K24P 0861

Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree (C. B. S. S. – Supple. (One Time Mercy chance)/Imp.) Examination, April 2024
(2017 to 2022 Admission)
MATHEMATICS
MAT 2C 06 : Advanced Abstract Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. Show that $\alpha = 1 + i$ is algebraic over $\mathbb{Q}$ by finding $f(x) \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$.
2. Prove that an ideal $\langle p \rangle$ in a PID is maximal if and only if p is an irreducible.
3. Let E be an algebraic extension of a field F. Prove that there exist a finite number of elements $\alpha_1, \ldots, \alpha_n \in E$ such that $E = F(\alpha_1, \ldots, \alpha_n)$ if and only if E is a finite extension of F.
4. Prove that doubling the cube is impossible.
5. Show that for a prime p, the splitting field over $\mathbb{Q}$ of $x^p - 1$ is of degree $p - 1$ over $\mathbb{Q}$.
6. If E is a finite extension of F, prove that $\{E : F\}$ divides $[E : F]$. **(4×4=16)**

PART – B

Answer **four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let F be a field and let f(x) be a nonconstant polynomial in F[x]. Prove that there exist an extension field E of F and an $\alpha \in E$ such that $f(\alpha) = 0$.

   b) Let $\alpha = \sqrt{2} + i$. Find $irr(\alpha, \mathbb{Q})$ and $deg(\alpha, \mathbb{Q})$ for the algebraic number $\alpha \in \mathbb{C}$.

P.T.O.

8. a) State and prove Gauss's lemma.
   b) If D is a UFD, prove that D[x] is a UFD.

9. a) State and prove Euclidean algorithm.
   b) Prove that the norm function N hold following properties for $\alpha, \beta \in \mathbb{Z}[i]$:
      i) $N(\alpha) \geq 0$
      ii) $N(\alpha) = 0$ if and only if $\alpha = 0$
      iii) $N(\alpha\beta) = N(\alpha)N(\beta)$.

Unit – II

10. a) If E is a finite extension field of a field F and K is a finite extension field of E, prove that K is a finite extension of F and [K : F] = [K : E] [E : F].
    b) Prove that a field F is algebraically closed if and only if every nonconstant polynomial in F[x] factors in F[x] into linear factors.
    c) Prove that $\mathbb{Q}\left(\sqrt{3}+\sqrt{7}\right) = \mathbb{Q}\left(\sqrt{3}, \sqrt{7}\right)$.

11. a) Let α and β be two constructible numbers. Prove that $\alpha + \beta$, $\alpha - \beta$, $\alpha\beta$ and α/β if $\beta \neq 0$ are constructible.
    b) Prove that trisecting an angle is impossible.

12. a) Prove that a finite field $GF(p^n)$ of p^n elements exists for every prime power p^n.
    b) Prove that the set of all automorphisms of $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$ leaving $\mathbb{Q}$ fixed is isomorphic to Klein 4-group.

Unit – III

13. State and prove isomorphism extension theorem.

14. a) Prove that a field E, where $F \leq E \leq \overline{F}$, is a splitting field if and only if every automorphism of $\overline{F}$ leaving F fixed maps E onto itself and thus induces an automorphism of E leaving F fixed.
    b) State main theorem of Galois theory.

15. a) If E is a finite extension of F, prove that E is separable over F if and only if each α in E is separable over F.
    b) Prove that every finite field is perfect. **(4×16=64)**