K23P 3298

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree (CBSS-Supple. (One Time Mercy Chance)/Imp.) Examination, October 2023
(2017 to 2022 Admissions)
MATHEMATICS
MAT1C04 : Basic Topology

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries 4 marks.

1. Let A be a subset of a topological space $(X, \mathcal{T})$. Prove that $\overline{A} = A \cup A'$.

2. Let A and B be subsets of a topological space $(X, \mathcal{T})$. Prove that
   i) A is open if and only if A = int (A).
   ii) int (A) $\cup$ int (B) $\subseteq$ int $(A \cup B)$.

3. Let A be a subset of a topological space $(X, \mathcal{T})$. Prove that $\mathcal{T}_A$ is a topology on A.

4. Prove that every subspace of a separable metric space is separable.

5. Prove that a topological space $(X, \mathcal{T})$ is connected if and only if it cannot be expressed as the union of two nonempty sets that are separated in X.

6. Prove that a topological space is locally pathwise connected if and only if each path component of each open set is open.

PART – B

Answer **four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Prove that a family $\mathcal{B}$ of subsets of a set X is a basis for some topology on X if and only if :
   i) $X = \cup\{B : B \in \mathcal{B}\}$ and
   ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq B_1 \cap B_2$.

P.T.O.

b) Let $X = \{1, 2, 3\}$ and let $\mathscr{B} = \{\{2\}, \{1, 2\}, \{1, 3\}\}$ be a basis for $\mathscr{T}$. Find $\mathscr{T}$.

c) Let X be a set and let $\mathscr{S}$ be a collection of subsets of X such that $X = \cup\{S : S \in \mathscr{S}\}$. Prove that there is a unique topology $\mathscr{T}$ on X such that $\mathscr{S}$ is a sub-basis for $\mathscr{T}$.

8. a) Let (X, d) be a metric space, and define $\bar{d} : X \times X \to \mathbb{R}$ by $\bar{d}(x, y) = \min\{d(x, y), 1\}$. Prove that $\bar{d}$ is a metric on X and the topology induced by $\bar{d}$ is the topology induced by d.

b) Prove that every separable metric space is second countable.

c) Give an example of a separable space which is not second countable.

9. a) Let A be a subset of a metric space (X, d). Prove that the following statements are equivalent :

i) A is nowhere dense.

ii) If U is nonempty open subset of X, then there exists a nonempty open set V such that $V \subset U$ and $V \cap \bar{A} = \emptyset$.

iii) Every nonempty open set in X, contains an open ball whose closure is disjoint from A.

b) State and prove Baire Category theorem.

Unit – II

10. a) Let A be a closed subset of a topological space $(X, \mathscr{T})$. If C is closed in $(A, \mathscr{T}_A)$, prove that C is closed in $(X, \mathscr{T})$.

b) State and prove Pasting lemma.

c) Prove that the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, 0)$ is an embedding of $\mathbb{R}$ in $\mathbb{R}^2$.

11. a) Let (X_1, d_1) and (X_2, d_2) be metric spaces, for each $i = 1, 2$ let $\mathcal{T}_i$ be the topology on X_i generated by d_i and let $\mathcal{T}$ denote the product topology on $X = X_1 \times X_2$. Furthermore let $\mathcal{U}$ denote the topology on X generated by the product metric d. Prove that $\mathcal{T} = \mathcal{U}$.

b) Let $(X, \mathcal{T})$, $(Y_1, \mathcal{U}_1)$ and $(Y_2, \mathcal{U}_2)$ be topological spaces and let $f : X \to Y_1 \times Y_2$ be a function. Prove that f is continuous if and only if $\pi_i \circ f$ is continuous for each $i = 1, 2$.

c) Prove that the function $h : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $h(x) = (x^2 + 4, x^3 - 2x + 6)$ is continuous.

12. a) Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be an indexed family of topological spaces, and for each $\alpha \in \Lambda$, let $(A_\alpha, \mathcal{T}_{A_\alpha})$ be a subspace of $(X_\alpha, \mathcal{T}_\alpha)$. Then prove that the product topology on $\prod_{\alpha \in \Lambda} A_\alpha$ is the same as the subspace topology on $\prod_{\alpha \in \Lambda} A_\alpha$ is determined by the product topology on $\prod_{\alpha \in \Lambda} X_\alpha$.

b) Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a collection of topological spaces, let X be a set, and for each $\alpha \in \Lambda$ let $f_\alpha : X \to X_\alpha$ be a function. Let $\mathcal{T}$ be the weak topology on X iduced by $\{f_\alpha : \alpha \in \Lambda\}$ and let $(Y, \mathcal{U})$ be a topological space. Prove that a function $f : Y \to X$ is continuous if and only if for each $\alpha \in \Lambda$, $f_\alpha \circ f : Y \to X_\alpha$ is continuous.

Unit – III

13. a) Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Prove that the following conditions are equivalent :

i) The subspace $(A, \mathcal{T}_A)$ is connected.

ii) The set A cannot be expressed as the union of two nonempty sets that are separated in X.

iii) There does not exist $U, V \in \mathcal{T}$ such that $U \cap A \neq \varnothing$, $V \cap A \neq \varnothing$, $U \cap V \cap A = \varnothing$ and $A \subseteq U \cap V$.

b) Prove that $(\mathbb{R}, \mathcal{T})$ is connected, where $\mathcal{T}$ is the usual topology on $\mathbb{R}$.

c) Prove that $I = [0, 1]$ has the fixed point property.

14. a) Prove that the topologist's sine curve is connected but not pathwise connected.

b) Give an example of a pathwise connected space which is not locally connected.

15. a) Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a collection of topological spaces, and suppose that for each $\alpha \in \Lambda$, $X_\alpha \neq \emptyset$. Let $X = \prod_{\alpha \in \Lambda} X_\alpha$ and let $\mathcal{T}$ be the product topology on X. Prove that $(X, \mathcal{T})$ is connected if and only if, for each $\alpha \in \Lambda$, $(X_\alpha, \mathcal{T}_\alpha)$ is connected.

b) Define Cantor set. Prove that the Cantor set is totally disconnected.