set S⊆V(G) is called the sub graph of G induced by S and is denoted by G[S].

Definition 8

Let G be a graph and $v \in V(G)$, then the number of edges incident with v in the graph G is called the degree of the vertex and it is denoted by d(v).

- The minimum degree of a graph G is denoted by $\delta(G)$.
- The maximum degree of a graph G is denoted by $\Delta(G)$.

Definition 9

A maximal independent set of G is an independent set that is not a proper subset of another independent set of G.

Theorem 1 (Euler theorem)

The sum of the degrees of vertices of a graph is equal to twice the number of edges.

Chapter 1

DOMINATION IN GRAPH

Definition 1.1

A vertex v in a graph G is said to dominate itself and each of its neighbours.We say in other words that v dominates the vertices of its closed neighborhood N[v].

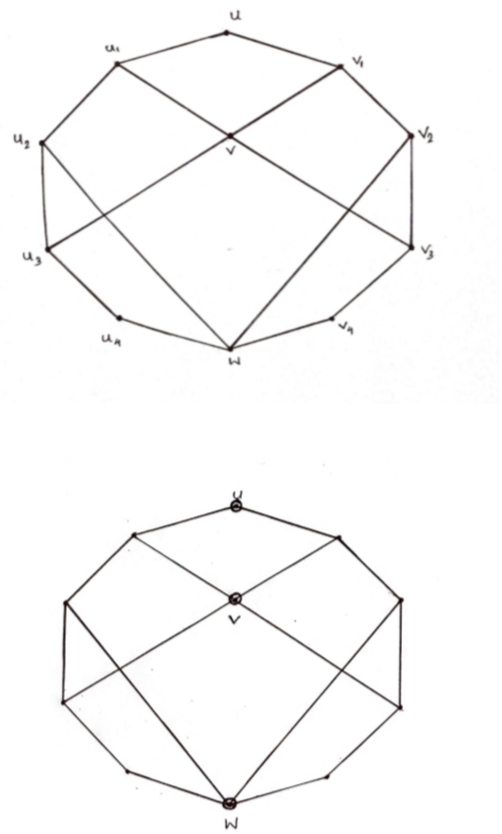


Figure 1.1 : Two dominating sets in a graph G

Definition 1.2

Let G be a graph. A set S$\subset$V is called a dominating set of G if every vertex u $\in$ V|S has a neighbor v$\in$S. Equivalently, every vertex of G is either in S or in the neighbor set N(S)=U v$\in$S N(V) of S in G. A vertex u is said to be dominated by a vertex v$\in$G if either u = v or uv$\in$E (G).

Definition 1.3

A γ-set of G is a minimum dominating set of G, which is a dominating set of G whose cardinality is minimum. A dominating set S of G is minimal if S properly contains no dominating set S of G.

Definition 1.4

The domination number of G is the cardinality of a minimum dominating set of G.It is denoted by γ(G).

Example 1.1

For the Petersen graph P,γ(p)=3.In Figure 1.2, $\{v_1, v_8, v_9\}$ is a γ-set of P while the set $\{v_1$,v_2,v_3,v_4,$v_5\}$ is a minimal dominating set of P.

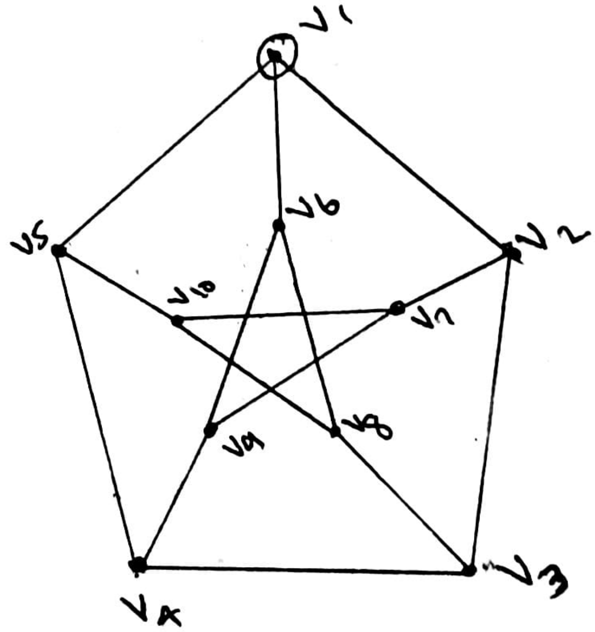


Figure 1.2 Petersen graph P for which γ(p)

Example 1.2

Determine the minimum number of queens that can be placed on the chessboard (see figure 1.3) such that every square is either occupied by one of the queens in a single move.

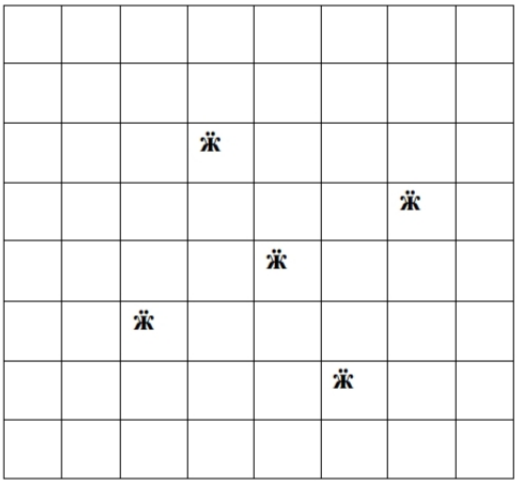


Figure 1.3 Queen Domination

Two queens on a chessboard are attacking if the square occupied by one of the queens can be reached by the other queen in a single move. Otherwise they are non-attacking queens.The minimum number of non-attacking queens that dominate all the squares of a chessboard is the minimum cardinality of an independent dominating set in G. The answer of this example is 5. Figure 1.3 gives one set of dominating locations for the five queens.

