K24P 0863

Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree (C.B.S.S. – Supple. (One Time Mercy Chance)/Imp.) Examination, April 2024
(2017 to 2022 Admissions)
MATHEMATICS
MAT2C08 : Advanced Topology

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Define compact topological space. Give an example.
2. Define uniformly continuous functions on metric spaces. Give an example of a continuous but not uniformly continuous function with justification.
3. Show by an example that the open continuous image of a Hausdorff space need not be Hausdorff.
4. Prove that the Moor plane is not normal.
5. Define homotopy. Give an example.
6. Let (X, τ) be a topological space and let $x_0 \in X$. Furthermore, let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Omega(X, x_0)$ and suppose $\alpha_1 \simeq_p \alpha_2$ and $\beta_1 \simeq_p \beta_2$. Then prove that $\alpha_1 * \beta_1 \simeq_p \alpha_2 * \beta_2$.

PART – B

Answer **any four** questions from this Part without omitting any **Unit**. **Each** question carries **16** marks. **(4×16=64)**

Unit – I

7. a) State the Bolzano-Weierstrass property on a topological space. Let (X, d) be a metric space that has the Bolzano-Weierstrass property. Then prove that (X, d) is totally bounded.

b) Define T_1 space. Let (X, τ) be a T_1 space, let $A \subseteq X$ and let p be a limit point of A. Then prove that every neighbourhood of p contains an infinite number of distinct members of A.

c) Let (X, τ) be a T_1 – space. Then prove that X is countably compact if and only if it has the Bolzano-Weierstrass property.

8. a) Prove that every closed subset of a compact space is compact.

b) Prove that a topological space (X, τ) is compact if and only if every family of closed subsets of X with the finite intersection property has a nonempty intersection.

c) Let (X, τ) be a topological space and let B be a basis for τ. Then prove that (X, τ) is compact if and only if every cover of X by members of B has a finite subcover.

9. a) With detailed explanation, give an example of a countably compact topological space which is not compact.

b) Prove that every closed subspace of locally compact Haus-dorff space is locally compact.

c) With suitable example, show that the continuous image of a locally compact space need not be locally compact.

Unit – II

10. a) Let (X, τ) be a topological space. Then prove that the following statements are equivalent.

   i) (X, τ) is a T_1 space.

   ii) For each $x \in X$, $\{x\}$ is closed.

   iii) If A is any subset of X, Then $A = \cap \{ U \in F : A \subseteq U\}$.

b) Let (X, τ) be a topological space, let (Y, U) be a Hausdorff space and let $f : X \to Y$ be continuous. Then prove that $\{(x_1, x_2) \in X \times X : f(x_1) = x_2\}$ is a closed subset of $X \times X$.

c) Prove that completely regular is a topological property.

11. a) Prove that a Hausdorff space (X, τ) is locally compact if and only if for each $p \in X$ and each neighborhood V of p there is a neighborhood U of p such that $\bar{U}$ is compact and $\bar{U} \subseteq V$.

b) Prove that every subspace of regular space is regular.

c) Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be a family of topological spaces, and let $X = \prod_{\alpha \in \Lambda} X_\alpha$. Then prove that (X, τ) is regular if and only if (X_α, τ_α) is regular for each $\alpha \in \Lambda$.

12. a) Let (X, τ) be a topological space with a dense subset D and a closed, relatively discrete subset C such that $P(D) \leq C$. Then prove that (X, τ) is not normal.

b) Prove that every second countable regular space is normal.

Unit – III

13. By proving the necessary lemmas, show that every normal space is completely regular.

14. State and prove Alaxander Subbase theorem.

15. a) Let (X, τ) be a topological space and let D be a dense subset of I. Suppose that for each $t \in D$, there is an open set U_t in X such that :

1) if $t_1 < t_2$ then $\overline{U_{t_1}} \subseteq U_{t_2}$ and

2) $X = \cup_{t \in D} U_t$. Define $f : X \to I$ by $f(x) = \text{glb}\{t \in D : x \in U_t\}$ for each $x \in X$. Then prove that f is continuous.

b) Let (X, d) be a compact metric space, let (Y, U) be a Hausdorff space and let $f : X \to Y$ be a continuous function that maps X onto Y. Then prove that (Y, U) is metrizable.