K18P 1430

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree (Reg./Suppl./Imp.) Examination, October 2018
(2017 Admn. Onwards)
MATHEMATICS
MAT1C02 : Linear Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries 4 marks :

1. Let T be a linear operator on $\mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 - x_2 + x_3)$, is T invertible ? If so find T^{-1}.

2. Let $\alpha_1 = (1, 0, -1, 2)$ and $\alpha_2 = (2, 3, 1, 1)$ and let W be the subspace of $\mathbb{R}^4$ spanned by α_1 and α_2, which linear functional of the form $f(x_1, x_2, x_3, x_4) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ are in the annihilator of W?

3. Let W_1 and W_2 be subspaces of a finite dimensional vector space V. Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

4. Let V be an n-dimensional vector space and let T be a linear operator on V. Suppose that there exist some positive integer k so that $T^k = 0$. Prove that $T^n = 0$.

5. Let T be a linear operator on a finite dimensional vector space over the field of complex numbers prove that T is diagonalizable iff T is annihilated by some polynomial over $\mathbb{C}$ which has distinct roots.

6. Prove that every finite dimensional inner product space has an orthonormal basis.

P.T.O.

PART – B

Answer 4 questions from this part without omitting any Unit. **Each** question carries **16** marks.

Unit – 1

7. a) Let V be the space of all polynomial functions from $\mathbb{R}$ into $\mathbb{R}$ of the form $f(x)= c_0 + c_1x_1+ c_2x_2+ c_3x_3$. If D is the differentiation operator on V find the matrix of D in the ordered basis $B = \{f_1,f_2,f_3,f_4\}$ for V where $f_k(x) = x^{k-1}$, $k = 1,2,3,4$.

   b) Let V be a finite dimensional vector space over the field F and let $B = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, $B' = \{\alpha'_1, \alpha'_2, ..., \alpha'_n\}$ be ordered bases for V . Suppose T is a linear operator on V. If $P = [P_1,P_2,..., P_n]$ is the $n \times n$ matrix with columns $P_j=[\alpha'_j]_B$ then prove that $[T]_{B'}= P^{-1}[T]_BP$. Alternatively if U is the invertible operator on V defined by $U\alpha_j= \alpha'_j$, $j =1,2,...,n$, then prove that $[T]_{B'} = [U]_B^{-1}[T]_B [U]_B$.

8. a) If f is a non zero linear functional on the vector space V then prove that the null space of f is a hyperspace in V. Conversely, every hyperspace in V is the null space of a (not unique) non zero linear functional on V.

   b) Let V and W be vector spaces over the field F and let T be a linear transformation from V into W, then prove that there exist a unique linear transformation T^t from W^* in to V^* such that $(T^tg)(\alpha)=g(T\alpha)$ for every g in W^* and α in V.

9. a) If W_1 and W_2 are subspaces of a finite dimensional vector space then prove that $W_1 = W_2$ iff $W_1^0 = W_2^0$.

   b) If W is the subspace of $\mathbb{R}^5$ which is spanned by the vectors $\alpha_1 =(2, -2, 3, 4, -1)$, $\alpha_2 = (-1, 1, 2, 5, 2)$, $\alpha_3 = (0, 0, -1, -2, 3)$ and $\alpha_4 = (1, -1, 2, 3, 0)$ then find the annihilator of W.

Unit – 2

10. a) Let V be a finite dimensional vector space over the field F and let T be a linear operator on V. Then prove that T is triangulable iff the minimal polynomial for T is a product of linear polynomials over F .

    b) Let V be a finite dimensional vector space over the field F and let T be a linear operator on V. Then prove that T is diagonalizable iff the minimal polynomial for T has the form $P = (x - c_1) (x - c_2)...(x - c_k)$.

11. a) Let V be a finite dimensional vector space over the field F. Let $\mathcal{F}$ be a commuting family of triangulable linear operators on V. Then prove that there exist an ordered basis for V such that every operator in $\mathcal{F}$ is represented by a triangular matrix in that basis.

b) Let $\mathcal{F}$ be a commuting family of diagonalizable linear operators on a finite dimensional vectorspace V. Then prove that there exist an ordered basis for V such that every operator in $\mathcal{F}$ is represented in that basis by a diagonal matrix.

12. Let T be a linear operator on a finite dimensional vectorspace V. If f is the characteristic polynomial for T, then prove that f(T) = 0.

Unit – 3

13. a) Let T be a linear operator on the finite dimensional vector space V over the field F. Suppose that the minimal polynomial for T decomposes over F in to a product of linear polynomials. Then prove that there is a diagonalizable operator D on V and a Nil potent operator N on V such that

i) T = D + N

ii) DN = ND and D and N are uniquely determined by i) and ii) and each of them is a polynomial in T.

b) State and prove generalized Cayley-Hamilton Theorem .

14. a) Apply Gram-Schmidt process to the vectors, $\beta_1 = (1, 0, 1)$, $\beta_2 = (1,0, -1)$, $\beta_3 = (0, 3, 4)$ to obtain an orthonormal basis for $\mathbb{R}^3$ with the standard inner product.

b) Let W be the subspace of the inner product space V and let β in V prove that if α is a best approximation to β by vectors in W then $\beta - \alpha$ is orthogonal to every vector in W.

15. a) Prove that an orthogonal set of non zero vectors in an inner product space is linearly independent.

b) Let V be an inner product space and $\beta_1, \beta_2, ..., \beta_n$ be linearly independent vectors in V. Prove that there exist orthogonal vectors $\alpha_1, \alpha_2,..., \alpha_n$ in V such that for i =1, 2,...,n the set $\{\alpha_1, \alpha_2,..., \alpha_k\}$, k = 1, 2,...,n is a basis for the subspace spanned by $\{\beta_1, \beta_2, ..., \beta_k\}$.