Theorem 1.1

Let S be a dominating set of a graph G.Then S is a minimal dominating set if and only if for each vertex u of S, one of the following two conditions hold:

i. u is an isolated vertex of G[S],the sub graph induced by S in G.

ii. There exist a vertex v∈V|S such that u is the only neighbor of v in S.

Proof

Suppose that S is a minimal dominating set of G.Then for each vertex u of S, S∈{u} is not a dominating set of G.Hence there exist V|(S|u) such that v is dominated by no vertex of S|{u}.

If v = u, then u is an isolated vertex of G[S],and hence condition (i) holds.

If v≠u, as S is a dominating set of G, then condition (ii) holds.

Conversely, assume that S is not a minimal dominating set of G.Then there up that in example 1.1 the set S=$\{v_1, v_2, v_3, v_4, v_5\}$ is a minimal dominating set of the Petersen graph P in which no vertex is an isolated vertex of P[S] and for each i=$\{1, 2, 3, 4, 5\}$, v_i is the only vertex of S that is adjacent to $v_i + 5$.

Definition 1.5

Let S be a dominating set of a graph G, and u $\in$ S.The private neighborhood of u relative to S in G is the set of vertices which are in the closed neighborhood of u, but not in the closed neighborhood of any vertex in S |{u}.

Thus, the private neighborhood P_N(u,s) of u with respect to S is given by P_N(u,s)=N[u]|$U_v \in S|$\{u\}N[v].Note that u∈ P_N(u,s) if and only if u is an isolated vertex of G[S] in G.

Theorem 1.1 can be restated as follows:

Theorem 1.2

A dominating set S of a graph G is a minimal dominating set of G if and only if P_N(u,s)$\neq \phi$ for every u $\in$ s.

Corollary 1.1

Let G be a graph having no isolated vertices. If S is a minimal dominating set of G then V|S is a dominating set of G.

Proof

As S is a minimal dominating set, by theorem 1.2, P_N(u,s) $\neq \phi$ for every u$\in$S. This means that for every u$\in$S, there exists v$\in$V|S such that uv$\in$E(G), and consequently, V|S is a dominating set of G.

Corollary 1.2

Let G be a graph of order n$\geq$2. If δ(G)$\geq$1, then γ(G)$\leq\frac{n}{2}$.

Proof

As $\delta(G) \geq 1$, G has no isolated vertices. If S is a minimal dominating set of G, by corollary 1.1 both S and $V \mid S$ are dominating sets of G. Certainly, atleast one of them is of cardinality at most $\frac{n}{2}$.
Since, $\gamma(G) \leq min\{\mid S \mid, \mid V - S \mid\} \leq \frac{n}{2}$.

Corollary 1.3

If G is a connected graph of order $n \geq 2, \gamma(G) \leq \frac{n}{2}$.

Proof

As G is a connected graph and n$\geq$2, G has no isolated vertices.Now, by applying corollary 1.2 we get $\gamma(G) \leq \frac{n}{2}$.

Definition 1.6

There are a number of variations of domination, we consider the best known of these. In this variation, we could resist domination so that a vertex u is only permitted to dominate a vertex v if v is a neighbour of u. In order to distinguish this kind of domination from ordinary domination we refer to this kind of domination as open domination, although the term total domination is used as well.

- If w $\in$ N(v) then we say here that v openly dominates w. That is a vertex v openly dominates the vertices in its open neighborhood N(v).

Definition 1.7

A set S of vertices in a graph G is an open dominating set of G if every vertex of G is adjacent to atleast one vertex of S. Therefore, a graph G contains an open dominating set if and only if G can contain no isolated vertices.
Furthermore, if S is an open dominating set of G, then the sub graph induced by S contains no isolated vertices.

Definition 1.8

The minimum cardinality of an open dominating set is the open domination number $\gamma\circ(G)$ of G. An open dominating set of cardinality $\gamma\circ(G)$ is a minimum open dominating set for G.

Example 1.3

For the graph G of Figure 1.4, determine a) The domination number of G. b) The open domination number of G.

