K21P 1070

Reg. No. : ....

Name : ....

III Semester M.Sc. Degree (CBSS – Reg./Suppl./Imp.) Examination, October 2021
(2018 Admission Onwards)
MATHEMATICS
MAT 3C13 : Complex Function Theory

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks.

1. Prove that the period module of a function f(z) which is nonconstant and meromorphic in the whole plane is discrete.

2. Prove that the sum of residues of an elliptic function is zero.

3. Show that the complex plane $\mathbb{C}$ and the disc $D = \{z : |z| < 1\}$ are homeomorphic.

4. Define
   i) function element
   ii) germ
   iii) analytic continuation along a path.

5. Show that any two harmonic conjugates of a given harmonic function in a simply connected region differ by a constant.

6. Define subharmonic and superharmonic functions. Also state the maximum principle for subharmonic functions. **(4×4=16)**

P.T.O.

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Prove that a discrete module consists either of zero alone, of the integral multiples nw of a single complex number $w \neq 0$, or of all linear combinations $n_1w_1 + n_2w_2$ with integral coefficients of two numbers w_1, w_2 with nonreal ratio w_2/w_1.

   b) If $a_1, a_2, \ldots, a_n$ are zeros and $b_1, b_2, \ldots, b_n$ are poles of an elliptic function in a period parallelogram, prove that $(a_1 + a_2 + \ldots + a_n) - (b_1 + b_2 + \ldots + b_n)$ is a period.

8. a) Define the Weierstrass sigma function $\sigma(z)$ and show that any elliptic function with periods w_1 and w_2 can be written as $C\prod_{k=1}^{n}\frac{\sigma(z-a_k)}{\sigma(z-b_k)}$, where C is a constant.

   b) With usual notations, prove that the Weierstrass P function satisfies the differential equation $P'(z)^2 = 4\,P(z)^3 - g_2\,P(z) - g_3$.

9. a) Show that the series $\sum_{n=1}^{\infty} n^{-z}$ represents an analytic function z in the half plane $\text{Re}z > 1$.

   b) Derive Reimann's functional equation $\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\,\zeta(1-z)\sin\left(\frac{\pi z}{2}\right)$ for $-1 < \text{Re}z < 0$.

   c) State and prove Euler's theorem.

Unit – II

10. State and prove Runge's theorem.

11. a) State and prove Mittag-Leffler's theorem.

    b) Find a meromorphic function in the plane with a pole at every integer.

12. a) With usual assumptions, when is a function element (f, D) said to admit unrestricted analytic continuation in G ? Also state and prove the monodromy theorem.

    b) Let (f, D) be a function element which admits unrestricted continuation in the simply connected region G. Prove that there is an analytic function $F : G \to \mathbb{C}$ such that $F(z) = f(z)$ for all z in D.

Unit – III

13. a) If $u : G \to \mathbb{C}$ is harmonic, then prove that u is infinitely differentiable.

    b) Let G be a region and suppose that u is a continuous real valued function on G with the MVP. If there is a point a in G such that $u(a) \geq u(z)$ for all z in G, then prove that u is a constant function.

    c) Define the Poisson Kernel $P_r(\theta)$. Prove that

    i) $P_r(\theta) = \text{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1-2r\cos\theta + r^2}$

    ii) $\int_{-\pi}^{\pi} P_r(\theta)d\theta = 2\pi$.

14. a) If $u : G \to \mathbb{R}$ is a continuous function which has the MVP, then prove that u is harmonic.

    b) State and prove Harnack's theorem.

15. a) Let G be a region and $f : \partial_\infty G \to \mathbb{R}$ be a continuous function. Prove that $u(z) = \sup\{\varphi(z) : \varphi \in P(f, G)\}$ defines a harmonic function u on G.

    b) Derive Jensen's formula. **(4×16=64)**