K21P 4212

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.) Examination, October 2021
(2018 Admission Onwards)
MATHEMATICS
MAT1C04 : Basic Topology

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks.

1. Let X = {a, b, c}. Give an example of a collection of subsets of X which is a topology on X. Further, give an example of a collection of subsets of X which is not a topology on X.

2. Is int (A∪B) = int (A) ∪ int(B) ? Justify your answer.

3. Describe the weak topology on $\mathbb{R}$ induced by the family of constant functions from $\mathbb{R}$ to $\mathbb{R}$, where the co-domain has the usual topology ? Justify your answer.

4. Let $(A, \mathcal{T}_A)$ be a subspace of $(X, \mathcal{T})$. Is a set open in $(A, \mathcal{T}_A)$ be necessarily open in $(X, \mathcal{T})$? Justify your answer.

5. Prove that the closed unit interval has the fixed point property.

6. Is connectedness a hereditary property ? Justify your answer.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Define finite complement topology on a set X. Show that finite complement topology is a topology on X.

   b) Let X = {a, b, c} and B = {{a, b}, {b, c}, X}. Can B be a basis for a topology on X. Justify your answer.

   c) Give an example of a basis B for a space X. Show that this B satisfies the conditions for a collection of sets to be a basis.

P.T.O.

8. a) Let X be a set and S be a collection of subsets such that $X = \cup \{S : S \in \mathcal{S}\}$. Prove that there is a unique topology $\mathcal{T}$ on X for which S is a sub basis.

   b) Is $\mathbb{R}$ with finite complement topology a first countable space ? Justify your answer.

   c) Prove that every second countable space is separable.

9. a) Show that, in a Hausdorff space a convergent sequence has a unique limit.

   b) Let (X, d) be a metric space, $< x_n >$ a Cauchy sequence in X and let $A = \{x_n : n \in \mathbb{N}\}$. Prove that A is bounded.

   c) Let $(X, \mathcal{T})$ be a topological space, (Y, d) a metric space, $f : X \to Y$ a function and $f_n : X \to Y$ a continuous function for each $n \in \mathbb{N}$ such that $< f_n >$ converges uniformly to f. Prove that f is continuous.

Unit – II

10. a) Define subspace topology on A, where A is a subset of a topological space X. Show that subspace topology is a topology on the subset A.

    b) Is separability a hereditary property ? Justify your answer.

    c) Let $(X, \mathcal{T})$, (Y_1, U_1), (Y_2, U_2) be topological spaces. Prove that $f : X \to Y_1 \times Y_2$ is continuous if and only if $\pi_i \circ f$ is continuous for each i = 1, 2.

11. a) Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ be topological spaces and $(X_1 \times X_2, \mathcal{T})$ be the product space. Show that the product topology on $X_1 \times X_2$ is the smallest topology for which both the projections, from the product space to the factor spaces, are continuous.

    b) Let (X_1, d_1) and (X_2, d_2) be metric spaces and let $X = X_1 \times X_2$. Prove that the product topology on X is same as the topology on X generated by the product metric.

    c) Define weak topology. Define product topology for an arbitrary collection of topological spaces in terms of weak topology.

12. a) Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a family of topological spaces and $(A_\alpha, \mathcal{T}_{A\alpha})$ a subspace of $(X_\alpha, \mathcal{T}_\alpha)$ for each $\alpha \in \Lambda$. Prove that the product topology on $\prod_{\alpha\in\Lambda} A_\alpha$ is same as the subspace topology on $\prod_{\alpha\in\Lambda} A_\alpha$ determined by the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$.

b) Let $(X, \mathcal{T})$ be a topological space and let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a collection of topological spaces and let $f_\alpha : X \to X_\alpha$ be a continuous function for each $\alpha \in \Lambda$. Prove that $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a basis for $\mathcal{T}$ if and only if $\{f_\alpha : \alpha \in \Lambda\}$ separates points from closed sets.

Unit – III

13. a) Let $\{(A_\alpha, \mathcal{T}_{A\alpha}) : \alpha \in \Lambda\}$ be a collection of connected subspaces of $(X, \mathcal{T})$ and $A = \cup_{\alpha\in\Lambda} A_\alpha$. If $\cap_{\alpha\in\Lambda} A_\alpha \neq \phi$, prove that $(A, \mathcal{T}_A)$ is connected.

b) Let $(X, \mathcal{T})$ be a connected space, O an open cover of X and a, b be distinct points of X. Prove that there is a simple chain consisting of members of O that connects a and b.

14. a) Prove that a topological space is locally pathwise connected if and only if each path component of each open set is open.

b) Let $h : X \to Y$ be a homeomorphism between two connected spaces $(X, \mathcal{T})$ and (Y, U). If p is a cut point of X, prove that h(p) is a cut point of Y.

c) Prove that every pathwise connected space is connected.

15. a) Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a collection of topological spaces and let $X = \prod_{\alpha\in\Lambda} X_\alpha$ with product topology $\mathcal{T}$. Prove that the $(X, \mathcal{T})$ is locally connected if and only if $(X_\alpha, \mathcal{T}_\alpha)$ is locally connected for each $\alpha \in \Lambda$ and for all but a finite number of $\alpha \in \Lambda$, $(X_\alpha, \mathcal{T}_\alpha)$ is connected.

b) Define a 0-dimensional space. Prove that every 0-dimensional T_0 space is totally disconnected.
