K20P 0347

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (CBSS-Reg./Suppl./Imp.) Examination, April 2020
(2017 Admission Onwards)
MATHEMATICS
MAT 2C 07 : Measure and Integration

Time : 3 Hours   Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Define Lebesgue outer measure of a set. Prove that outer measure is translation invariant.
2. Let $\{E_i\}$ be a sequence of measurable sets. If $E_1 \subseteq E_2 \subseteq \ldots$, prove that $m(\lim E_i) = \lim m(E_i)$.
3. $f(x)$, $0 \le x \le 1$, is defined by $f(x) = 0$ for x is rational and if x is irrational, $f(x) = n$, where n is the number of zeros immediately after the decimal point, in the representation of x on the decimal scale. Show that f is measurable and find $\int_0^1 f\, dx$.
4. Let f and g be integrable functions. Prove that $f + g$ is integrable and $\int(f + g)\, dx = \int f\, dx + \int g\, dx$.
5. Define a σ – ring. Prove that every algebra is a ring and every σ – algebra is a σ – ring. Is the converse true ? Justify.
6. Let $f, g \in L^p(\mu)$ and let a and b be constants, prove that $af + bg \in L^p(\mu)$.

PART – B

Answer **any four** questions from this Part without omitting a any Unit. **Each** question carries **16** marks. **(4×16=64)**

Unit – I

7. a) Define a measurable set. Prove that the class of measurable sets is a σ – algebra.
   b) Prove that there exists a non-measurable set.
8. a) Prove that a set E is measurable if and only if for $\epsilon > 0$, there exists an open set $O \supseteq E$ such that $m^*(O - E) \le \epsilon$.
   b) Prove that every interval is measurable.
9. a) State and prove Fatou's Lemma.
   b) Show that $\int_1^\infty \frac{dx}{x} = \infty$.

Unit – II

10. a) If f is continuous on the finite interval [a, b], then prove that f is integrable. Also prove that $F(x) = \int_a^x f(t)\,dt$, $(a < x < b)$ is a differentiable function such that $F'(x) = f(x)$.

    b) If f is Riemann integrable and bounded over finite interval [a, b], then prove that f is integrable and $R\int_a^b f\,dx = \int_a^b f\,dx$.

    c) Show that $\lim \int_0^\infty \frac{dx}{(1+x/n)^n x^{\frac{1}{n}}} = 1$.

11. a) Let f be a bounded function defined on the finite interval [a, b], then prove that f is Riemann integrable over [a, b] if and only if f is continuous a.e.

    b) State and prove Lebesgue's Dominated Convergence Theorem.

12. a) Let $\{A_i\}$ be a sequence in a ring R, prove that there exists a sequence $\{B_i\}$ of disjoint sets of R such that $B_i \subseteq A_i$ for each i and $\bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} B_i$ for each N so that $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} = B_i$.

    b) Let μ^* be an outer measure on H(R) and let S^* denote the class of μ^*-measurable sets. Prove that S^* is a σ –ring and μ^* restricted to S^* is a complete measure.

Unit – III

13. a) Let $\{f_n\}$ be a sequence of measurable functions $f_n : X \to [0, \infty]$, prove that $\int \sum_{n=1}^{\infty} f_n\,d\mu = \sum_{n=1}^{\infty} \int f_n\,d\mu$.

    b) Let [[X, S, μ]] be a measure space and f a non-negative measurable function. Prove that $\phi(E) = \int_E f\,d\mu$ is a measure on the measurable space [[X, S]]. Also, if $\int f\,d\mu < \infty$, prove that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $A \in S$ and $\mu(A) < \delta$, then $\phi(A) < \epsilon$.

14. a) State and prove Holder's inequality.

    b) If $1 \le p < \infty$ and $\{f_n\}$ is a sequence in $L^p(\mu)$ such that $\|f_n - f_m\|_p \to 0$ as $m, n \to \infty$, prove that there exists a function f and a subsequence $\{n_i\}$ such that $\lim f_{n_i} = f$ a.e. Also prove that $f \in L^p(\mu)$ and $\lim \|f_n - f\|_p = 0$.

15. a) State and prove Minkowski's inequality.

    b) Let $f_n \in L^2(a, b)$, $n = 0, 1, 2, \ldots$, $f \in L^2(a, b)$ and let $\lim \|f_n - f\|_2 = 0$.

       i) Show that $\int_a^b f^2\,dx = \lim \int_a^b f_n^2\,dx$.

       ii) Verify (i) for $(a, b) = (-\pi, \pi)$.