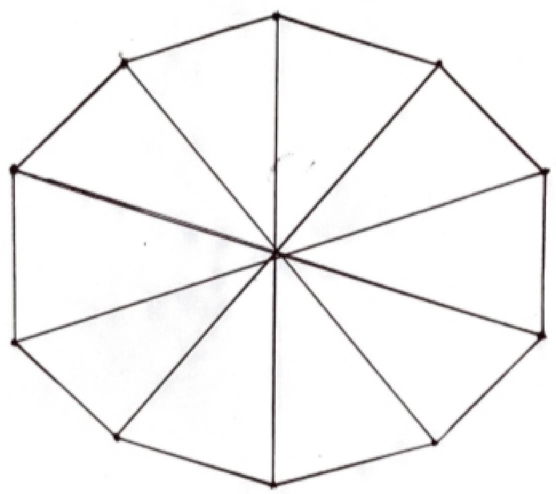


Figure 1.4

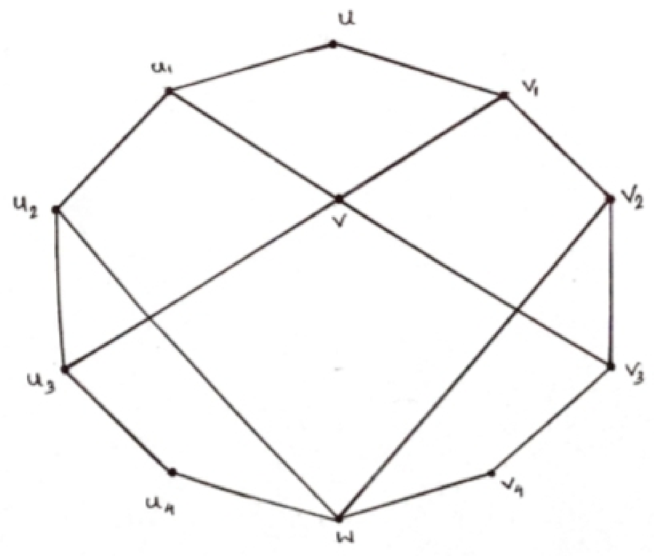
Solution

$\gamma(G)=3$

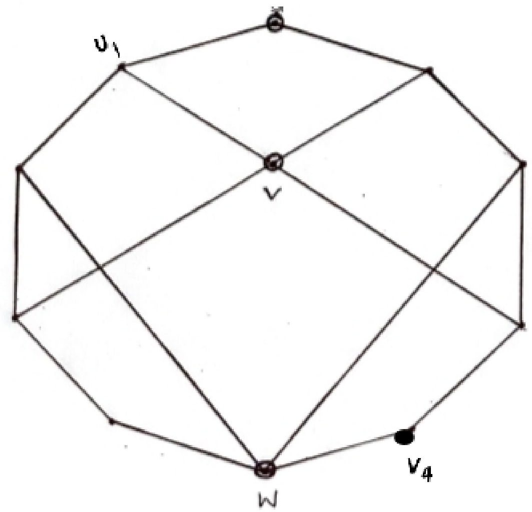
$\gamma\circ(G)=4$

Example 1.4

In figure 1.5, the set S=$\{u_1$,v_1,w, $v_4\}$ is a minimum open dominating set of G and so,$\gamma\circ$(G) = 4.



G



Chapter 2

BOUNDS FOR THE DOMINATION NUMBER

In this section we present lower and upper bounds for the domination number γ(G).

Observation 2.1

(a) The vertex v dominates N(v) and |N(v)|$\leq \Delta$G.

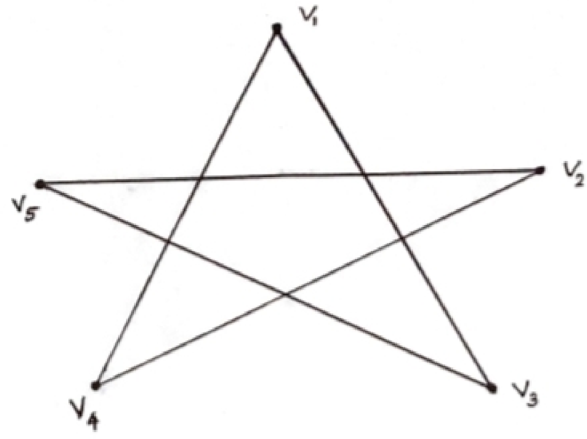
(b) Let v be any vertex of G. Then v|N(v) is a dominating set of G.

These two observations yield the following lower and upper bounds for γ(G).

Theorem 2.1

For any graph G, $\frac{n}{1+\Delta(G)} \leq \gamma(G) \leq n - \Delta(G)$.

Illustration of Theorem 2.1



G

Figure 2.1

The graph G in Figure.2.1 be a graph with a dominating set $S = \{v_3, v_4\}$.

Here , the domination number $\gamma(G) = 2$, $n = 5$ and $\Delta(G) = 2$.

$\left[\frac{n}{1+\Delta(G)}\right] \leq \gamma(G) \leq n - \Delta(G) \Rightarrow \left[\frac{5}{1+2}\right] \leq 2 \leq 5 - 2$

$$\Rightarrow \frac{5}{3} < 2 < 3$$

Bounds for the size m in terms of order n and domination number $\gamma(G)$

Theorem 2.2

Let G be a graph of order n , size m and domination number γ . Then

$$m \leq [\tfrac{1}{2}(n-\gamma)(n-\gamma+2)]\text{.............................(a)}$$

Proof

If $\gamma = 1$

$\frac{1}{2}(n-\gamma)(n-\gamma+2) = \frac{1}{2}(n-1)(n-1+2)$

$= \frac{1}{2}(n-1)(n+1)$

$= \frac{1}{2}(n^2-1)$

While the maximum value for $m = \frac{n(n-1)}{2}$ (when $G = K_n$)

Since $m = \frac{1}{2}n(n-1) \leq \frac{1}{2}(n^2-1)$, and the result is true.

If $\gamma = 2$,

$\frac{1}{2}(n-\gamma)(n-\gamma+2) = \frac{1}{2}(n-2)(n-2+2)$

$= \frac{1}{2}n(n-2)$

Now when $\gamma = 2$, by the theorem 2.1, $\Delta \leq n-2$

(since $\gamma \leq n-\Delta$, then, $2 \leq n-\Delta$) and by Euler's Theorem ($\sum d_i = 2m$)

We get , $2m \leq \Delta$

Then, $2m \leq \Delta \leq n-2$ gives $2m \leq n-2$

$$m \leq \tfrac{1}{2}(n-2)$$

$$m \leq \tfrac{1}{2}n(n-2)$$

$$(\tfrac{1}{2}(n-2)) \leq \tfrac{1}{2}n(n-2)$$

and then the result is true.

Thus the result is true for $\gamma = 1$ and $\gamma = 2$.

We now assume that $\gamma \geq 3$.

We apply induction on n.

Let G be a graph of order n and size m and $\gamma \geq 3$.

