K19P 0172

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (Reg.) Examination, April 2019
(2017 Admission Onwards)
MATHEMATICS
MAT4C15 : Operator Theory

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. Let X be a normed space over K. If $A, B \in BL(X)$ and $k \neq 0$, then prove that $k \in \sigma(AB)$ if and only if $k \in \sigma(BA)$.

2. $x_n \to^w x$ and $y_n \to^w y$ in a normed space X then show that $x_n + y_n \to^w x + y$.

3. Interpret uniform convexity geometrically.

4. Define numerical range of an operator on a Hilbert space and prove or disprove that it is closed subset of K.

5. Let E be a measurable subset of $\mathbb{R}$ and $H = L^2(E)$. Fix z in $L^\infty(E)$ and define $A(x) = zx$, $x \in H$. Show that A is normal.

6. Let $u_1, u_2, \ldots$ constitute an orthonormal basis for H. Suppose that $A \in BL(H)$ is defined by a matrix M with respect to $u_1, u_2, \ldots$. Assume that M is triangular. Then show that A is normal if and only if M is diagonal. **(4×4=16)**

PART – B

Answer **four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let X be a normed space and $A \in BL(X)$ be of finite rank. Then show that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$.

   b) Let X a Banach space. If $A, B \in BL(X)$, A is invertible and $\epsilon = \|(A-B)A^{-1}\| < 1$, then show that B is invertible, $B^{-1} = A^{-1}\sum_{n=0}^{\infty}\left[(A-B)A^{-1}\right]^n$, $\|B^{-1}\| \leq \frac{\|A^{-1}\|}{1-\epsilon}$ and $\|B^{-1} - A^{-1}\| \leq \frac{\|A^{-1}\|\epsilon}{1-\epsilon}$.

P.T.O.

8. a) State and prove Spectral radius formula.

   b) Let X be a normed space. Then prove that if X′ is separable, so is X.

9. a) Show that the dual of c_0 with the norm $\|.\|_\infty$ is linearly isometric to l^1.

   b) Let X be a normed space and $\{x_n\}$ be a sequence in X. Then prove that $\{x_n\}$ is weak convergent in X if and only if

      i) (x_n) is a bounded sequence in X and

      ii) there is some $x \in X$ such that $x'(x_n) \to x'(x)$ for every x' in some subset of X′ whose span is dense in X′.

Unit – II

10. a) Let X be a Banach space which is uniformly convex in some equivalent norm. Then prove that X is reflexive.

    b) Define compact linear map and give an example.

11. a) Let X and Y be normed spaces and F : ∈ BL(X, Y). If F ∈ CL(X, Y), then prove that $F' \in CL(Y'.X')$. Also prove the converse if Y is a Banach space.

    b) Let X be normed space and A ∈ CL(X), and $0 \neq k \in K$. If (x_n) is a bounded sequence in X such that $A(x_n) - kx_n \to y$ in X, then prove that there is a subsequence (x_n) of (x_n) such that $x_{nj} \to x$ in X and $A(x) - kx = y$.

12. a) Let X be a linear space, $A : X \to X$ linear and $A(x_n) = k_n x_n$ for some $0 \neq x_n \in X$ and $k_n \in K$, n = 1, 2.... Let $k_n \neq k_m$ whenever $n \neq m$. Then prove that $\{x_1, x_2, ...\}$ is linearly independent subset of X.

    b) Let X be a normed space and A ∈ CL(X). Then prove that every eigenspace of A corresponding to a nonzero eigenvalue of A is finite dimensional.

Unit – III

13. a) Define invertible operator. Also give an example of an invertible operator.

    b) Let H be a Hilbert space. Consider A ∈ BL(H). Then prove that $Z(A) = R(A^*)^\perp$ and $Z(A^*) = R(A)^\perp$. Also prove that A is injective if and only if $R(A^*)$ is dense in H, and A^* is injective if and only if R(A) is dense in H.

    c) Define self-adjoint operator and give an example.

14. a) Let H be a Hilbert space. Consider $A \in BL(H)$ and A be self adjoint. Then prove that $\|A\| = \sup\{|\langle A(x), x\rangle| : x \in H, \|x\| \leq 1\}$.

b) Let H be a Hilbert space and (A_n) be a sequence in BL(H) and $A \in BL(H)$ be such that $\|A_n - A\| \to 0$ as $n \to \infty$. If each A_n is self adjoint unitary or normal, then prove that A is self adjoint, unitary or normal respectively.

15. a) Let H be a Hilbert space and $A \in BL(H)$. Then prove that $\sigma_e(A) \subset \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \cup \left\{k : \bar{k} \in \sigma_e(A^*)\right\}$.

b) Let H be a finite dimensional Hilbert space over K and $A \in BL(H)$. Suppose that there is an orthonormal basis for H consisting of eigen values of A. Then prove that A is a normal operator. If $K = \mathbb{R}$, then prove that A is in fact a self adjoint operator. **(4×16=64)**