K21P 0786

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (CBSS – Reg./Suppl. (Including Mercy Chance)/Imp.) Examination, April 2021
(2017 Admission Onwards)
MATHEMATICS
MAT2C09 : Foundations of Complex Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks.

1. If γ and σ are closed rectifiable curves having the same initial points, show that $n(\gamma + \sigma ; a) = n(\gamma ; a) + n(\sigma, a)$ for every $a \notin \{\gamma\} \cup \{\sigma\}$.
2. Does an analytic function map closed sets onto closed sets ? Justify your answer.
3. Define i) isolated singularity ii) removable singularity. Illustrate with examples.
4. Suppose f has a pole of order m at z = a and let $g(z) = (z-a)^m f(z)$, then show that $\text{Res}(f ; a) = \frac{1}{(m-1)!} g^{(m-1)}(a)$.
5. Show that H(G) is closed in $C(G, \mathbb{C})$.
6. Show that $\lim\limits_{z \to 0} \frac{\log(1+z)}{z} = 1$. (4×4=16)

PART – B

Answer **any four** questions from this Part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let G be a connected open set and let $f : G \to \mathbb{C}$ be an analytic function. Prove that $f \equiv 0$ if and only if the set $\{z \in G : f(z) = 0\}$ has a limit point in G.

   b) Let f be an entire function and suppose that there is a constant M, an $R > 0$ and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\leq n$.

P.T.O.

8. a) State and prove the first version of Cauchy's integral formula.

   b) Let G be a region and let $f : G \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every triangular path T in G. Prove that f is constant in G.

9. a) Define fixed end point homotopy. Also state and prove independence path theorem.

   b) Suppose f is analytic in B(a ; R) and let $\alpha = f(a)$. If $f(z) - \alpha$ has a zero of order m at z = a, prove that there is an $\varepsilon > 0$ and $\delta > 0$ such that for $|\beta - \alpha| < \delta$, the equation $f(z) = \beta$ has exactly m simple roots in B (a ; ε).

   c) Let G be a region and suppose that f is a non constant analytic function on G. For any open set U in G prove that f(U) is open.

Unit – II

10. a) State and prove the theorem on Laurent series development of a function which is analytic in an annulus.

    b) State and prove Casaroti-Weierstrass theorem.

11. a) State and prove the residue theorem.

    b) Use residue theorem to show that $\int_0^{\pi} \frac{d\theta}{a + \cos\theta} = \frac{\pi}{\sqrt{a^2 - 1}}$.

12. a) Let G be a region in $\mathbb{C}$ and f an analytic function on G. Suppose that there is a constant M such that $\lim_{z \to a} \sup |f(z)| \leq M$ for all a in $\partial_\infty G$. Prove that $|f(z)| \leq M$ for all z in G.

    b) State and prove Schwarz's lemma.

Unit – III

13. a) Define the set $C(G, \Omega)$ and show that it can be considered as a metric space.

    b) Define equicontinuity at a point and equicontinuity over a set. If $F \subset C(G, \Omega)$ is equicontinuous at each point of G, prove that F is equicontinuous over each compact subset of G.

14. a) Prove that a family F in H(G) is normal if and only if F is locally bounded.

b) Let G be a simply connected region which is not the whole plane and let $a \in G$. Then prove that there is a unique analytic function $f : G \to \mathbb{C}$ having the following properties :

i) $f(a) = 0$ and $f'(a) = 0$

ii) f is one-one

iii) $f(G) = \{z : |z| < 1\}$.

15. a) Define an infinite product and show that a necessary condition for the convergence of an infinite product is that the n^{th} term must go to 1.

b) Let G be a region and let $\{a_j\}$ be a sequence of distinct points in G with no limit point in G and let $\{m_j\}$ be a sequence of integers. Prove that there is an analytic function f defined on G whose only zeros are at the points a_j and further a_j is a zero of f of multiplicity m_j. **(4×16=64)**