If v is a vertex of maximum degree Δ of G , again by theorem 2.1,

$| N(v) |= \Delta \leq n - \gamma$ and hence $\Delta = n - \gamma - r$, where $0 \leq r$.

Given that,

$| N(v) |= n - \gamma - r$ (see fig 2.2)

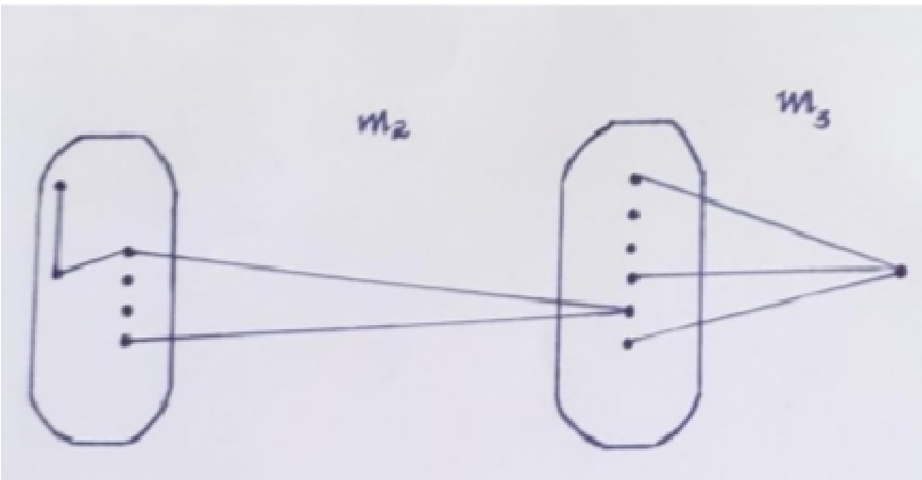
Let $S = V \setminus N[v]$.

Then $| S |=| V | - | N(v) | -1$

$$= n - (n - \gamma - r) - 1$$

$| S |= \gamma + r - 1$...(b)

Fig 2.2 the set S in the proof of 2.2



S

Let m_1 be the size of G [S], m_2 be the number of edges between S and N(v), and m_3 be the size of G [N [v]].

Clearly, $m = m_1 + m_2 + m_3$. If D is a γ–set of G [S], then $D \cup \{v\}$ is a dominating set of G.

Hence,

$\gamma(G) = \gamma \ \leq \mid D \mid +1$ ....(c)

By the induction hypothesis, this implies, by virtue of (b) and (c), that

$$m_1 \ \leq \ [\, \tfrac{1}{2}\, (\mid S \mid - \mid D \mid)\, (\mid S \mid - \mid D \mid +2)\,]$$

$$\leq [\, \tfrac{1}{2}\, (\, \gamma + r - 1\,) \ - (\, \gamma - 1\,)\,]\, [\, (\, \gamma + r - 1\,) \ - (\, \gamma - 1) \ + 2\,]$$

$$= [\, \tfrac{1}{2}\, [\, \gamma + r - 1 - \gamma + 1]\, [\, \gamma + r - 1 - \gamma + 1 + 2]]$$

$$= [\, \tfrac{1}{2}\, r\, (\, r + 2\,)\,]$$

If $u \in N(v)$, then $(\, S \setminus N(v)\,) \cup \{u, v\}$ is dominating set of G.

Therefore , $\gamma \ \leq \mid S \setminus N(v) \mid +2$

$$= \mid S \mid - \mid S \cap N(u) \mid +2$$

$$\leq \ (\, \gamma + r - 1\,) - \mid S \cap N(u) \mid +2 \qquad \text{[by (b)]}$$

i.e , $\mid S \cap N(u) \mid \leq \ \gamma + r - 1 + 2 - \gamma$

$$= \gamma + 1$$

This in turn implies that for each vertex $u \in N(v)$,

$\mid S \cap N(u) \mid \leq r + 1.$

Consequently,

$$m_2 = \text{the number of edges between N(v) and S}$$

$$\leq |\ N(v)\ |\ (\ r+1\ )$$

$$= \Delta\ (\ r+1\ ) \qquad \text{...(e)}$$

Now the sum of the degrees of vertices of N [v] $\leq\ (\ \Delta + 1\ )$.

As there are m_2 edges between N (v) and S.

The sum of the degrees of the vertices of N [v] in G [N [v]]

$$= (\text{ the sum of the degrees of the vertices of N (v) in G }) - m_2$$

$$\leq\ (\ \Delta + 1\ )\ - m_2$$

Thus,

$$m_3\ \leq\ \tfrac{1}{2}\,[\,\Delta(\,\Delta + 1\,) - m_2\,] \text{...(f)}$$

From (d), (e), and (f), we get

$$m = m_1 + m_2 + m_3$$

$$\leq\ \tfrac{1}{2}\,r(\,r+2\,) + m_2 + \tfrac{1}{2}\,[\,\Delta(\,\Delta+1\,) - m_2\,]$$

$$= \tfrac{1}{2}\,r(\,r+2\,) + \tfrac{1}{2}\,[\,\Delta(\,\Delta+1\,)\,]\ + m_2 - \tfrac{m_2}{2}$$

$$= \tfrac{1}{2}\,r(\,r+2\,) + \tfrac{1}{2}\,[\,\Delta(\,\Delta+1)\,] + \tfrac{m_2}{2}$$

$$= \tfrac{1}{2}\,r(\,r+2\,) + \tfrac{1}{2}\,[\,\Delta(\,\Delta+1\,)\ + m_2\,]$$

$$\leq\ \tfrac{1}{2}\,r(\,r+2\,) + \tfrac{1}{2}\,[\,\Delta(\,\Delta+1\,) + \Delta(r+1\,)]$$

$$\leq\ \tfrac{1}{2}\,(\,n-\gamma-\Delta\,)\,(n-\gamma-\Delta+2) + \tfrac{1}{2}\,[\,\Delta(\Delta+1) + \Delta(r+1)\,]$$

