K18P 0228

Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree (Regular) Examination, March 2018
MATHEMATICS
(2017 Admn.)
MAT 2C 06 : Advanced Abstract Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any 4** questions. **Each** question carries **4** marks.

1. Give an example of a principal ideal domain. Justify your claim.
2. Prove that if N is a multiplicative norm on an integral domain D, then N(u) = 1 for every unit u in D.
3. Prove that these exist algebraic extensions which are not finite extensions.
4. Prove that every finite field is an algebraic extension of $\mathbb{Z}_p$ for some prime p.
5. Find all isomorphisms of $\mathbb{Q}(3\sqrt{2})$ onto a subfield of $\overline{\mathbb{Q}}$. Which of them are automorphisms ?
6. If $f(x) \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$, prove that all zeros of f(x) have multiplicity one. (4×4=16)

PART – B

Answer 4 questions without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. Prove that if D is a unique factorization domain, then D[x] is also a unique factorization domain. 16
8. a) Prove that if F is a field and x and y are indeterminates, then F[x, y] is not a PID. 5
   b) Prove that if D is a PID, then any two non-zero elements a and b in D. Gave a gcd and that any gcd of a and b can be expressed as $\lambda a + \mu b$ for some $\lambda, \mu \in D$. 7
   c) Find all the units in $\mathbb{Z}\left[\sqrt{-5}\right]$. 4
9. a) What is $\mathbb{Z}[i]$? Prove that $\mathbb{Z}[i]$ is a Euclidean domain. 12
   b) State Kronecker's theorem. How would you construct an extension field of $\mathbb{Q}$ contain a root of the polynomial $x^3 + 2x^2 + 4x + 6$? 4

P.T.O.

Unit – II

10. a) Prove that if E is a finite extension of F and K is a finite extension of E, then K is a finite extension of F. 10

b) Prove that $\sqrt[3]{2}$ is not a member of $\mathbb{Q}(\sqrt{2})$. Also obtain $\left[\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) : \mathbb{Q}\right]$. 6

11. a) Prove that if α and $\beta \neq 0$ are constructible real numbers, then α/β is also constructible. 4

b) Prove that 'squaring the circle is impossible. 4

c) Prove that if F is a finite field and n is any positive integer, then these is an irreducible polynomial in F[x] of degree n. 8

12. a) If $\{\sigma_i : i \in I\}$ is a collection of automorphisms of a field E, prove that the set of all elements in E, left fixed by σ_i, for all $i \in I$, is a subfield of E. 6

b) Describe all automorphisms of the field :

i) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$

ii) $\mathbb{Z}_2(\alpha)$, where α is the root of $x^2 + x + 1$, in the algebraic closure of $\mathbb{Z}_2$. 10

Unit – III

13. a) Prove that if $F \leq E \leq \bar{F}$ and if every automorphism of $\bar{F}$ leaving F fixed induces an automorphism of E, then E is a splitting field over F. 8

b) Prove that any two algebraic closures of a field are isomorphic. 4

c) Find for which finite extensions F of $\mathbb{Q}$, the following is true.
$[F : \mathbb{Q}] = \{F : \mathbb{Q}\} = |G(F/\mathbb{Q})|$. 4

14. a) Let F be a field, E be a finite extension of F and K be a finite extension of E. Prove that K is separable over F if and only if K is separable over E and E is separable over F. 6

b) Prove that any finite field is perfect. 10

15. a) Let K be a finite normal extension of F and let E be a field such that $F \leq E \leq K$. Prove that

i) K is a finite normal extension of E.

ii) $[K : E] = |G(K/E)|$ and $[E : F]$ = the number of left cosets of G(K/E) in G(K/F).

iii) The lattice diagram of subgroups of G(K/F) is the inverted lattice of intermediate fields of K over F. (3+4+4)

b) Prove that for every positive integer n, there exists a finite normal extension $F \leq K$ such that $G(K/F) = \mathbb{Z}_n$. 5