K17P 1596

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree (Regular) Examination, October 2017
MATHEMATICS
(2017 Admission)
MAT1C03 : Real Analysis

Time : 3 Hours Max. Marks : 80

Instructions : *1) Answer* ***any four*** *questions from Part –* ***A****.*
2) ***Each*** *question carries* ***4*** *marks.*
3) Answer ***any four*** *questions from Part –* ***B*** *without omitting* ***any*** *Unit.*
4) ***Each*** *question carries* ***16*** *marks.*

PART – A

1. Let X be an infinite set. For $p, q \in X$, define $d(p, q) = 1$ if $p \neq q$, $d(p, q) = 0$ if $p = q$. Show that d is a metric on X.

2. Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

3. Let f be defined for all real x and suppose that $|f(x) - f(y)| \leq (x - y)^2$ for all real x and y. Prove that f is constant.

4. Suppose α increases on [a, b], $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$ and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in R(\alpha)$ and $\int f \, d\alpha = 0$.

5. If $f \in R(\alpha)$ on [a, b], then prove that $|f| \in R(\alpha)$ and $\left|\int_a^b f \, d\alpha\right| \leq \int_a^b |f| \, d\alpha$.

6. Determine whether f defined by $f(x) = x^2 \sin\left(1/x\right)$, if $x \neq 0$, $f(0) = 0$ is of bounded variation on [0, 1].

P.T.O.

PART – B

Unit – I

7. a) Let A be the set of all sequences whose elements are the digits 0 and 1. Prove that A is uncountable.

   b) Let $\{E_n\}$, n = 1, 2, 3, ..., be a sequence of countable sets. Prove that $\bigcup_{n=1}^{\infty} E_n$ is countable.

   c) Let X be a metric space. Define a neighborhood of a point $p \in X$ and prove that every neighborhood is an open set.

8. a) Prove that every k-cell is compact.

   b) Let E be a set in $\mathbb{R}^k$. If every infinite subset of E has a limit point in E, then prove that E is closed and bounded.

9. a) Let f be a continuous mapping of a compact metric space X into a metric space Y. Prove that f is uniformly continuous on X.

   b) Define (i) discontinuity of the second kind (ii) monotonic function and prove that monotonic functions have no discontinuities of the second kind.

Unit – II

10. a) State and prove the generalized mean value theorem. Also deduce the mean value theorem.

    b) Show that the mean value theorem fails to hold for complex valued functions.

    c) Suppose $f'(x)$, $g'(x)$ exist and $g'(x) \neq 0$ and f(x) = g(x) = 0. Prove that $\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$.

11. a) Let $\bar{f}$ be a continuous mapping of [a, b] into $\mathbb{R}^k$ and $\bar{f}$ is differentiable in (a, b). Prove that there exists $x \in (a, b)$ such that $|\bar{f}(b) - \bar{f}(a)| \leq (b - a)|\bar{f}'(x)|$.

    b) Prove that $f \in R(\alpha)$ on [a, b] if and only if for every $\varepsilon > 0$, there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

12. a) If f is monotonic on [a, b] and if α is continuous on [a, b], prove that $f \in R(\alpha)$.

b) Let α be increasing monotonically and $\alpha' \in R$ on [a, b] and f be a bounded real function on [a, b]. Prove that $f \in R(\alpha)$ if and only if $f\alpha' \in R$ on [a, b] and that $\int_a^b f\,d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx$.

Unit – III

13. a) Let $f \in R$ on [a, b]. For $a \le x \le b$, define $F(x) = \int_a^x f(t)\,dt$. Prove that F is continuous on [a, b]. Further if f is continuous at $x_0 \in [a, b]$, prove that F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

b) Define the Riemann-Stieltjes integral of a mapping $\bar{f} = (f_1, ..., f_k)$ of [a, b] into $\mathbb{R}^k$. If $\bar{f}$ maps [a, b] into $\mathbb{R}^k$ and if $\bar{f} \in R(\alpha)$ for some monotonically increasing function α on [a, b], prove that $|\bar{f}| \in R(\alpha)$ and $\left|\int_a^b \bar{f}\,d\alpha\right| \le \int_a^b |\bar{f}|\,d\alpha$.

14. a) When is a function f said to be a bounded variation on [a, b] ? Also define the total variation of f on [a, b].

b) If f is of bounded variation on [a, b], prove that f is bounded on [a, b].

c) Let f be of bounded variation on [a, b] and let $c \in (a, b)$. Prove that f is of bounded variation on [a, c] and on [c, b] and $V_f(a, b) = V_f(a, c) + V_f(c, b)$.

15. a) Let f be a bounded variation on [a, b]. Define V on [a, b] by $V(x) = V_f(a, x)$ if $a < x \le b$ and $V(a) = 0$. Prove that

i) V is an increasing function on [a, b]

ii) $V - f$ is an increasing function on [a, b].

b) Let f be of bounded variation on [a, b]. If $x \in (a, b]$, let $V(x) = V_f(a, x)$ and put $V(a) = 0$. Prove that every point of continuity of f is also a point of continuity of V. Further, prove that the converse is also true.

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