(As $\Delta = n - \gamma - r$)

$$= \frac{1}{2}\left[\,(n-\gamma)(n-\gamma+2) + (n-\gamma)(-\Delta) - \Delta(n-\gamma-\Delta+2)\right] + \frac{\Delta}{2}[(\Delta+1)+(r+1)]$$

$$= \frac{1}{2}\left[\,(n-\gamma)(n-\gamma+2)\right] - \frac{\Delta}{2}\left[(n-\gamma)+(n-\gamma-\Delta+2)\right] + \frac{\Delta}{2}[(\Delta+1)+(r+1)]$$

$$= \frac{1}{2}(n-\gamma)(n-\gamma+2) - \frac{\Delta}{2}[(n-\gamma) + (n-\gamma-\Delta+2) - (\Delta+1)(r+1)]$$

$$= \frac{1}{2}(n-\gamma)(n-\gamma+2) - \frac{\Delta}{2}[(n-\gamma) + (n-\gamma+2) - \Delta - (\Delta+1)(r+1)]$$

(since $\Delta = n - \gamma - r$)

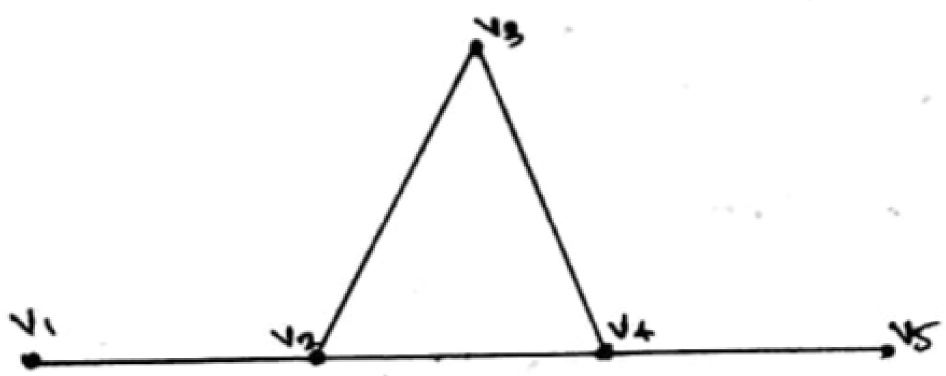
$$= \frac{1}{2}(n-\gamma)(n-\gamma+2) + \frac{\Delta}{2}[(\Delta+r) + (\Delta+r+2) - \Delta - \Delta - 1 - r - 1]$$

[$n - \gamma = \Delta + r$]

$$= \frac{1}{2}(n-\gamma)(n-\gamma+2) - \frac{\Delta}{2}(\gamma)$$

$$\leq \frac{1}{2}(n-\gamma)(n-\gamma+2)$$

Illustration of theorem 2.2



Let G be a graph with domination number $\gamma(G)$, order $n = 5$, size $m = 5$.

Then ,

$$m \leq \ [\,\tfrac{1}{2}(n-\gamma)(n-\gamma+2)] \ \Rightarrow 5 \ \leq \ [\tfrac{1}{2}(\,5-2)\,(5-2+2)]$$

$$\Rightarrow \ 5 \ \leq \ 7.5$$

Chapter 3

VARIETIES OF DOMINATION

Definition 3.1

In this topic, we will consider a variety of conditions that can be imposed either on the dominated set V - S , or on V , or on the method by which the vertices in V - S are dominated . These include the following.

i) **Multiple domination** in which we insist that each vertex in V - S be dominated by at least k vertices in S for a fixed positive integer k.

ii) **Locating domination** in which we insist that each vertex in V - S has a unique set of vertices in S which dominate it.

iii) **Strong domination** in which we insist that each vertex v in V - S be dominated by at least one vertex in S whose degree is greater than or equal to the degree of v.

iv) **Weak domination** specifies that each vertex v in V - S dominated by at least one vertex in S whose degree is less than or equal to the degree of v.

v) **Global domination** in which we insist that the domination set

S also dominates the vertices V - S in the complement of G.

vi) **Directed domination** in digraphs in which we insist that for each vertex v in V - S , there is a directed edge from u to v for at least one vertex u in S.

MULTIPLE DOMINATION

Let S be a dominating set in a graph G = (V , E).

If we view the dominating set or a set that either monitors or controls the vertices in V - S , then the removal or failure , of an edge may result in a set which is no longer dominating.

If this is an undesirable situation , then it may be necessary to increase the level of domination of each vertex so that , even if an edge fails , the set S will still be a dominating set.

The idea of dominating each vertex in V-S multiple times originated with Fink and Jacobson.

Theorem 3.1

If S is a γ-set of a graph G , then at least one vertex in V - S is dominated by no more than two vertices in S.

Proof

Let S be a minimum dominating set in G.

Assume that every vertex in V - S is dominated by three or more vertices.

Let u $\in$ V - S and let v and w be two vertices in S which dominate u.

It follows from our assumption that every vertex in V - S is dominated by at least one vertex in S - { v , w }.

Therefore, the set V - S = S - {v,w } $\cup$ {u} is a dominating set.

But since |V - S |<| S |, we contradicts the assumption that S is a minimum dominating set.

Theorem 3.2

If G is a graph with $\Delta(G) \geq k \geq 2$,then $\gamma_k(G) \geq \gamma(G) + K - 2$.

