K22P 1604

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (CBSS – Reg./Sup./Imp.) Examination, October 2022
(2019 Admission Onwards)
MATHEMATICS
MAT1C04 : Basic Topology

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Prove that every 0-dimensional T_0 space is totally disconnected.

2. Let X be a set with at least two members and let $\mathcal{T}$ be the trivial topology on X. Then show that (X, $\mathcal{T}$) is not metrizable.

3. Define usual topology and lower limit topology on $\mathbb{R}$.

4. Let (X, $\mathcal{T}$) be a topological space, let A be a subset of X and let $\mathcal{B}$ be a basis for $\mathcal{T}$. Then prove that $\{B \cap A : B \in \mathcal{B}\}$ is a basis for the subspace topology on A.

5. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be Hausdorff spaces and let $\mathcal{T}$ be the product topology on $X = X_1 \times X_2$. Then prove that (X, $\mathcal{T}$) is a Hausdorff space.

6. Examine whether $\mathbb{R} - \{0\}$ with usual topology is connected or not.

PART – B

Answer **any four** questions from this Part without omitting any Unit. **Each** question carries **16** marks. **(4×16=64)**

Unit – I

7. a) Let d be the usual metric for $\mathbb{R}^n$. Then show that

   $A = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n$: for each i = 1, 2, ..., n, x_i is rational$\}$ is a countable dense subset of $\mathbb{R}^n$.

   b) Prove that every complete metric space is of the second category.

P.T.O.

c) Let $(X, \mathcal{T})$ be a topological space, let (Y, d) be a metric space, let $f : X \to Y$ be a function and for each $n \in \mathbb{N}$, let $f_n : X \to Y$ be a continuous function such that the sequence $\langle f_n \rangle$ converges uniformly to f. Then prove that f is continuous.

8. a) Prove that a family $\mathcal{B}$ of subsets of a set X is a basis for some topology on X if and only if : (1) $X = \cup \{B : B \in \mathcal{B}\}$ and (2) if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq B_1 \cap B_2$.

b) Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on a set X and let $\mathcal{B}$ and $\mathcal{B}'$ be bases for $\mathcal{T}$ and $\mathcal{T}'$ respectively. Then prove that the following conditions are equivalent :

i) $\mathcal{T}'$ is finer than $\mathcal{T}$.

ii) For each $x \in X$ and each $B \in \mathcal{B}$ such that $x \in B$, there is a member B' of $\mathcal{B}'$ such that $x \in B'$ and $B' \subseteq B$.

c) Show that the lower-limit topology on $\mathbb{R}$ is not the usual topology on $\mathbb{R}$.

9. a) Let A be a subset of a topological space $(X, \mathcal{T})$, and let $x \in X$. Then prove that $x \in \bar{A}$ if and only if every neighborhood of x has a nonempty intersection with A.

b) Let A be a subset of a topological space $(X, \mathcal{T})$. Then prove that $\bar{A} = A \cup A'$.

c) Prove that every second countable space is separable.

Unit – II

10. a) Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be an indexed family of topological spaces, and for each $\alpha \in \Lambda$, let $(A_\alpha, \mathcal{T}_{A\alpha})$ be a subspace of $(X_\alpha, \mathcal{T}_\alpha)$. Then prove that the product topology on $\prod_{\alpha \in \Lambda} A_\alpha$ is the same as the subspace topology on $\prod_{\alpha \in \Lambda} A_\alpha$ determined by the product topology on $\prod_{\alpha \in \Lambda} X_\alpha$.

b) Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be an indexed family of first countable spaces, and let $X = \prod_{\alpha \in \Lambda} X_\alpha$. Then prove that $(X, \mathcal{T})$ is first countable if and only if $\mathcal{T}_\alpha$ is the trivial topology for all but a countable number of α.

11. a) Let $(A, \mathcal{T}_A)$ be a subspace of a topological space $(X, \mathcal{T})$. Prove that a subset C of A is closed in $(A, \mathcal{T}_A)$ if and only there is a closed subset D of $(X, \mathcal{T})$ such that $C = A \cap D$.

b) Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, let $f : X \to Y$ be a function, and let $\{U_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets of X such that

$X = \bigcup_{\alpha \in \Lambda} U_\alpha$ and $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for each $\alpha \in \Lambda$. Then prove that f is continuous.

c) Prove that the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, 0)$ for each $x \in \mathbb{R}$ is an embedding of $\mathbb{R}$ in $\mathbb{R}^2$.

12. a) Let $(X, \mathcal{T})$, $(Y_1, \mathcal{U}_1)$ and $(Y_2, \mathcal{U}_2)$ be topological spaces and let $f : X \to Y_1 \times Y_2$ be a function. Then prove that f is continuous if and only if $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

    b) Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be Hausdorff spaces, and let $\mathcal{T}$ denote the product topology on $X = X_1 \times X_2$. Then prove that $(X, \mathcal{T})$ is Hausdorff.

    c) Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces, and let π_1 and π_2 denote the projection maps. Then prove that $S = \left\{ \pi_1^{-1}(U) : U \in \mathcal{T}_1 \right\} \cup \left\{ \pi_2^{-1}(V) : V \in \mathcal{T}_2 \right\}$ is a subbasis for the product topology on $X_1 \times X_2$.

Unit – III

13. a) Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a collection of topological spaces, and let $\mathcal{T}$ be the product topology on $X = \prod_{\alpha \in \Lambda} X_\alpha$. Then prove that $(X, \mathcal{T})$ is locally connected if and only if for each $\alpha \in \Lambda$, $(X_\alpha, \mathcal{T}_\alpha)$ is locally connected and for all but a finite number of $\alpha \in \Lambda$, $(X_\alpha, \mathcal{T}_\alpha)$ is connected.

    b) Prove that a topological space $(X, \mathcal{T})$ is locally connected if and only if each component of each open set is open.

    c) Let $(X, \mathcal{T})$ be a topological space and suppose $X = A \cup B$, where A and B are nonempty subsets that are separated in X. If H is a connected subspace of X, then prove that $H \subseteq A$ or $H \subseteq B$.

14. a) Let $(X, \mathcal{T})$ be a topological spaces and let $A \subseteq X$. Then prove that the following conditions are equivalent :

    i) The subspace $(A, \mathcal{T}_A)$ is connected.

    ii) The set A cannot be expressed as the union of two nonempty sets that are separated in X.

    iii) There do not exist $U, V \in \mathcal{T}$ such that $U \cap A \neq \varnothing$, $V \cap A \neq \varnothing$, $U \cap V \cap A \neq \varnothing$ and $A \subseteq U \cup V$.

    b) Prove that the closed unit interval I has the fixed-point property.

    c) Let $(X, \mathcal{T})$ be a topological space and suppose $X = A \cup B$, where A and B are nonempty subsets that are separated in X. If H is a connected subspace of X, then prove that $H \subseteq A$ or $H \subseteq B$.

15. a) Prove that each path component of a topological space is pathwise connected.

    b) Show that the topologist's sine curve is not pathwise connected.

    c) Define path product of two paths in a topological space.
