K21P 1069

Reg. No. : ....

Name : ....

III Semester M.Sc. Degree (CBSS – Reg./Suppl./Imp.) Examination, October 2021
(2018 Admission Onwards)
MATHEMATICS
MAT3C12 : Functional Analysis

Time : 3 Hours

Max. Marks : 80

PART – A

Answer **four** questions from this **Part**. **Each** question carries **4** marks.

1. Give an example of an element in l^1 but not in c_{00} and prove your claim.
2. Explain Riesz lemma for the normed space $\mathbb{R}^2$ and its subspace $\{(x, x) \in \mathbb{R}^2\}$ geometrically.
3. Show that the norms $\|.\|_2$ and $\|.\|_\infty$ on K^n, n = 1, 2, ... are equivalent.
4. Show that c_{00} is not a Banach space.
5. Show that a closed map need not be continuous.
6. Let $H = l^2$ and for n = 1, 2, ... and $u_n = (0, \ldots 0, 1, 0, 0, \ldots)$ where 1 occurs only in the n^{th} entry. Show that $\{u_n : n = 1, 2, \ldots\}$ is an orthonormal basis for H.

PART – B

Answer **4** questions from this Part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Define L^p - space for $1 \leq p \leq \infty$.

   b) Let Y be a closed subspace of a normed space X. For x + Y in the quotient space X/Y let $|||x + Y||| = \inf\{||x + y|| : y \in Y\}$. Then prove that $|||.|||$ is a norm on X/Y. Also shows that a sequence x_n + Y) converges to x + Y in X/Y if and only if there is a sequence (y_n) in Y such that $(x_n + y_n)$ converges to x in X.

P.T.O.

8. a) Let $||.||$ and $||.||'$ be norms on a linear space X. Then prove that the norm $||.||$ is stronger than the norm $||.||'$ if and only if there is some $\alpha > 0$ such that $||x||' \leq \alpha||x||$ for all $x \in X$.

   b) Let X and Y be normed spaces and $F : X \to Y$ be a linear map such that the range R(F) of F is finite dimensional. Then prove that F is continuous if and only if the zero space Z(F) of F is closed in X.

9. a) Let X be a linear space over $\mathbb{C}$. Regarding X as a linear space over $\mathbb{R}$, consider a real linear functional $u : X \to \mathbb{R}$. Define $f(x) = u(x) - iu(ix)$, $x \in X$. Then prove that f is a complex linear functional on X.

   b) State and prove Hahn-Banach extension theorem.

Unit – II

10. a) State and prove Uniform boundedness principle.

    b) State closed graph theorem.

11. a) Show that the closed graph theorem may not hold if the domain of the linear map is not a Banach space.

    b) Let X and Y be normed spaces and $F : X \to Y$ be linear. Then prove that F is an open map if and only if there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $||x|| \leq \gamma ||y||$.

12. a) Prove that there are scalars $k_n \in K$, $n = 0, \pm1, \pm2, \ldots$ such that $k_n \to 0$ as $n \to \pm\infty$, but there is no $x \in L^1([-\pi, \pi])$ such that $\hat{x} = k_n$ for all $n = 0, \pm1, \pm2, \cdots$.

    b) State two norm theorem.

Unit – III

13. a) State and prove Schwarz inequality.

b) Let $\{u_\alpha\}$ be an orthonormal set in an innerproduct space X and $x \in X$. Let $E_\alpha = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$. Then prove that E_α is a countable set. Also prove that $\langle x, u_\alpha \rangle \to 0$ as $n \to \infty$ if E_α is a denumerable set.

14. a) Let E be an orthogonal subset of X and $0 \notin E$. Then prove that E is independent. Also prove that $||x - y|| = \sqrt{2}$ for all $x \neq y$ in E if E is orthonormal.

b) State and prove projection theorem.

15. a) Define weak boundedness and give an example.

b) Prove that a subset of a Hilbert space is weak bounded if and only it is bounded.