Proof

Let S be a minimum k - dominating set in G.

Let u $\in$ V $-$ S

And let $v_1, v_2, ..., v_k$ be distinct vertices in S which dominate u.

Notice that , since $\Delta(G) \geq k \geq 2$.

We know that V $-$ S $\neq \phi$ because there is always a k - dominating set, each vertex in V $-$ S is dominated by at least one vertex in

S $-\{v_1, v_2, ..., v_k\}$.

Therefore, since u dominates each vertex in $\{v_2, ...v_k\}$.

We know that the set S = $S - \{v_2, ..., v_k\} - \{u\}$ is a dominating set in G .

Therefore, $\gamma(G) \leq | V - S | = \gamma_K(G) - (K-1) + 1 = \gamma_k(G) - k + 2$.

Theorem 3.3

For any graph G ,

$$\gamma_k(G) \geq \frac{kn}{(\Delta(G)+k)}$$

Proof

Let S be a minimum k - dominating set.

Let t denote the number of edges between S and V − S.

Since the degree of each vertex in S is at most Δ ,

$$t \leq \Delta(G)\gamma_k(G)$$

But since each vertex in V S is adjacent with at least k vertices in S.

We know that t $\geq k[n - \gamma_k(G)]$.

Combining these two inequalities produces

$$\gamma_k(G) \geq \frac{kn}{(\Delta(G)+k)}$$

LOCATING DOMINATION

In the main, conditions on a dominating set S put restrictions either on the sub graph induced by S , or on the number of times each vertex v in V (G) must be dominated.

For example, each vertex v must be dominated at least k times, at most k times,exactly k times , at least deg (v) times or as a certain parity of times.

When considering locating sets in graphs one can think of selecting a minimum set S of vertices of a graph G to achieve the function " location through triangulation " .

For a given k- tuple of vertices $(v_1, v_2, ..., v_k)$ assign to each vertex v ∈ V(G) the k-tuple of its distance to these vertices.

$$\text{f(v)} = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$$

Letting S = $v_1, v_2, ..., v_k)$, the k - tuple f (v) is called S - location of v.

A set S is a locating set for G if no two vertices have the same S - location and the location number L(G) is the minimum cardinality of a locating set.

Definition 3.2

i) The $open\, k - neighborhood$ of a vertex v ∈ V, is the set

$$N_k(v) = \{u : u \in v \text{ and } d(u, v) \leq k\}$$

ii) The set $N_k[v] = N_k(v) \cup \{v\}$ is caled $the\, closed\, k - neighborhood$ of v.

iii) Every vertex w ∈ $N_k[v]$ is said to be $k\ -\ adjacent$ to v. Thus we define the k - degree $deg_k(v)$ of a vertex as | $N_K(V)$.

The $minimum\, k -\ degree$ ð equals min $\{deg_k(v) : v \in V\ \}$, while the $maximum\, k - degree$ $\Delta_k(G)$ equals max $\{deg_k(v) : v \in V\}$.

iv) Finally , for a set S of vertices , we define $N_k(S)$ to be the union of the open k - neighborhoods of vertices in S , while $N_k[S]$ is the union of the closed k- neighborhoods of vertices in S.

v) A set is a $distance - k\, dominating$ set if $N_k[S] = V$.

vi) The distance - k domination number $\gamma_k(G)$ equals the minimum cardinality of a distance - k dominating set in G.

STRONG AND WEAK DOMINATION

Given two adjacent vertices u and v we say that u strongly dominates v if deg (u) $\geq$ deg (v).
Similarly , we say that v weakly dominates if deg (u) $\leq$ deg (v).
A set S $\leq$ V(G) is a strong dominating set of G if every vertex in V$-$S is strongly dominated by at least one vertex in S .
Similarly, S is a weak - dominating set if every vertex in V $-$ S is weakly dominated by at least one vertex in S.
The strong (weak) domination number $\gamma_s(G)$ (respectively $\gamma_w(G)$) is the minimum cardinality of a strong (weak) dominating set of G.

GLOBAL AND FACTOR DOMINATION

The notion of a dominating set S in a graph G = (V , E) can be extended in a natural way to a set which is a dominating set in both G and the complement $\overline{G} = (\overline{V, E})$of G.

This concept was introduced independently by Sampathkumar , who used the term global domination , and by Brigham and Dutton , who used the more general term of factor domination.

A graph H = (V, E) is said to have a t - factoring into factors F(H) = $\{G_1, G_2, ..., G_t$ if each graph $G_i = (V_i, E_i)$ has the same vertex set , $V_i = V$, and the edge sets $\{E_1, E_2, ..., E_t\}$ form a partition of E.

Given a t- factoring F of H, a subset $D_f \leq$ V is a factor dominating set if D_f is a dominating set of G_i , for $1 \leq i \leq t$.

The factor domination number $\gamma_{f_t}(F(H))$ is the minimum cardinality of a factor dominating set of F (H).

We will write γ_{f_t} when the graph H and the factoring F (H) are understood , and γ_i will denote $\gamma(G_i)$.

DOMINATION IN DIRECTED GRAPHS

A directed graph (also called a digraph) D = (V, A) consists of a finite set of vertices and a set A of directed edges , called arcs , where A $\leq V \times V$.

An arc (u, v) is said to be directed from u to v in which case u is said to be a *predecessor* of v , v is *successor* of u and u dominates v.

The *outset* of a vertex u is the set O(u) = $\{v : (u, v) \in A\}$, while the *inset* is the set I(u) = $\{w : (w, u) \in A\}$. We also define O[u] = $O(u) \cup \{u\}$ and I[u] = $I(u) \cup \{u\}$.

