K21P 0783

Reg. No. : ...............................

Name : ...................................



# II Semester M.Sc. Degree (CBSS – Reg./Suppl. (Including Mercy Chance)/ Imp.) Examination, April 2021
## (2017 Admission Onwards)
## MATHEMATICS
## MAT 2C 06 : Advanced Abstract Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any 4** questions. **Each** question carries **4** marks :

1. Argue that $\mathbb{Z}[x]$ is a UFD.
2. Find all the units in $\mathbb{Z}[i]$.
3. Find $\text{irr}\left(\sqrt{2-i\sqrt{2}}, \mathbb{Q}\right)$. Justify your claim.
4. Prove that every finite field has $p^n$ elements for some prime p and a positive integer n.
5. Prove that if $\alpha, \beta \in \overline{F}$ are both separable over F, then $\alpha + \beta$ is separable over F.
6. Describe the group of the polynomial $x^4 - 1$ over $\mathbb{Q}$. **(4×4=16)**

PART – B

Answer **4** questions without omitting **any** Unit. **Each** question carries **16** marks :

Unit – I

7. a) If D is a PID, prove that every element in D which is neither zero nor a unit in D is a product of irreducibles. 9

   b) Let D be a UFD, F be a field of quotients of D and let $f(x) \in D[x]$ be an irreducible in D[x] with degree of $f(x) > 0$. Prove that f(x) is irreducible in F[x] also. 7

P.T.O.

8. a) State and prove the Euclidean algorithm. 10

b) Illustrate the Euclidean algorithm on $\mathbb{Z}$ by computing the gcd of 12, 249 and 2006. 6

9. a) Let p be an odd prime in $\mathbb{Z}$. Prove that $p = a^2 + b^2$ for some integers a and b if and only if $p \equiv 1 \pmod 4$. 14

b) Prove that there exist an extension field of $\mathbb{Q}$ containing a zero of $x^3 + 3x^2 + 6x + 15$. 2

## Unit – II

10. a) Prove that every finite extension is an algebraic extension. 5

b) Prove that the field $\mathbb{C}$ of complex numbers is algebraically closed. 5

c) Prove or disprove : A field F is algebraically closed if and only if F has no proper algebraic extension. 6

11. a) Prove that if $\alpha$ and $\beta$ are constructible real numbers, then $\alpha\,\beta$ is also constructible. 4

b) Prove that squaring the circle is impossible. 4

c) Prove that if F is a finite field and n is any positive integer, then there exist irreducible polynomials of degree n is F[x]. 8

12. a) State and prove conjugation isomorphism theorem. 10

b) Describe all automorphisms of the field $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{2}\right)$. 6

## Unit – III

13. a) Let E be a finite extension of F, $\sigma$ be an isomorphism of F onto a field F′ and let $\overline{F'}$ be an algebraic closure of F′. Prove that the number of extensions of $\sigma$ to an isomorphism of E onto a subfield of $\overline{F'}$ is finite and independent of $\sigma$, F′ and $\overline{F'}$. 8

b) Prove that if $F \leq E \leq K$, where K is a finite extension of F, then $\{K : F\} = \{K : E\} \{E : F\}$. 4

c) Prove that $\mathbb{Q}\left(\sqrt[4]{2}\right)$ is not a splitting field extension of $\mathbb{Q}$. 4

14. a) Prove that every finite field is perfect. 10

b) Give an example of an extension $F \leq E$, where E is not a separable extension of F. 6

15. a) Define (finite) normal extension of a field. 2

b) Let K be a finite normal extension of F and $F \leq E \leq K$. Prove that

i) K is a normal extension of E.

ii) E is a normal extension of F if and only if the Galois group G(K/E) is a normal subgroup of G(K/F).

iii) $[K : E] = | G(K/E) |$. **(3+8+3)**