K22P 1601

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (CBSS – Reg./Sup./Imp.) Examination, October 2022
(2019 Admission Onwards)
MATHEMATICS
MAT1C01 : Basic Abstract Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks.

1. List the elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Find the order of each the elements.

2. Let X be a G-set. For $x_1, x_2 \in X$, let $x_1 \sim x_2$ if and only if there exists $g \in G$ such that $gx_1 = x_2$. Prove that ~ is an equivalence relation on X.

3. Let N be a normal subgroup of G and H be any subgroup of G. Prove that $H \vee N = HN = NH$.

4. Let H^*, H and K be subgroups of G with H^* normal in H. Show that $H^* \cap K$ is normal in $H \cap K$.

5. Let $f(x) = 2x^2 + 3x + 4$, $g(x) = 3x^2 + 2x + 3$ in $\mathbb{Z}_6[x]$. Find $f(x) + g(x)$ and $f(x)g(x)$.

6. Let R be a ring with unity 1. Prove that the map $\phi : \mathbb{Z} \to R$ given by $\phi(n) = n \cdot 1$ for $n \in \mathbb{Z}$ is a homomorphism of $\mathbb{Z}$ into R.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let $G_1, G_2, \ldots, G_n$ be groups. For $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ in $\prod_{i=1}^{n} G_i$, define $(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n)$ to be the element $(a_1b_1, a_2b_2, \ldots, a_nb_n)$. Prove that $\prod_{i=1}^{n} G_i$ is a group under this operation.

   b) State Fundamental theorem of finitely generated Abelian groups.

   c) Find all abelian groups of order 16 up to isomorphism.

8. a) Prove that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime.

   b) Let X be a G-set and let $x \in G$. Prove that $|G_x| = (G : G_x)$. Also if $|G|$ is finite, show that $|G_x|$ is a divisor of $|G|$.

9. a) State and prove First Sylow theorem.

   b) Prove that no group of order 96 is simple.

Unit – II

10. Prove that any integral domain D can be embedded in a field F such that every element of F can be expressed as a quotient of two elements of D.

11. a) State and prove Third isomorphism theorem.

    b) Define free Abelian group. Prove that $\mathbb{Z}_n$ is not free Abelian.

    c) Let $G \neq \{0\}$ be a free abelian group with a finite basis. Prove that every basis of G is finite and all bases of G have the same number of elements.

12. State and prove Schreier theorem.

Unit – III

13. a) State and prove division algorithm for F[x].

    b) State Factor theorem and factorize $x^4 + 3x^3 + 2x + 4 \in \mathbb{Z}_5[x]$.

14. a) Let $f(x) \in F[x]$ and let f(x) be of degree 2 or 3. Prove that f(x) is reducible over F if and only if it has a zero in F.

    b) State and prove Eisenstein criterion for irreducibility.

    c) Prove that $25x^5 - 9x^4 - 3x^2 - 12$ is irreducible over $\mathbb{Q}$.

15. a) Let $R = \{a + b\sqrt{2} / a, b \in \mathbb{Z}\}$ and let R′ consists of all 2 × 2 matrices of the form $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{Z}$. Show that R is a subring of $\mathbb{R}$ and R′ is a subring of $M_2(\mathbb{Z})$. Also show that $\phi : R \to R'$, where $\phi(a + b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ is an isomorphism.

    b) Let R be a commutative ring with unity. Prove that M is a maximal ideal of R if and only if R/M is a field.