The *outdegree* of a vertex u is od(u) =| $O(u)$ | and the *indegree* of u is in(u) =| $I(u)$ |.

A set $S \leq V$ is independent if no two vertices of S are joined by an arc .

The independence number $\beta \circ (D)$ is the maximum cardinality of an independent set in D.

A set $S \leq V$ is called *absorbent* if for every vertex $v \in V - S$, there exists a vertex $u \in S$ which is a successor of v , that is , v $\rightarrow$ u is an arc in A.

A set $S \leq V$ is a dominating set of D if every vertex $v \in V - S$ is dominated by at least one vertex $u \in S$.

The domination number $\gamma(D)$ of a digraph D is the *minimum cardinality* of a dominating set in D.

A set $S \leq V$ is a kernel if it is both independent and absorbent.

Example 3.1

We observe that not every digraph has a kernel.

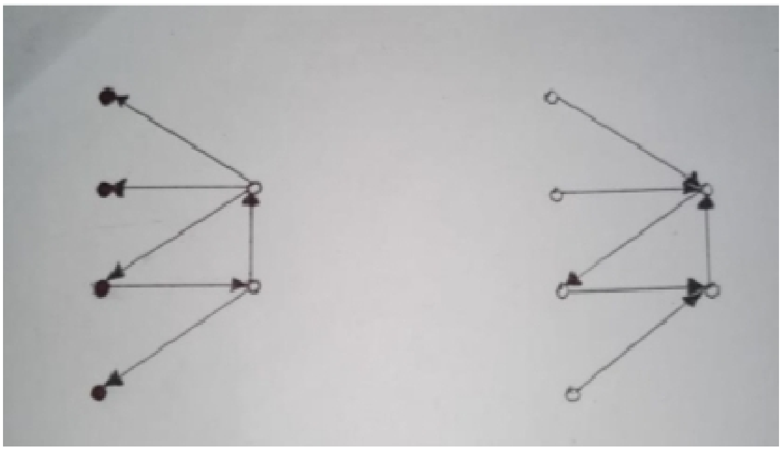


Fig 3.1 : Directed graph with and without kernals

The shaded vertices in the digraph D_1 in the figure above form a kernel of D_1 ; while the graph D_2 , which is obtained from D_1 by reversing the directions of three arcs ,has no kernel .

Thus we observe that not every digraph has a kernel.

Chapter 4

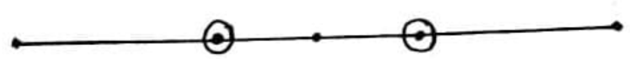
INDEPENDENT DOMINATION AND IRREDUNDANCE

Definition 4.1

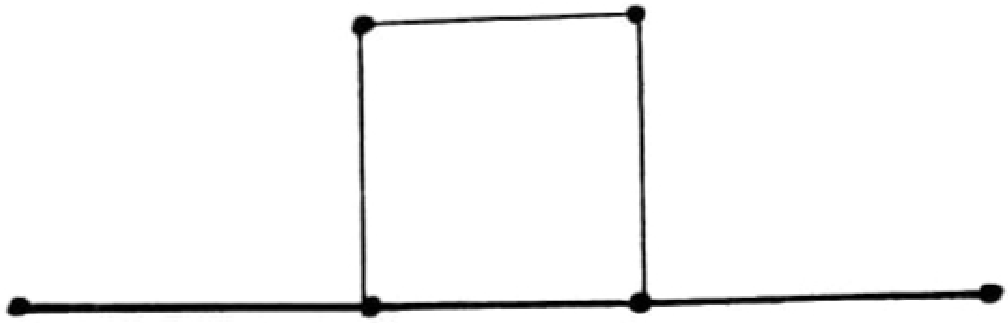
A subset S of the vertex set of a graph G is an independent set of G if S is both an independent and a dominating set. The independent domination number i(G) of G is the minimum cardinality of an independent dominating set of G .

It is clear that $\gamma(G) \leq i(G)$ for any graph G .
For the path 5, $\gamma(5) = i(5) = 2$, (see fig.3.1(a)) while the graph G of fig.3.1(b), $\gamma(G = 2)$ and i(G) =2.

In fact, $\{v_2, v_5\}$ is a γ-set for G , while $\{v_1, v_3, v_5\}$ is a minimum dominating set of G.



(a)



(b)

Figure 3.1

(a) $\gamma(G) = 2 = i(G)$

(b) $\gamma(G) = 2$ while i(G) = 3

Theorem 4.1

Every maximal independent set of a graph G is a minimal dominating set.

Proof

Let S be a maximal independent set of G . Then S must be a dominating set of G.

If not there exist a vertex $v \in V \mid S$ that is not dominated by S , and so $S \cup \{u\}$ is an independent set of G , violating the maximality of S. Further, S must be a minimal dominating set of G . If not, there exist a vertex u of S such that $T = S - \{u\}$ is also a dominating set of G .

This means, as $u \notin T$, u has a neighbor in T and hence S is not independent , a contradiction.

Definition 4.2

A set $S \subset V(G)$ is called irredundant if every vertex V of S has atleast one private neighbor.

This definition means that either v is an isolated vertex of G [S] or else v has a private neighbor in $V \mid S$; that is , there exist at least one vertex $w \in V \mid S$ that is adjacent only to v in S .

