K23P 3296

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree (CBSS – Supple. (One Time Mercy Chance)/Imp.) Examination, October 2023
(2017 to 2022 Admissions)
MATHEMATICS
MAT1C02 : Linear Algebra

Time : 3 Hours　　　　　　　　　　　　　　　　Max. Marks : 80

PART – A

Answer **four** questions from this part. Each question carries 4 marks.

1. Let A be an m × n matrix with entries in the field F. Prove that row rank (A) = column rank (A).

2. Let T be a linear operator on R^3, the matrix of which in the standard ordered basis is $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$. Find a basis for the range of T and a basis for the null space of T.

3. Let F be a field and let $A = \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$ be a 3 × 3 matrix over F. Find the minimal polynomial for A.

4. Let T be a linear operator on an n-dimensional vector space V. Prove that the characteristic and minimal polynomials for T have the same roots except for multiplicities.

P.T.O.

5. Let T be the diagonalizable operator on R^3 whose matrix representation with respect to standard basis is $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. Use the Lagrange polynomials to write the representing matrix A in the form $A = E_1 + 2E_2$, $E_1 + E_2 = I$, $E_1E_2 = 0$.

6. State and prove polarization identities.

PART – B

Answer **four** questions from this part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose that V is finite dimensional. Prove that rank (T) + nullity (T) = dim V.

   b) Let V be the space of polynomial functions f from F into F, given by $f(x) = c_0 + c_1x + \ldots + c_kx^k$. Describe the range and null space for the differentiation transformation from V into V.

8. a) Let V and W be vector spaces over the field F. Prove that L(V, W) is a vector space over F with the addition and scalar multiplication defined as $(T + U)(\alpha) = T\alpha + U\alpha$ and $(cT)(\alpha) = c(T\alpha)$, where $T, U \in L(V, W)$ and $c \in F$.

   b) Let T be the linear operator on C^3 for which $T\epsilon_1 = (1, 0, i)$, $T\epsilon_2 = (0, 1, 1)$, $T\epsilon_3 = (i, 1, 0)$. Is T invertible.

9. a) Let V be a finite dimensional vector space over the field F, and let $\mathscr{B} = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for V. Prove that there is a unique dual basis $\mathscr{B}^* = \{f_1, \ldots, f_n\}$ for V* such that $f_i(\alpha_j) = \delta_{ij}$. Also prove that for each linear functional f on V we have $f = \sum_{i=1}^{n} f(\alpha_i)f_i$ and for each vector α in V we have $\alpha = \sum_{i=1}^{n} f(\alpha)\alpha_i$.

   b) Let $\mathscr{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be basis for C^3 defined by $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1)$, $\alpha_3 = (2, 2, 0)$. Find the dual basis of $\mathscr{B}$.

Unit – II

10. a) Let T be a linear operator on a finite dimensional space V and let c be a scalar. Prove that the following statements are equivalent :

   i) c is a characteristic value of T.

   ii) The operator (T – cI) is singular.

   iii) det (T – cI) = 0.

   b) Let T be a linear operator on R^3 which is represented in the standard ordered basis by the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. Prove that T is diagonalizable by exhibiting a basis for R^3, each vector of which is a characteristic vector of T.

11. State and prove Cayley Hamilton theorem.

12. a) Let V be a finite dimensional vector space over the field F and let T be a linear operator on V. Prove that T is diagonalizable if and only if the minimal polynomial of T has the form $p = (x - c_1) \ldots (x - c_k)$, where $c_1, \ldots, c_k$ are distinct elements of F.

   b) Let V be a finite dimensional vector space and let W_1 be any subspace of V. Prove that there is a subspace W_2 of V such that $V = W_1 \oplus W_2$.

Unit – III

13. State and prove Primary decomposition theorem.

14. a) State Cyclic decomposition theorem. Let T be the linear operator on R^3 which is represented in the standard ordered basis by $\begin{bmatrix} 3 & -4 & -4 \\ -1 & 3 & 2 \\ 2 & -4 & -3 \end{bmatrix}$. Find nonzero vectors $\alpha_1, \ldots, \alpha_r$ satisfying conditions of the cyclic decomposition theorem.

   b) Let W be a subspace of an inner product space V and β be a vector in V. Prove that the vector α in W is a best approximation to β by vectors in W if and only if $\beta - \alpha$ is orthogonal to every vector in W.

15. a) State and prove Gram Schmidt orthogonalization process.

   b) Prove that every finite dimensional inner product space has an orthonormal basis.