K21P 0784

Reg. No. : ...............................

Name : ...................................



# II Semester M.Sc. Degree (CBSS – Reg./Suppl. (Including Mercy Chance)/Imp.) Examination, April 2021
## (2017 Admission Onwards)
## MATHEMATICS
## MAT 2C 07 : Measure and Integration

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks.

1. Show that if F is measurable and $m^*(F\Delta G) = 0$, then G is measurable.
2. Show that the Lebesgue measure of the set of irrationals in [0, 1] is 1.
3. Prove that outer measure is translation invariant.
4. If $f_n(x) = \frac{\log(x+n)}{n} e^{-x} \cos x$, then show that $\int_0^1 f_n(x)dx = 0$.
5. Let A, B be subsets of a set C, let A, B, C $\in \mathscr{R}$ and let $\mu$ be a measure on $\mathscr{R}$. Show that if $\mu(A) = \mu(C) < \infty$, then $\mu(A \cap B) = \mu(B)$.
6. Let $p > 0$ and $f \in L^p(\mu)$ where $f \geq 0$, and let $f_n = \min(f, n)$. Show that $f_n \in L^p(\mu)$ and $\lim \|f_n - f\|_p = 0$. **(4×4=16)**

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

UNIT – I

7. a) For any sequence of sets $\{E_i\}$, prove that $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$.

   b) Show that, for any set A and any $\varepsilon > 0$, there is an open set O containing A and such that $m^*(O) \leq m^*(A) + \varepsilon$.

   c) If $m^*(E) < \infty$ then prove that E is measurable if and only if, $\forall\, \varepsilon > 0$, $\exists$ disjoint finite intervals, $I_1, I_2, \ldots, I_n$ such that $m^*\left(E \Delta \bigcup_{i=1}^{n} I_i\right) < \varepsilon$.

P.T.O.

8. a) Prove that the class of Lebesgue measurable sets $\mathscr{M}$ is a $\sigma$-algebra.

   b) Let $E \subseteq M$ where M is measurable and $m(M) < \infty$. Show that E is measurable if and only if $m(M) = m^*(E) + m^*(M - E)$.

9. a) Prove that not every measurable set is a Borel set.

   b) Let f be a non negative measurable function. Then prove that there exists a sequence $\{\varphi_n\}$ of simple functions such that, for each x, $\varphi_n(x) \uparrow f(x)$.

UNIT – II

10. a) Define an integrable function. Prove that if f and g are integrable then f + g is integrable and $\int f\,dx + \int g\,dx = \int (f + g)\,dx$.

    b) Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \uparrow f$. Show that $\int f\,dx = \lim \int f_n dx$.

    c) Let f be a non negative integrable function on [0, 1]. Then prove that there exists a measurable function $\varphi(x)$ such that $\varphi f$ is integrable on [0, 1] and $\varphi(0 +) = \infty$.

11. a) Let f be a bounded function defined on the finite interval [a, b], then prove that if f is Riemann integrable over [a, b] if, and only if, it is continuous a.e.

    b) Let f be bounded and measurable on a finite interval [a, b] and let $\varepsilon > 0$. Then prove that there exist

       (i) a step function h such that $\int_a^b |f - h|\,dx < \varepsilon$.

       (ii) a continuous function g such that g vanishes outside a finite interval and $\int_a^b |f - g|\,dx < \varepsilon$.

12. a) Let $\mu^*$ be an outer measure on $\mathscr{H}(\mathscr{R})$ and let $S^*$ denote the class of $\mu^*$ measurable sets. Then prove that $S^*$ is a $\sigma$-ring and $\mu^*$ restricted to $S^*$ is a complete measure.

    b) Prove that the outer measure $\mu^*$ on $\mathscr{H}(\mathscr{R})$ defined $\mu$ on R and the corresponding outer measure defined by $\bar{\mu}$ on $S(\mathscr{R})$ and $\bar{\mu}$ on $S^*$ are the same.

UNIT – III

13. a) Let E and F be measurable sets, $f \in L(E)$ and $\mu(E \Delta F) = 0$ then prove that $f \in L(F)$ and $\int_E f = \int_F f$.

b) Let f be a measurable function and let $f = g$ a.e. $(\mu)$, where $\mu$ is a complete measure. Then prove that g is measurable. Further show that complete of $\mu$ is necessary.

c) Let $[\![X, S, \mu]\!]$ be a measure space and f a non negative measurable function. Then prove that $\phi(E) = \int_E f\, d\mu$ is a measure on the measurable space $[\![X, S]\!]$. Further prove that, if $\int f\, d\mu < \infty$, then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that, if $A \in S$ and $\mu(A) < \delta$, then $\phi(A) < \varepsilon$.

14. a) State and prove Holder's inequality. State and prove necessary and sufficient condition for equality occurs in Holder's inequality.

b) Let $0 < p < 1$ and $f \geq 0$, $g \geq 0$, $f, g \in L^P(\mu)$. Show that $||f + g||_p \geq ||f||_p + ||g||_p$.

15. a) Prove that if $1 \leq p < \infty$ and $\{f_n\}$ is a sequence in $L^p(\mu)$ such that $||f_n - f_m||_p \to 0$ as $n, m \to \infty$, then there exists a function f and a sequence $\{n_i\}$ such that $\lim f_{n_i} = f$ a.e. Further prove that $f \in L^p(\mu)$ and $||f_n - f||_p \to 0$.

b) Prove that if $\{f_n\}$ is a sequence in $L^\infty(\mu)$ such that $||f_n - f_m||_\infty \to 0$ as n, $m \to \infty$, then there exists a function f such that $\lim f_n = f$ a.e., $f \in L^\infty(\mu)$ and $\lim ||f_n - f||_\infty = 0$. **(4×16=64)**