K20P 0117

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (CBSS – Reg./Suppl./Imp.) Examination, April 2020

(2017 Admission Onwards)

MATHEMATICS

MAT4C16 : Differential Geometry

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks :

1. Sketch the level sets and graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = x_1 - x_2$.

2. Show that the set of all unit vectors at all points of $\mathbb{R}^2$ forms a 3-surface in $\mathbb{R}^4$.

3. Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all t in I.

4. If $\mathbb{X}$ and $\mathbb{Y}$ are smooth vector fields tangent to an n-surface S in $\mathbb{R}^{n+1}$ along a parametrized curve $\alpha : I \to S$, verify that $(\mathbb{X} . \mathbb{Y})' = \mathbb{X}' . \mathbb{Y} + \mathbb{X} . \mathbb{Y}'$, where $\mathbb{X}'$ denote the covariant derivative of $\mathbb{X}$.

5. Find the length of the parametrized curve $\alpha : [0, 2] \to \mathbb{R}^2$ defined by $\alpha(t) = (t^2, t^3)$.

6. Define the differential $d\varphi$ of a smooth map $\varphi : U \to \mathbb{R}^m$ where U is open in $\mathbb{R}^n$. Also with usual notations, show that the value of $d\varphi$ ($\mathbb{V}$) does not depend on the choice of the parametrized curve. **(4×4=16)**

P.T.O.

PART – B

Answer **any four** questions without omitting any Unit. **Each** question carries 16 marks :

Unit – I

7. a) Show that for a smooth vector field $\mathbb{X}$ on an open set U of $\mathbb{R}^{n+1}$ there exists a maximal integral curve of $\mathbb{X}$ through each point p of U.

   b) Consider the vector field $\mathbb{X}(x_1, x_2) = (x_1, x_2, 1, 0)$ on $\mathbb{R}^2$. For $t \in \mathbb{R}$ and $p \in \mathbb{R}^2$, let $\varphi_t(p) = \alpha_p(t)$ where α_p is the maximal integral curve of $\mathbb{X}$ through p.

      i) Show that for, each t, φ_t is a one to one transformation from $\mathbb{R}^2$ onto itself.

      ii) Show that φ_0 = identity, $\varphi_{t_1 + t_2} = \varphi_{t_1} \circ \varphi_{t_2}$ for all $t_1, t_2 \in \mathbb{R}$ and $\varphi_{-t} = \varphi_t^{-1}$ for all $t \in \mathbb{R}$.

8. a) Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f and let c = f (P). Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is $[\nabla f(p)]^{\perp}$.

   b) Define an n-surface in $\mathbb{R}^{n+1}$. Let $f : U \to \mathbb{R}$ be smooth where U is open in $\mathbb{R}^{n+1}$. Show that graph (f) is an n-surface in $\mathbb{R}^{n+1}$.

9. a) State and prove Lagrange multiplier theorem.

   b) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\mathbb{X}$ be a smooth tangent vector field on S and $p \in S$. Prove the existence of the maximal integral curve of $\mathbb{X}$ through p.

   c) Find two orientations on the n-sphere $x_1^2 + ... + x_{n+1}^2 = 1$.

Unit – II

10. a) Define spherical image of an oriented n-surface in $\mathbb{R}^{n+1}$ and illustrate with an example.

    b) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p \in S$ and let $\mathbb{V} \in S_p$. Prove the existence of the maximal geodesic in S passing through p with initial velocity $\mathbb{V}$.

11. a) What is meant by Levi-Civita parallelism ? State any four properties of Levi-Civita parallelism.

b) Let S be an n-surface in $\mathbb{R}^{n+1}$, let p, q $\in$S. Let α be a piecewise smooth parametrized curve from p to q. Prove that the parallel transport $P_\alpha : S_p \to S_q$ along α is a vector space isomorphism which preserves dot products.

12. a) Prove that the Weingarten map is self-adjoint.

b) Prove that local parametrizations of plane curves are unique up to reparametrization.

c) Let $\alpha(t) = (x(t), y(t))$, $t \in I$ be a local parametrization of an oriented plane curve C. Show that $k \circ \alpha = (x' y'' - y' x'')/\left(x'^2 + y'^2\right)^{3/2}$, k(p) denotes the curvature of C at $p \in C$.

Unit – III

13. a) Let C be a connected oriented plane curve and let $\beta : I \to C$ be a unit speed global parametrization of C. Prove that β is either one to one or periodic.

b) Let η be the 1-form on $\mathbb{R}^2 - \{0\}$ defined by
$\eta = \left(-x_2 / \left(x_1^2 + x_2^2\right)\right) dx_1 + \left(x_1 / \left(x_1^2 + x_2^2\right)\right) dx_2$. Prove that for $\alpha : [a, b] \to \mathbb{R}^2 - \{0\}$ any closed piecewise parametrized curve in $\mathbb{R}^2 - \{0\}$, $\int_\alpha \eta = 2\pi k$ for some integer k.

14. a) On each compact oriented n-surface S in $\mathbb{R}^{n+1}$, prove that there exists a point p such that the second fundamental form at p is definite.

b) Find the Gaussian curvature of the ellipsoid $x_1^2 + \left(x_2^2 / 4\right) + \left(x_3^2 / 9\right) = 1$, oriented by its outward normal.

15. a) Find the Gaussian curvature of the parametrized torus Ψ in $\mathbb{R}^3$ defined by $\Psi(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$.

b) Let S be an n-surface in $\mathbb{R}^{n+1}$ and let $f : S \to \mathbb{R}^k$. If $f \circ \varphi$ is smooth for each local parametrization $\varphi : U \to S$, then prove that f is smooth. **(4×16=64)**