K17U 0241

Reg. No. : ....

Name : ....

VI Semester B.Sc. Degree (CCSS – Supple./Improv.) Examination, May 2017
CORE COURSE IN MATHEMATICS
6B10 MAT : Analysis and Topology
(2009-2013 Admns.)

Time : 3 Hours

Weightage : 30

1. Fill in the blanks :

a) $\int_1^5 \frac{t}{1+t^2}\, dt =$ ____________

b) If F_1 and F_2 are antiderivatives of $f : I \to R$ on an interval I, $F_1 - F_2 =$ ________

c) Let $I = [0, 1]$ and let $f : I \to R$ be continuous. If $\int_0^x f = \int_x^1 f$ for all $x \in I$ then $f(x) =$ ____________

d) If $g(x) = x$ on $[0, 1]$ and $P_n = \left(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right)$ then $\lim_{n\to\infty} (U(P_n, g) - L(P_n, g)) =$ ____________ **(Wt. : 1)**

Answer **any 6** from the following 9 questions (Wt. **1 each**) :

2. Let X be a non-empty set and define a real valued function d on X as follows

For any ordered pair (x, y) of elements of X

$d(x, y) = 1$ if $x \neq y$

$= 0$ if $x = y$

Show that d is a metric on X.

3. If X is a metric space and $x, y \in X$, show that $\exists$ disjoint open spheres centred at x and y.

4. Is the following statement true ?

"Intersection of any collection of open sets is open". Justify your claim.

P.T.O.

5. If T_1 and T_2 are 2 topologies on a non-empty set X, show that $T_1 \cap T_2$ is also a topology on X.

6. If X and Y are topological spaces and $f : X \to Y$ is a one-to-one onto continuous function, then f is a homomorphism. Prove or disprove.

7. Show that a constant function is Riemann integrable.

8. Let $I = [a, b]$ and $f : I \to R$ be integrable on I. If $f(x) \geq 0$ for all $x \in I$, is it true that $\int_a^b f \geq 0$? Justify.

9. State Mean Value theorem for integrals. If f is continuous on $I = [a, b]$, show that $\exists\, c \in I$ such that $\int_a^b f = f(c)(b-a)$.

10. If $\sum a_n x^n$ and $\sum b_n x^n$ converge on some interval $(-r, r)$, $r > 0$ to the same function f, then prove that $a_n = b_n$ for all $n \in N$. **(Wt : 1×6=6)**

Answer **any 7** from the following 10 questions (Weightage **2 each**) :

11. Let X be a metric space. Show that a subset G of X is open iff it is a union of open spheres.

12. Let Y be a subspace of a metric space and let A be a subset of the metric space Y. Show that A is open as a subset of Y, iff it is the intersection with Y of a set which is open in X.

13. Let $I = [a, b]$ and $f : I \to R$ be monotone on I. Show that f is integrable on I.

14. Show that a Cauchy sequence is convergent iff it has a convergent subsequence.

15. Let X be a topological space, Y a metric space and A a subspace of X. If f is a continuous mapping of A into Y, show that f can be extended in atmost one way to a continuous mapping of $\overline{A}$ into Y.

16. Show that any intersection of closed sets is closed. Hence show that $\overline{\overline{A}} = \overline{A}$.

17. Let $f : I \to R$ be bounded, P a partition of I and Q a refinement of P. Show that
i) $L(P, f) \leq L(Q, f)$ ii) $U(Q, f) \leq U(P, f)$ $(I = [a, b])$

18. Show that a sequence (f_n) of bounded functions on $A \subseteq R$ converges uniformly on A to f iff $||f_n - f||_A \to 0$.

19. Let (f_n) be a sequence of functions that are integrable on [a, b] and suppose (f_n) converges uniformly on [a, b] to f, show that f is integrable on [a, b] and

$$\int_a^b f(x)dx = \lim \int_a^b f_n(x)dx$$

20. Check for uniform convergence of the sequence $\{f_n\}$ of functions given by

$f_n(x) = \frac{1}{n(1+x^2)}, x \in \mathbb{R}$. **(Wt : 2×7=14)**

Answer **any 3** from the following 5 questions (Wt. **3 each**) :

21. State and prove Cauchy's criterion for uniform convergence.

22. Let R be the set of all real numbers. Define d_1 and d_2 on R by $d_1(x, y) = |x - y|$ and $d_2(x, y) = \frac{|x-y|}{1+|x-y|}$. Show that d_1 and d_2 are metrics.

23. Prove the following :

"If $\{f_n\}$ is a sequence of continuous functions on a set $A \subseteq R$ converging uniformly on A to a function $f : A \to R$, then f is continuous.

Is the statement true if we replace uniform convergence by pointwise convergence ?

24. Let A, B be two subsets of a metric space V. Prove the following :

a) $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$ b) $\text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B)$

Give an example of sets A and B such that $\text{int}(A) \cup \text{int}(B) \neq \text{int}(A \cup B)$.

25. Let X be a non-empty set and define an operation $\mathcal{C}$ on the collection of subsets of X satisfying the following

i) $\mathcal{C}(\phi) = \phi$ ii) $A \subset \mathcal{C}(A)$ where $A \subseteq X$

iii) $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$ where $A \subseteq X$ iv) $\mathcal{C}(A \cup B) = \mathcal{C}(A) \cup \mathcal{C}(B)$, $A, B \subseteq X$

Let $\tau = \{B \subset X;\ \mathcal{C}(X \setminus B) = X \setminus B\}$

Show that τ is a topology on X. With this topology, show that for any $A \subset X$, $\overline{A} = \mathcal{C}(A)$

(Wt : 3×3=9)