K21P 1071

Reg. No. : ....

Name : ....

III Semester M.Sc. Degree (CBSS – Reg./Suppl./Imp.) Examination, October 2021
(2018 Admission Onwards)
MATHEMATICS
MAT3C14 : Advanced Real Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. Define uniform bounded functions and give an example.

2. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

3. Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$. For what values of x does the series converge absolutely ?

4. Show that $\lim_{x \to +\infty} x^n e^{-x} = 0$ for every n.

5. Show that $\log \Gamma$ is convex on $(0, \infty)$.

6. Find $\lim_{n \to \infty} \frac{n}{\log n}\left[n^{\frac{1}{n}} - 1\right]$. **(4×4=16)**

PART – B

Answer 4 questions from this Part without omitting **any Unit**. **Each** question carries **16** marks.

Unit – I

7. a) Compare pointwise convergence and uniform convergence.

   b) If $\{f_n\}$ is a sequence of continuous function on E and if $f_n \to f$ uniformly on E, then show that f is continuous on E.

P.T.O.

8. a) Suppose K is compact, and

i) $\{f_n\}$ is a sequence of continuous functions on K.

ii) $\{f_n\}$ converges pointwise to a continuous function f on K.

iii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$ I, n = 1, 2, ...

Then show that $f_n \to f$ uniformly on K.

b) Let α be monotonically increasing on [a, b]. Suppose $f_n \mathscr{R}(\alpha)$ on [a, b], for n = 1, 2, ... and suppose $f_n \to f$ uniformly on [a, b] then prove that $f \in \mathscr{R}(\alpha)$ on [a, b] and $\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha$.

9. a) Suppose $\{f_n\}$ is a sequence of functions, differentiable on [a, b] and such that $\{f_n(x_0)\}$ converges for some point x_0 on [a, b]. If $\{f'_n\}$ converges uniformly on [a, b] then show that $\{f_n\}$ converges uniformly on [a, b], to a function f, and $f'(x) = \lim_{n \to \infty} f'_n(x)\ (a \leq x \leq b)$.

b) Define algebra and give an example.

Unit – II

10. a) State and prove Taylor's theorem.

b) Suppose $a_0, \cdots a_n$ are complex numbers $n \geq 1, a_n \neq 0, P(z) = \sum_{k=0}^{\infty} a_k z^k$. Then prove that P(z) = 0 for some complex number z.

11. a) Define orthogonal system of functions.

b) If $\{\phi_n\}$ be orthonormal on [a, b] and if $f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$, then prove that $\sum_{n=1}^{\infty} |c_n|^2 \leq \int_a^b |f(x)|^2 \, dx$.

c) If f(x) = 0 for all x in some segment J, then prove that $\lim S_N(f ; x) = 0$ for every $x \in J$.

12. a) Define beta function.

b) State and prove Stirling's formula.

Unit – III

13. a) Show that a linear operator A on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X.

b) If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and c is a scalar, then prove that $||A + B|| \leq ||A|| + ||B||$, $||cA|| = |c|\,||A||$. With the distance between A and B is defined as $||A - B||$, prove that $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space.

c) Let Ω be the set of all invertible linear operators on $\mathbb{R}^n$. Then prove that Ω is an open subset of $L(\mathbb{R}^n)$ and the mapping $A \to A^{-1}$ is continuous on Ω.

14. a) Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ and f is differentiable at a point $x \in E$. Then prove that the partial derivatives $(D_j f_i)(x)$ exist and $f'(x)e_j = \sum_{i=1}^{m} (D_j f_i)(x) u_i \ (1 \leq j \leq n)$, where $\{e_1, \cdots e_n\}$ and $\{u_1, \ldots u_m\}$ are the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

b) Suppose f maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in E and there is a real number M such that $\|f'(x)\| \leq M$ for every $x \in E$. Then prove that $|f(b) - f(a)| \leq M\,|b - a|$ for all $a \in E, b \in E$.

15. State and prove inverse function theorem. **(4×16=64)**