K24P 0864

Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree (C.B.S.S. – Supple. (One Time Mercy Chance)/Imp.) Examination, April 2024
(2017 to 2022 Admissions)
MATHEMATICS
MAT 2C 09 : Foundations of Complex Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Attempt **any four** questions from this part. **Each** question carries **4** marks :

1. Given that γ and σ are closed rectifiable curves having the same initial points. Prove that $n(\gamma + \sigma, a) = n(\gamma, a) + n(\sigma, a)$ for every $a \notin \{\gamma\} \cup \{\sigma\}$.

2. Let f be analytic on $B(0, 1)$ and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show that $|f'(0)| \leq 1$.

3. Does the function $f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$ has an essential singularity at $z = 0$? Justify your answer.

4. Using residue Theorem, prove that $\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$.

5. Define the set $C(G, \Omega)$ and show that it is non-empty.

6. State the Weierstrass Factorization theorem.

PART – B

Answer **any four** questions from this part without omitting any Unit. **Each** question carries **16** marks :

Unit – I

7. a) Prove the following : If G is simply connected and $f : G \to C$ is analytic in G then f has a primitive in G.

   b) State and prove The Open Mapping Theorem.

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8. State and prove the Third Version of Cauchy's Theorem.

9. Prove the following : let G be a connected open set and let $f : G \to C$ be an analytic function. Then the following conditions are equivalent.

   a) $f \equiv 0$;

   b) there is a point a in G such that $f^n(a) = 0$ for each $n \geq 0$;

   c) $\{z \in G : f(z) = 0\}$ has a limit point in G.

Unit – II

10. a) Show that for $a > 1$, Show that $\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}}$.

    b) State and prove the Residue theorem.

11. State and prove the Laurent Series Development.

12. Prove the following :

    a) If $|a| < 1$ then $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ is a one-one map of $D = \{z : |z| < 1\}$ on to itself ; the inverse of φ_a is φ_{-a}. Furthermore, φ_a maps ∂D on to ∂D, $\varphi'_a(0) = 1 - |a|^2$ and $\varphi'_a(a) = (1-|a|^2)^{-1}$.

    b) Let $f(z) = \frac{1}{z(z-1)(z-2)}$; give the Laurent series of f(z) in each of the following annuli :

       i) ann(0 ; 0, 1),

       ii) ann (0 ; 1, 2),

       iii) ann (0 ; 2, ∞).

Unit – III

13. a) Prove the following : If G is open in C then there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \cup_{n=1}^{\infty} K_n$. Moreover the sets K_n can be chosen to satisfy the following conditions :

       i) $k_n \subset \text{int } K_{n+1}$.

       ii) $K \subset G$ and K is compact implies $K \subset K_n$ for some n.

       iii) Every component of $C_\infty - K_n$ contains a component of $C_\infty - G$.

    b) State and prove Hurwitz's theorem.

14. a) With the usual notations, prove that $|1 - E_p(z)| \le |z|^{p+1}$ for $|z| \le 1$ and $p \ge 0$.

b) Discuss the convergence of the infinite product $\prod_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 0$.

15. a) Show that $\prod(1+z_n)$ converges absolutely iff $\prod(1+|z_n|)$ converges.

b) Prove the following : If $\text{Re} z_n > 0$ then the product $\prod z_n$ converges absolutely iff the series $\Sigma(z_n - 1)$ converges absolutely.

c) Prove the following : Let $\text{Re} z_n > 0$ for all $n \ge 1$. Then $\prod_{n=1}^{\infty} z_n$ converges to a non zero number iff the series $\sum_{n=1}^{\infty} \log z_n$ converges.