K21P 0239

Reg. No. : ...............................

Name : ...............................



# IV Semester M.Sc. Degree (C.B.S.S. – Reg./Suppl. (Including Mercy Chance)/Imp.) Examination, April 2021
## (2017 Admission Onwards)
## MATHEMATICS
## MAT 4C 15 : Operator Theory

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. Let X be a normed space and A be a bounded operator on X. If $k \in \sigma_a(A)$, then prove that there is a sequence $\{x_n\}$ in X with $||x_n|| = 1$ for every n such that $||(A - kI)(x_n)|| \to 0$ as $n \to \infty$.

2. Let X, Y and Z be normed spaces. If $F \in BL(X, Y)$ and $G \in BL(Y, Z)$ then prove that $(GF)' = F'G'$, where $F'$ denotes the transpose of the operator F.

3. State true or false and justify. "Every weak convergence sequence in the dual of a normed space is weak* convergent."

4. State true or false and justify. "Every finite dimensional normed space is reflexive."

5. State true or false and justify. "Every continuous linear map on a normed space is compact."

6. Let A be unitary operator on a Hilbert space H. Then prove that $||A|| = 1$.

7. Prove that the numerical range of a bounded operator on a Hilbert space is bounded.

8. Define Hilbert Schmidt operator on a separable Hilbert space and give an example.

P.T.O.

PART – B

Answer **four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

**Unit – I**

9. Let X be a Banach space and $A \in BL(X)$. Then prove that

   a) A is invertible if and only if A is bounded below and the range of A is dense in X.

   b) The set of all invertible operators is open in BL(X).

10. Let X be a Banach space over $\mathbb{C}$ and $A \in BL(X)$. Then prove that

    a) $\sigma_e(A) \subseteq \sigma_a(A) \subseteq \sigma(A)$.

    b) $\sigma(A)$ is non-empty.

11. Let X and Y be normed spaces and $F \in BL(X, Y)$.

    a) If $X'$ is separable, then prove that X is separable.

    b) Prove that $\|F'\| = \|F\| = \|F''\|$ and $F''J_X = J_Y F$, where $J_x$ and $J_y$ are the canonical embedding of X and Y into $X''$ and $Y''$ respectively.

12. Let X be a normed space.

    a) If X is finite dimensional, then prove that the weak convergence and norm convergence are the same.

    b) If X is separable, then prove that every bounded sequence in $X'$ has a weak* convergent subsequence.

**Unit – II**

13. Let X be a normed space.

    a) If X is reflexive, then prove that every bounded sequence in X has a weak convergent subsequence.

    b) If X is uniformly convex and $\{x_n\}$ is a bounded sequence in X such that $\|x_n\| \to 1$ and $\|x_n + x_m\| \to 2$ as $n, m \to \infty$, then prove that $\{x_n\}$ is a Cauchy sequence.

14. Let X be a normed space.
    a) If X is reflexive, then prove that its dual $X'$ is also reflexive.
    b) If X is Banach and uniformly convex, then prove that X is reflexive.

15. Let X and Y be normed spaces and $F : X \to Y$ be linear.
    a) Prove that F is a compact map if and only if for every bounded sequence $\{x_n\}$ in X, $\{F(x_n)\}$ has a subsequence which converges in Y.
    b) If F is compact, then prove that $F'$ is also a compact map.

16. Let X be a normed space and $A \in CL(X)$. Then prove that
    a) The eigenspectrum and the spectrum of A are countable sets and zero is the only possible limit point of it.
    b) $\sigma(A) = \sigma(A')$.

## Unit – III

17. Let H be a Hilbert space and $A \in BL(H)$.
    a) Prove that there is a unique $B \in BL(H)$ such that $\langle A(x), y \rangle = \langle x, B(y) \rangle$, for every $x, y \in H$. Can we drop the completeness of H ? Justify.
    b) If $R(A) = H$, then prove that $A^*$ is bounded below, where $A^*$ is the adjoint of A.

18. Let H be a Hilbert space over $\mathbb{C}$ and $A \in BL(H)$.
    a) If A is self-adjoint, then prove that $||A|| = \sup \{ |\langle A(x), x \rangle| : x \in H, ||x|| \leq 1 \}$.
    b) Prove that there are unique self adjoint operators B and C on H such that $A = B + iC$.

19. Let H be a Hilbert space over $\mathbb{C}$ and $A \in BL(H)$.
    a) Prove that $\sigma(A) = \sigma_a(A) \cup \{k : \bar{k} \in \sigma_e(A^*)\}$.
    b) If A is a self-adjoint operator, then prove that $A^2 \geq 0$ and $A \leq ||A||I$, where $I$ is the identity operator on H.

20. Let H be a Hilbert space and $A \in BL(H)$.
    a) If A is compact, then prove that $A^*$ is also a compact operator.
    b) If A is a Hilbert-Schmidt operator, then prove that $A^*$ is also a Hilbert-Schmidt operator.