K19P 0359

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (Reg./Suppl./Imp.) Examination, April 2019
(2017 Admission Onwards)
MATHEMATICS
MAT2 C09 : Foundations of Complex Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks :

1. Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic and $a \in G$ such that $|f(a)| \leq |f(z)|$ for all z in G. Show that either $f(a) = 0$ or f is constant.

2. Let f be analytic in $B(a ; \mathbb{R})$ and suppose that $f(a) = 0$. Show that a is a zero of multiplicity m if and only if $f^{(m-1)}(a) = \ldots = f(a) = 0$ and $f^{(m)}(a) \neq 0$.

3. Find the Laurent development of $f(z) = \dfrac{1}{(z-1)(z-2)}$ valid in the annulus $(0 ; 1, 2)$.

4. Using residue theorem, show that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$.

5. Define the set $C(G, \Omega)$ and show that it is non-empty.

6. Show that a necessary condition for the convergence of an infinite product is that the n^{th} term must go to 1. **(4×4=16)**

P.T.O.

PART – B

Answer **any four** questions from this Part without omitting any Unit. **Each** question carries **16** marks :

Unit – I

7. a) Define winding number and prove that it is an integer.

   b) Let r be a closed rectifiable curve in $\mathbb{C}$. Prove that :

      i) $n(-r, a) = -n(r, a)$ for every $a \notin \{r\}$,

      ii) $n\{r, a\}$ is constant for a belonging to a component of $G = \mathbb{C} - \{r\}$,

      iii) $n\{r, a\} = 0$ for a belonging to the unbounded component of G.

8. a) Let r be a rectifiable curve and suppose φ is a function defined and continuous on $\{r\}$. For each $m \geq 1$, let $F_m(z) = \int_r \frac{\varphi(w)}{(w-z)^m} dw$ for $z \notin \{r\}$. Prove that each F_m is analytic on $\mathbb{C} - \{r\}$ and $F_m'(z) = m F^{m+1}(z)$.

   b) State and prove the first version of Cauchy's integral formula.

9. a) If r_0 and r_1 are two closed rectifiable curves in G and $r_0 \sim r_1$, prove that $\int_{r_0} f = \int_{r_1} f$ for every function f analytic on G.

   b) State and prove open mapping theorem.

Unit – II

10. a) If f has an isolated singularity at a, then prove that the point z = a is a removable singularity if and only if $\lim_{z \to a}(z-a) f(z) = 0$.

    b) State and prove Casaroti-Weierstrass theorem.

11. a) Use residue theorem to show that $\int_0^\infty \frac{\log x}{1+x^2} dx = 0$.

    b) State and prove Rouche's theorem.

12. a) State and prove Schwarz's lemma.

b) If $|a| < 1$, define $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$, prove that φ_a is a one-one map of $D = \{z : |z| < 1\}$ onto itself ; the inverse of φ_a is φ_{-a}. Also prove that φ_a maps ∂D onto ∂D, $\varphi_a(a) = 0$, $\varphi_a'(0) = 1-|a|^2$ and $\varphi_a'(a) = (1-|a|^2)^{-1}$.

Unit – III

13. a) With usual notations, prove that C (G, Ω) is a complete metric space.

b) Prove that a set $F \subset C(G, \Omega)$ is normal if and only if for every compact set $K \subset G$ and $\delta>0$ there are functions $f_1, ..., f_n$ in F such that for $f \in F$ there is at least one k, $1 \leq k \leq n$ with $\sup\{d(f(z), f_K(z)) : z \in K\} < \delta$.

14. a) If $\{f_n\}$ is a sequence in H(G) and f belongs to $C(G, \mathbb{C})$ such that $f_n \to f$, then prove that f is analytic and $f_n^{(k)} \to f^{(k)}$ for each $k \geq 1$.

b) State and prove Reimann mapping theorem.

15. a) Let $\operatorname{Re} z_n > 0$ for all $n \geq 1$. Prove that $\prod_{n=1}^{\infty} z_n$ converges to a non-zero number if and only if the series $\sum_{n=1}^{\infty} \log z$ converges.

b) Let $\operatorname{Re} z_n > -1$; then prove that the series $\Sigma \log(1 + z_n)$ converges absolutely if and only if the series Σz_n converges absolutely.

c) State and prove the Weierstrass factorization theorem. **(4×16=64)**