K18P 1033

Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree (Reg.) Examination, October 2018
MATHEMATICS
(2017 Admn. Onwards)
MAT3C12 : Functional Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. Show that if $x_n \to x$ in l^1 then $x_n \to x$ in l^2.
2. Give an example of an element in $L^1(\mathbb{R})$ but not in $L^2(\mathbb{R})$ and prove your claim.
3. Show that the norms $||.||_1$ and $||.||_2$ on K^n, n = 1, 2, . . . are equivalent.
4. If X is an infinite dimensional space then prove that it contains a hyperspace which is not closed.
5. Let X be a normed linear space and (x_n) be a sequence in X. Prove or disprove : (x_n) converges in X if and only if $f(x_n)$ converges in K for every $f \in X'$.
6. Give an example of a function on $K^n \times K^4$ which is linear in the first variable and conjugate symmetric but not an inner product. Also prove your claim.

PART – B

Answer **4** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) State and prove Jenson's inequality.

   b) State and prove Riesz Lemma.

P.T.O.

8. a) Show that a linear map F from a normed space X to a normed space Y is a homeomorphism if and only if there are $\alpha, \beta > 0$ such that $\beta\|x\| \leq \|F(x)\| \leq \alpha\|x\|$ for all $x \in X$. In case there is a linear homeomorphism from X onto Y then prove that X is complete if and only if Y is complete.

   b) Let X denote a subspace of B(T) with the sup norm, $1 \in X$ and f be a linear functional on X. If f is continuous and $\|f\| = f(1)$, then prove that f is positive. Conversely, if $\text{Re}\, x \in X$ whenever $x \in X$ and if f is positive, then prove that f is continuous and $\|f\| = f(1)$.

9. a) Let X and Y be normed spaces and $X \neq \{0\}$. Then prove that BL(X, Y) is a Banach space in the operator norm if and only if Y is a Banach space.

   b) Let X be a normed space and Y be a Banach space. Let X_0 be a dense subspace of X and $F_0 \in BL(X_0, Y)$. Then prove that there is a unique $F \in BL(X, Y)$ such that $F|X_0 = F_0$ and $\|F\| = \|F_0\|$.

Unit – II

10. a) Let X be a normed space and E be a subset of X. Then prove that E is bounded in X if and only if f(E) is bounded in K for every $f \in X'$.

    b) Define closed map. If a closed map F is bijective then prove that its inverse F^{-1} is also a closed map.

11. a) State and prove closed graph theorem.

    b) Define open map and give an example.

12. a) Let X and Y be normed spaces and $F : X \to Y$ be linear. Then prove that F is an open map if and only if there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma\|y\|$.

    b) Show that the open mapping theorem may not hold if the range of the linear map is not a Banach space.

Unit – III

13. a) State and prove parallelogram law.

    b) Let u_α be an orthonormal set in a Hilbert space H. Then prove that the following conditions are equivalent.

       i) $\{u_\alpha\}$ is an orthonormal basis for H.

       ii) For every $x \in H$, we have

       $x = \Sigma_n \langle x, u_n \rangle u_n$, where $\{u_1, u_2, ...\} = \{u_\alpha : \langle x, u_n \rangle u_n u_n$, where

       $\{u_1, u_2, ...\} = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$.

iii) For every $x \in H$, we have $\|x\|^2 = \Sigma_n |\langle x, u_n\rangle|^2$, where $\{u_1, u_2, ...\} = \{u_\alpha : \langle x, u_\alpha\rangle \neq 0\}$.

iv) Span $\{u_\alpha\}$ is dense in H.

v) If $x \in H$ and $\langle x, u_\alpha\rangle = 0$ for all α, then $x = 0$.

14. a) Let X be an inner product space and $f \in X'$. Let $\{u_1, u_2, ...\}$ be an orthogonal set in X. Then prove that $\sum_n |f(u_n)|^2 \leq \|f\|^2$.

b) Prove that the projection theorem does not hold for an incomplete inner product space.

15. a) Let (x_n) be a bounded sequence in a Hilbert space H then prove that it has a weak convergent subsequence.

b) Let H be a Hilbert space over K. If F_1 and F_2 are closed subspaces of H, then prove that $(F_1 + F_2)^\perp$ equals the closure of $F_1^\perp + F_2^\perp$.