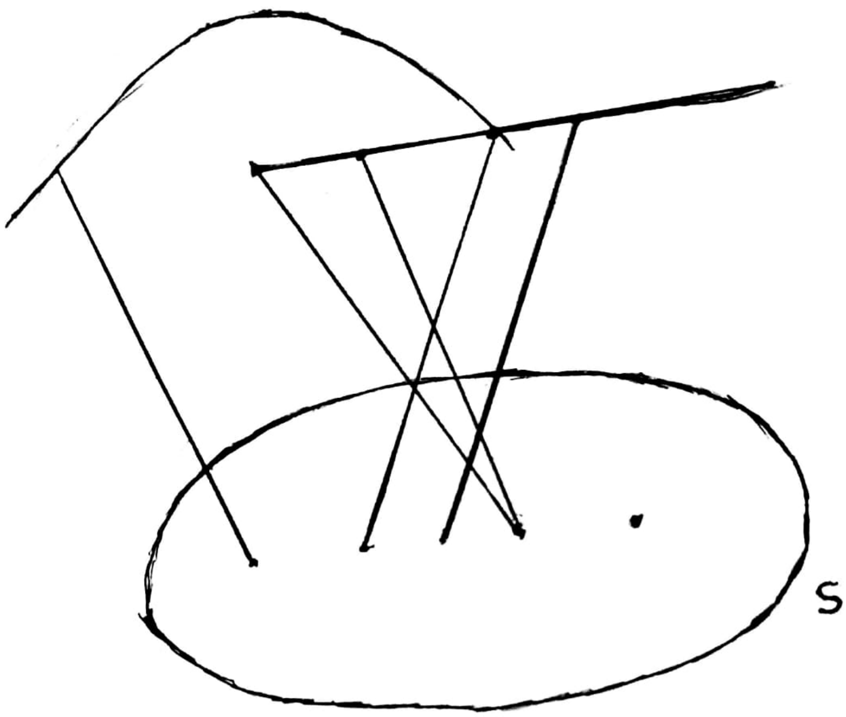


Fig 4.1

S is an irredundant set but not a dominating set. Hence an irredundant set S need not be dominating. If S is both irredundant and dominating, then it is minimal dominating and vice versa.

Theorem 4.2

A set $S \subset V$ is a minimal dominating set of G if and only if S is both dominating and irredundant.

Proof

Assume that S is both a dominating and an irredundant set of G . If S was not a minimal dominating set, there exist $v^{'} \in S$ such that $S \mid \{v^{'}\}$ is also a dominating set. But as S is irredundant, $v^{'}$ has a private neighbor $w^{'}$ (may be equal to $w^{'}$).
Since $w^{'}$ has no neighbor in $S \mid \{v^{'}\}$ is not a dominating set of G .
Thus, S is a minimal dominating set of G .
The proof of the converse is similar.

We define below a few more well-known graph parameters:

i. The minimum cardinality of a maximal irredundant set of a graph G is known as the irredundance number and is denoted by ir(G).
ii. The maximum cardinality of an irredundant set is known as the upper irredundance number and is denoted by IR(G).
iii. The maximum cardinality of a minimal dominating set is known as the upper domination number and is denoted by $\Gamma(G)$.

Theorem 4.3

For any graph G the following inequality chain holds:

$ir(G) \leq \gamma(G) \leq i(G) \leq \beta\circ(G) \leq \Gamma(G) \leq IR(G)$.

Proof

The inequality, $ir(G) \leq \gamma(G)$ is a consequence of the fact that every minimal dominating set of vertices of G is an irredundant set.

We have already observed that $\gamma(G) \leq i(G)$.
From the above two inequalities we get , $ir(G) \leq \gamma(G) \leq i(G)$.
Since an independent dominating set is independent, so
$i(G) \leq \beta\circ(G)$.

Moreover, every maximum independent set is a dominating set, so
$\beta\circ(G) \leq \Gamma(G)$.
Since every minimal dominating set is an irredundant set, it follows that $\Gamma(G) \leq IR(G)$.
Then by combining all the above inequalities, we get
$ir(G) \leq \gamma(G) \leq i(G) \leq \beta\circ(G) \leq \Gamma(G) \leq IR(G)$.

APPLICATIONS OF DOMINATION IN GRAPH THEORY

Domination in graphs has application to several fields. Domination arises in the facility problems, where the number of facilities (eg: hospitals, fire stations...) is fixed and one attempts to minimize the distance that a person needs to travel to get to the closet facility.

Facility location problems: The dominating sets in graphs are natural models for facility location problems in operational research. Facility location problems are concerned with the location of one or more facilities in a way that optimizes a certain objectives such as minimizing transportation cost, providing equitable service to customers and capturing the largest market share.

Radio stations: We have a collection of small villages in a remote part of the world. We locate the radio stations in some of these villages so that the messages can be broadcast to all the villages in that region. The radio station has a limited broadcasting range. We have to place several stations to reach all the villagers. It is very costly, we use domination graphs to locate as few as possible so that the villagers got benefited.

Modelling Biological Networks: Using graph theory as a modelling tool in biologicl networks allow the utilization of the most graphical invariants in such a way that is possible to identify secondary RNA motifs numerically. Those graphical invariants are variations of the domination number of a graph. The results of the research carried out in show that the variations of the domination number can be used for correctly distinguished among the trees that represent native structures and those that are not likely candidates to represent RNA.

CONCLUSION

In this thesis my research mainly focused on domination in graphs. Domination has many applications in the areas such as facility location problems, planning of defence strategies, surveillance related problems etc.

In the four chapters of the thesis, first chapter is based on the concept of Domination, Open Domination and some examples. The main result of the chapter helps us to find Domination number by using various methods. In the second chapter we saw and discussed about the bounds for the Domination number with the help of suitable examples. In chapter three we discussed about the various varities of Domination. We also introduced a new kind of Domination in the fourth chapter namely Independent Domination.

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