K23P 0499

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.) Examination, April 2023
(2019 Admission Onwards)
MATHEMATICS
MAT 2C 07 : Measure and Integration

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any 4** questions. **Each** question carries **4** marks.

1. Show that every countable set has measure zero.
2. Define measurable function. Show that every continuous functions are measurable.
3. Let f(x) is function defined on [0, 2] defined by : f(x) = 1 for x rational, if x is irrational, f(x) = –1, then find $\int_0^2 f dx$.
4. If A and B are disjoint measurable sets, then show that $\int_{A \cup B} f dx = \int_A f dx + \int_B f dx$.
5. Show that $L^{\infty}(X, \mu)$ is a vector space over the real numbers.
6. State and prove Minkowski's inequality.

PART – B

Answer **any 4** questions without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Prove that Every interval is measurable.
   b) Define Borel sets. Show that every Borel set is measurable.
8. a) Show that collection of measurable function forms a vector space over real numbers.
   b) Show that Borel set is a proper subset of Lebesgue Measurable sets.
9. a) State and prove Fatou's Lemma.
   b) Let f and g be non-negative measurable functions. Then show that $\int f dx + \int g dx = \int (f + g) dx$.

P.T.O.

Unit – II

10. a) State and prove Lebesgue's Dominated Convergence theorem.

b) Let f be a bounded function defined on the finite interval [a, b], then prove that f is Riemann integrable over [a, b] if and only if it is continuous a.e.

11. a) Let μ^* be an outer measure on $\mathcal{H}(\mathcal{R})$ and let $\mathcal{S}^*$ denote the class of μ^* measurable sets. Then prove that $\mathcal{S}^*$ is a σ ring and μ^* restricted to $\mathcal{S}^*$ is a complete measure.

b) If μ is a σ-finite measure on a ring $\mathcal{R}$, then show that it has a unique extension to the σ-ring $\mathcal{S}(\mathcal{R})$.

12. a) Let f be bounded and measurable on a finite interval [a, b] and let $\in > 0$, then show that there exist a continuous function g such that g vanishes outside a finite interval and $\int_a^b |f - g| dx < \in$.

b) Define σ-finite and complete measure on a ring $\mathcal{R}$. Also show that Lebesgue measure m defined on $\mathcal{M}$, the class of measurable sets of $\mathbb{R}$ is σ-finite and complete.

Unit – III

13. a) Define L^p Space for $1 \le p \le \infty$. Also show that if $\mu(X) < \infty$ and $0 < p < q \le \infty$ then show that $L^q(\mu) \subseteq L^p(\mu)$.

b) State and prove Holder's Inequality. When does its equality occurs ?

14. a) Let f_n be a sequence of measurable functions, $f_n : X \to [0, \infty]$, such that $f_n(x) \uparrow$ for each x and let $f = \lim f_n$ then prove that $\int f dx = \lim \int f_n d\mu$.

b) Let [[X, S, μ]] be a measure space and f a non-negative measurable function. Then prove that $\phi(E) = \int_E f d\mu$ is a measure on the measurable space [[X, S]]. Also show that if $\int f d\mu < \infty$ then $\forall \in > 0, \exists \delta > 0$ such that if $A \in S$ and $\mu(A) < \delta$, then $\phi(A) < \in$.

15. a) If $1 \le p < \infty$ and $\{f_n\}$ is a sequence in $L^P(\mu)$ such that $||f_n - f_m||_p \to 0$ as $n, m \to \infty$ then show that there exists a function f and a sequence $\{n_i\}$ such that $\lim f_{n_i} = f$ a.e. and $f \in L^p(\mu)$.

b) Let f_n be a sequence in $L^\infty(\mu)$ such that $||f_n - f_m|| \to 0$ as $n, m \to \infty$. Then show that there exists a function f such that $\lim f_n = f$ a.e, $f \in L^\infty(\mu)$ and $\lim ||f_n - f||_\infty = 0$.