K17P 1597

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree (Regular) Examination, October 2017
(2017 Admission)
MATHEMATICS
MAT1 C04 : Basic Topology

Time : 3 Hours Max. Marks : 80

Instructions : *Answer **any four** questions from Part – **A**. **Each** question carries 4 marks.*

*Answer **any four** questions from Part – **B** without omitting **any** Unit. **Each** question carries **16** marks.*

PART – A

1. Let d be the discrete metric on a nonempty set X. Find the topology on X generated by d.
2. When is a metric space (X, d) said to be bounded ? Does boundedness depend on the metric ? Justify your answer.
3. Let $X = \{1, 2, 3\}$, $\mathfrak{T} = \{\phi, \{1\}, \{2, 3\}, X\}$, $Y = \{4, 5\}$ $\mathcal{U} = \{\phi, \{4\}, Y\}$. Find the defining subbasis for the product topology on $X \times Y$.
4. For each $n \in \mathbb{N}$, Let $X_n = \{1, 2\}$ and let $\mathfrak{T}_n$ be the discrete topology on X_n. Let $\mathfrak{T}$ be the product topology and $\mathcal{U}$ be the box topology on $\prod_{n \in \mathbb{N}} X_n$. Show that $\mathfrak{T} \neq \mathcal{U}$.
5. Let $\mathfrak{T}$ be the usual topology on $\mathbb{R}$. Show that $(\mathbb{R}, \mathfrak{T})$ is connected.
6. Define a totally disconnected space. Consider $\mathbb{R}$ with usual topology. Is $\mathbb{Q}$ with subspace topology totally disconnected ? Why ?

PART – B

Unit – I

7. a) Let X be an infinite set and $\mathfrak{T} = \{U \in \mathscr{P}(X) : U = \phi \text{ or } X - U \text{ is countable}\}$. Prove that $\mathfrak{T}$ is a topology on X. In case X is finite what is $\mathfrak{T}$?
   b) Let $X = \{1, 2, 3\}$. Determine whether $\mathscr{B} = \{\{1, 2\}, \{2, 3\}\}$ is a basis for a topology on X.
   c) State and prove a necessary and sufficient condition for a subset of $\mathscr{P}(X)$ to be a basis for a topology on X.

P.T.O.

8. a) Define a separable space. Give an example of a topological space which is not separable.

   b) Prove that every separable metric space is second countable.

   c) State and prove Baire category theorem.

9. a) Let $(X, \mathfrak{I})$ be a first countable space, let $\langle x_n \rangle$ be a sequence in X and let $x \in X$. Prove that $\langle x_n \rangle$ clusters at x if and only if there is a subsequence of $\langle x_n \rangle$ that converges to x.

   b) Let $(X, \mathfrak{I})$ and $(Y, \mathcal{U})$ be topological spaces. Prove that $f : X \to Y$ is continuous if and only if C is a closed subset of $(Y, \mathcal{U})$, then $f^{-1}(C)$ is a closed subset of $(X, \mathfrak{I})$.

   c) Let $(X, \mathfrak{I})$ be a topological space and (Y, d) be a metric space. Let for each $n \in \mathbb{N}$, $f_n : X \to Y$ be a continuous function such that $\langle f_n \rangle$ coverages uniformly to a function $f : X \to Y$, then prove that f is continuous.

Unit – II

10. a) Let A be a subset of a topological space $(X, \mathfrak{I})$. Define the subspace topology $\mathfrak{I}_A$ and show that $\mathfrak{I}_A$ is indeed a topology on A.

    b) Give an example of a topological space $(X, \mathfrak{I})$, a subspace $(A, \mathfrak{I}_A)$ of $(X, \mathfrak{I})$ and an open set in $(A, \mathfrak{I}_A)$ that is not open in $(X, \mathfrak{I})$.

    c) Define an embedding and show that the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, 0)$, for each $x \in \mathbb{R}$ is an embedding of $\mathbb{R}$ in $\mathbb{R}^2$, then $\mathbb{R}$ and $\mathbb{R}^2$ with their respective usual topologies.

11. a) Let $(X_1, \mathfrak{I}_1)$ and $(X_2, \mathfrak{I}_2)$ be topological spaces and let $(X_1 \times X_2, \mathfrak{I})$ be the product space. Prove that the projections $\pi_i : X_1 \times X_2 \to X_i$ (i = 1, 2) are continuous. Also prove that the product topology is the smallest topology for which both projections are continuous.

    b) Define the product space of a family of topological spaces $\{(X_\alpha, \mathfrak{I}_\alpha) : \alpha \in \Lambda\}$. Prove that the collection of all sets of the form $\prod_{\alpha \in \Lambda} U_\alpha$ where $U_\alpha \in \mathfrak{I}_\alpha$ for each $\alpha \in \Lambda$ and $U_\alpha = X_\alpha$ for all but a finite number of members of Λ is a basis for the product topology $\prod_{\alpha \in \Lambda} X_\alpha$.

    c) Prove that the product space of two Hansdorff spaces is a Hansdorff space.

12. a) Let $(C, \mathfrak{T}_C)$ and $(D, \mathfrak{U}_D)$ be subspaces of the topological spaces $(X, \mathfrak{T})$ and $(Y, \mathfrak{U})$ respectively. Prove that the product topology on $C \times D$ determined by $\mathfrak{T}_C$ and $\mathfrak{U}_D$ is same as the subspace topology on $C \times D$ determined by the product topology on $X \times Y$.

b) Let $\{(X_\alpha, \mathfrak{T}_\alpha) : \alpha \in \Lambda\}$ be an indexed family of topological spaces and let $X = \prod_{\alpha \in \Lambda} X_\alpha$. Prove that the product space $(X, \mathfrak{T})$ is second countable if and only if $(X_\alpha, \mathfrak{T}_\alpha)$ is second countable for all $\alpha \in \Lambda$ and $\mathfrak{T}_\alpha$ is the trivial topology for all but a countable number of α.

Unit – III

13. a) Prove that a topological space $(X, \mathfrak{T})$ is connected if and only if no nonempty proper subset of X is both open and closed.

b) Define fixed point property. Prove that the closed unit interval [0, 1] has the fixed point property.

c) Let $\mathfrak{T}$ be the lower limit topology on $\mathbb{R}$. Is $(\mathbb{R}, \mathfrak{T})$ connected ? Prove your answer.

14. a) Let $\{(X_\alpha, \mathfrak{T}_\alpha) : \alpha \in \Lambda\}$ be a collection of topological spaces and suppose that for each $\alpha \in \Lambda$, $X_\alpha \neq \phi$. Let $X = \prod_{\alpha \in \Lambda} X_\alpha$. Prove that the product space $(X, \mathfrak{T})$ is connected if and only if for each, $\alpha \in \Lambda$, $(X_\alpha, \mathfrak{T}_\alpha)$ is connected.

b) Prove that the fixed point property is a topological invariant.

15. a) Define a pathwise connected space. Show that the topologist's sine curve is not pathwise connected.

b) Define a locally pathwise connected space. Prove that a topological space is locally pathwise connected if and only if each path component of each open set is open.