K21P 0240

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (CBSS – Reg./Suppl. (Including Mercy Chance)/Imp.) Examination, April 2021
(2017 Admission Onwards)
Mathematics
MAT 4C16 : DIFFERENTIAL GEOMETRY

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks.

1. Sketch typical level sets of the function $f(x_1, x_2) = x_1^2 - x_2^2$.

2. Show that the graph of any function $f : \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F : \mathbb{R}^{n+1} \to \mathbb{R}$.

3. Find and sketch the gradient field of the function $f(x_1, x_2) = x_1 x_2$.

4. Show that the spherical image of an n-surface with orientation N is the reflection through the origin of the spherical image of the same n-surface with orientation – N.

5. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve in S. Let X and Y be smooth vector fields tangent to S along α. Show that $(X \cdot Y)' = X' \cdot Y + X \cdot Y'$.

6. Let $\alpha(t) = (x(t), y(t))$, $t \in I$ be a local parametrization of an oriented plane curve C. Show that $k \circ \alpha = (x'y'' - y'x'')/[(x')^2 + (y')^2]^{3/2}$.

7. Find the length of the parametrized curve $\alpha(t) = (t^2, t^3)$, $t \in [0, 2]$.

8. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ whose Gauss-Kronecker curvature is nowhere zero. Show that the Gauss map is a diffeomorphism.

P.T.O.

PART – B

Answer **any four** questions from this part without omitting any Unit. **Each** question carries **16** marks.

UNIT – I

9. a) Find the integral curve through $p = (0, 1)$ of the vector field $X(x_1, x_2) = (x_2, -x_1)$.

   b) Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f and let $c = f(p)$. Show that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

10. a) Show that the unit n-sphere $x_1^2 + \ldots + x_{n+1}^2 = 1$ is an n-surface in $\mathbb{R}^{n+1}$.

    b) Let S be an (n – 1)-surface in $\mathbb{R}^n$. Show that the cylinder over S is an n-surface in $\mathbb{R}^{n+1}$.

11. a) State Lagrange Multiplier theorem.

    b) Let $a, b, c \in \mathbb{R}$ be such that $ac - b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ on the unit circle $x_1^2 + x_2^2 = 1$ are the eigen values of the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

12. a) Let S be an n-surface in $\mathbb{R}^{n+1}$, let X be a smooth tangent vector field on S and let $p \in S$. Show that there exists a maximal integral curve of X through $p \in S$.

    b) Prove that each connected n-surface in $\mathbb{R}^{n+1}$ has exactly two orientations.

UNIT – II

13. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ for some $c \in \mathbb{R}$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in S$. Show that spherical image of S is the unit sphere S^n.

14. a) Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$.

    b) Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ with orientation N. Show that a parametrized curve $\alpha : I \to S$ is a geodesic in S if and only if it satisfies the differential equation $\ddot{\alpha} + (\dot{\alpha} \cdot N \circ \alpha) N \circ \alpha = 0$.

    c) Let S be an n-surface in $\mathbb{R}^{n+1}$. Show that the velocity vector field along a parametrized curve α in S is parallel if and only if α is geodesic in S.

15. a) Show that the Weingarten map is self-adjoint.

b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = x_1^2 - x_2^2$. Let $p = (1, 1)$ and $v = (p, \cos\theta, \sin\theta)$. Compute $\nabla_v f$.

16. a) Define the local parametrization of an oriented plane curve.

b) Let C be an oriented plane curve and $p \in C$. Show that there exists a local parametrization of C containing p.

c) Show that local parametrizations of plane curves are unique upto isomorphism.

UNIT – III

17. a) Show that the unit speed global parametrization of a connected oriented plane curve is either one to one or periodic.

b) Let U be an open set in $\mathbb{R}^{n+1}$. Show that for each 1-form ω on U there exists unique functions $f_i : U \to \mathbb{R}$, $i = 1, \ldots, n + 1$, such that $\omega = \sum_{i=1}^{n+1} f_i dx_i$.

c) Show that the integral of an exact 1-form over a closed curve is zero.

18. a) Let S be the sphere $x_1^2 + \ldots + x_{n+1}^2 = r^2, r > 0$, oriented by the inward unit normal. Let $p \in S$ and $v \in S_p$ be a unit vector. Find the normal curvature of S at p in the direction v.

b) Show that on a compact oriented n-surface S in $\mathbb{R}^{n+1}$ there exists a point p such that the second fundamental form at p is definite.

19. a) Define a parametrized n-surface in $\mathbb{R}n+k$ $(k \geq 0)$.

b) Give an example of a 2-surface in $\mathbb{R}^4$.

c) Find the Gaussian curvature of the parametrized torus
$\varphi(\theta, \phi) = ((a + b \cos \phi) \cos \theta, (a + b \cos \phi) \sin \theta, b \sin \phi)$, $a, b \in \mathbb{R}$.

20. a) Let S be an n-surface in $\mathbb{R}^{n+1}$ and let $p \in S$. Show that there exists an open set V about p in $\mathbb{R}^{n+1}$ and a parametrized n-surface $\varphi : U \to \mathbb{R}^{n+1}$ such that φ is one to one map from U onto $V \cap S$.

b) State and prove inverse function theorem for n-surfaces.
