K23P 0500

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.)
Examination, April 2023
(2019 Admission Onwards)
MATHEMATICS
MAT 2C08 : Advanced Topology

Time : 3 Hours　　　　　　　　　　　　　　　　　　Max. Marks : 80

PART – A

Answer **any 4** questions. **Each** question carries **4** marks.

1. Let $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

   a) Define a topology $\mathcal{T}_1$ on X such that (X, $\mathcal{T}_1$) is a compact space. Justify your answer.

   b) Define a topology $\mathcal{T}_2$ on X such that (X, $\mathcal{T}_2$) is not compact space. Justify your answer.

2. Prove or disprove : Every compact subset of a topological space is closed.

3. Prove that complete regularity is a topological property.

4. Give an example of Lindeloff space which is not compact.

5. Define Hilbert cube. Prove that a Hilbert cube is metrizable.

6. Prove that a normed space is completely regular.

PART – B

Answer **any 4** questions without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let $(X, \mathcal{T})$ be a T_1 space. Prove that X is a countably compact if and only if it has the Bollzano-Weierstrass property.

   b) Show that the condition that X is a T_1 space in part (a) is necessary. Justify your claim.

8. Prove that the product of any finite number of compact spaces is compact.

9. a) Prove or disprove : Local compactness is a topological property.

   b) Prove that every closed subspace of a locally compact Hausdorff space is locally compact.

   c) Give an example of a metric space which is locally compact but not sequentially compact.

Unit – II

10. a) Prove that every finite set in a T_1 space is closed.

    b) Prove that every second countable space is Lindeloff.

    c) Is the converse of part (b) true ? Justify your claim.

11. a) Define a completely normal topological space. Prove that a T_1 space $(X, \mathcal{T})$ is completely normal iff every subspace of X is normal.

    b) Prove that every second countable regular space is normal.

12. a) Let $\{(x_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a family of topological spaces and let $X = \prod_{\alpha\in\Lambda} X_\alpha$. Prove that X is completely regular iff $(X_\alpha, \mathcal{T}_\alpha)$ is completely regular for each $\alpha \in \Lambda$.

    b) Let $(X, \mathcal{T})$ be a topological space with a dense subset D and a closed, relatively discrete subset C such that $P(D) \leq C$. Then prove that $(X, \mathcal{T})$ is not normal.

    c) Give an example of a Lindeloff space that is not separable. Justify your answer.

Unit – III

13. a) Prove that a T_1 – space $(X, \mathcal{T})$ is normal if and only if whenever A is a closed subset of X and $f : A \to [-1, 1]$ is a continuous function, then there is a continuous function $F : X \to [-1, 1]$ such that $F|_A = f$.

    b) Using (a) part, state and prove Uryshon lemma.

14. State and prove Alexander sub base theorem.

15. a) State Urysohn metrization theorem. Using the Urysohn Metrization theorem prove the following :

    Let (X, d) be a compact metric space, let $(Y, \mathcal{U})$ be a Hausdorff space and let $f : X \to Y$ is a continuous function that maps X onto Y. Prove that $(Y, \mathcal{U})$ is metrizable.

    b) Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. Then show that homotopy ($\simeq$) is an equivalence relation on C(X, Y), the collection of continuous functions that maps X into Y.