M 7150

Reg. No. : ....

Name : ....

V Semester B.Sc. Degree (CCSS – Reg./Supple./Imp.) Examination, November 2014
CORE COURSE IN MATHEMATICS
5B07 MAT : Abstract Algebra

Time: 3 Hours Max. Weightage : 30

1. Mark **each** of the following **true** or **false** :

   a) A binary operation on a set S assigns at least one element of S to each ordered pair of elements of S.

   b) In a group each linear equation has a solution.

   c) Every cyclic group is abelian.

   d) Any group of prime order is cyclic. **(Wt. 1)**

Answer **any six** questions from the following (Weightage **one each**) :

2. If $(a * b)^2 = a^2 * b^2$ for a and b in a group G, show that $a * b = b * a$, where $a^2 = a * a$.

3. Prove that every cyclic group is abelian.

4. Obtain the group of symmetries of an equilateral triangle with vertices 1, 2 and 3.

5. Write the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}$ as a product of cycles.

6. If G is a group and H is a subgroup of G, prove that the relation '~' defined on G by a ~ b if and only if $a^{-1} b \in H$ is an equivalence relation.

P.T.O.

7. If $\varphi : G \to G'$ is a group homomorphism, show that Ker φ is a normal subgroup of G.

8. Prove that a factor group of a cyclic group is cyclic.

9. Define a ring homomorphism. Check whether $\varphi : \mathbb{Z} \to \mathbb{Z}$ defined by $\varphi(x) = 2x$ is a ring homomorphism.

10. Solve the equation $x^2 - 5x + 6 = 0$ in $\mathbb{Z}_{12}$.

11. Show that $\mathbb{Z}_p$ has no zero divisors if p is a prime. **(6×1=6)**

Answer **any seven** questions from the following (Weightage **2 each**) :

12. In a group G, prove that there is only one identity element and also prove that the inverse of every element is unique.

13. Prove that the intersection of two subgroups H and K of a group G is a subgroup of G.

14. If G is a group and $a \in G$, show that $H = \{a^n / n \in \mathbb{Z}\}$ is the smallest subgroup of G that contains 'a'.

15. If H is a subgroup of a finite group G, prove that order of H is a divisor of order of G. Also prove that the order of an element of a finite group divides the order of the group.

16. If A is a nonempty set and S_A is the collection of all permutations of A, prove that S_A is a group under permutation multiplication.

17. Define a homomorphism of a group G into a group G'. If $\varphi : G \to G'$ is a homomorphism of a group G onto a group G' and G is abelian, show that G' is also abelian.

18. Show that the mapping $\varphi : S_n \to \mathbb{Z}_2$ defined by

$$\varphi(\sigma) = \begin{cases} 0 \text{ if } \sigma \text{ is an even permutation} \\ 1 \text{ if } \sigma \text{ is an odd permutation} \end{cases}$$

is a homomorphism, where S_n is the symmetric group of n letters and $\sigma \in S_n$.

19. Prove that a group homomorphism $\varphi : G \to G'$ is a one-to-one map if and only if $\text{Ker } \varphi = \{e\}$, where e is the identity element of G.

20. Prove that the cancellation law hold in a ring R if and only if R has no zero divisors.

21. Show that every field is an integral domain. **(7×2=14)**

Answer **any three** questions from the following (Weightage **3 each**) :

22. Prove that a subgroup of a cyclic group is cyclic.

23. Show that every group is isomorphic to a group of permutations.

24. Prove that no permutation in S_n can be expressed both as a product of even number of transpositions and as a product of an odd number of transpositions.

25. If φ is a homomorphism from a group G into a group G', prove the following :

    i) $\varphi(e)$ is the identity element of G', where e is the identity element of G.

    ii) $\varphi(a^{-1}) = \varphi(a)^{-1}, a \in G$

    iii) $\varphi[H]$ is a subgroup of G', where H is a subgroup of G.

    iv) $\varphi^{-1}[K']$ is a subgroup of G, where K' is a subgroup of G'.

26. Show that every finite integral domain is a field. **(3×3=9)**

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